KONTINUUMMECHANIKA

Képletgyüjtemény

V. éves mérnök matematikus és doktorandusz hallgatóknak

Összeállította
Kozák Imre Kontinuummechanika című jegyzetéhez
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Alakváltozási tenzorok

Mozgásfüggvény | *Inverz mozgásfüggvény*

$$x^p = x^p(x^{o1}, x^{o2}, x^{o3}; t) \qquad x^{op} = x^{op}(x^1, x^2, x^3; t)$$

$$J_{x,x^o} = \left| \frac{\partial x^l}{\partial x^{ok}} \right| > 0$$

$$g^p_{\overset{o}{m}} = \mathbf{g}^p \cdot \mathbf{g}^o_m \qquad g^{\overset{o}{m}}_k = \mathbf{g}^{om} \cdot \mathbf{g}_m$$

$$\mathbf{r}(x^{o1}, x^{o2}, x^{o3}; t) = \mathbf{r}^o + \mathbf{u}^o(x^{o1}, x^{o2}, x^{o3}; t) \quad \mathbf{r}^o(x^1, x^2, x^3; t) = \mathbf{r} - \mathbf{u}(x^1, x^2, x^3; t)$$

Alakváltozási gradiens

$$\mathbf{F} = \mathbf{r} \otimes \nabla^o = F^p_{\overset{o}{k}} \mathbf{g}_p \otimes \mathbf{g}^{ok} = \mathbf{G}_L \otimes \mathbf{G}^{oL} = \mathbf{R} \cdot \mathbf{U}^o = \mathbf{V} \cdot \mathbf{R}$$

$$F^p_{\overset{o}{k}} = \frac{\partial x^p}{\partial x^{ok}} = R^p_{\overset{o}{l}} U^{ol}_{\ \ k} = g^p_{\overset{o}{m}} R^{om}_{\ \ l} U^{ol}_{\ \ k} = V^p_{\ l} R^l_{\ \overset{o}{k}} = V^p_{\ l} R^l_{\ m} g^m_{\overset{o}{k}}$$

$$\mathbf{F} = F^{\overset{o}{p}}_{\overset{o}{k}} \mathbf{g}^o_p \otimes \mathbf{g}^{ok} = \mathbf{I} + \mathbf{u}^o \otimes \nabla^o$$

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{r}^o \qquad dx^p = F^p_{\overset{o}{k}} dx^{ok}$$

Inverz alakváltozási gradiens

$$\mathbf{F}^{-1} = \mathbf{r}^o \otimes \nabla = \left(F^{-1}\right)^{\overset{o}{p}}_{\ \ k} \mathbf{g}^o_p \otimes \mathbf{g}^k = \mathbf{G}^o_L \otimes \mathbf{G}^L$$

$$\left(F^{-1}\right)^{\overset{o}{p}}_{\ \ k} = \frac{\partial x^{op}}{\partial x^k}$$

$$\mathbf{F}^{-1} = \left(F^{-1}\right)^{\overset{0}{p}}_{\ \ k} \mathbf{g}_{\overset{o}{p}} \otimes \mathbf{g}^k = \mathbf{I} - \mathbf{u} \otimes \boldsymbol{\nabla}$$

$$d\mathbf{r}^o = \mathbf{F}^{-1} \cdot d\mathbf{r} \qquad dx^{op} = \left(F^{-1}\right)^{\overset{o}{p}}_{\ \ k} dx^k$$

Jobboldali Cauchy Green | *Cauchy*

féle alakváltozási tenzor

$$\mathbf{C}^o = C^o_{kl} \mathbf{g}^{ok} \otimes \mathbf{g}^{ol} = \mathbf{F}^T \cdot \mathbf{F} = (\mathbf{U}^o)^2 \qquad \mathbf{B}^{-1} = (B)_{kl} \mathbf{g}^k \otimes \mathbf{g}^l = \left(\mathbf{F}^{-1}\right)^T \cdot \mathbf{F}^{-1} = \mathbf{V}^2$$

$$C^o_{kl} = F^p_{\overset{o}{k}} F_{p\overset{o}{l}} = \frac{\partial x^p}{\partial x^{ok}} \frac{\partial x^q}{\partial x^{ol}} g_{pq} \qquad \left(B^{-1}\right)_{kl} = \left(F^{-1}\right)^{\overset{o}{p}}_{\ \ k} \left(F^{-1}\right)_{\overset{o}{p}l} = \frac{\partial x^{op}}{\partial x^k} \frac{\partial x^{oq}}{\partial x^l} g^o_{pq}$$

$$d\mathbf{r}_i \cdot d\mathbf{r}_j = d\mathbf{r}^o_i \cdot \mathbf{C}^o \cdot d\mathbf{r}^o_j \qquad d\mathbf{r}^o_i \cdot d\mathbf{r}^o_j = d\mathbf{r}_i \cdot \mathbf{B}^{-1} \cdot d\mathbf{r}_j$$

Lagrange | *Euler*

féle alakváltozási tenzor

$$\mathbf{E}^o = E^o_{kl} \mathbf{g}^{ok} \otimes \mathbf{g}^{ol} = \frac{1}{2} (\mathbf{C}^o - \mathbf{I}) \qquad \mathbf{E} = E_{kl} \mathbf{g}^k \otimes \mathbf{g}^l = \frac{1}{2} \left(\mathbf{I} - \mathbf{B}^{-1}\right)$$

$$\mathbf{E}^o = \frac{1}{2} \left[(\nabla^o \otimes \mathbf{u}^o) + (\mathbf{u}^o \otimes \nabla^o) + (\nabla^o \otimes \mathbf{u}^o) \cdot (\mathbf{u}^o \otimes \nabla^o)\right]$$

$$\mathbf{E} = \frac{1}{2} \left[(\nabla \otimes \mathbf{u}) + (\mathbf{u} \otimes \nabla) - (\nabla \otimes \mathbf{u}) \cdot (\mathbf{u} \otimes \nabla)\right]$$

$$E^o_{kl} = \frac{1}{2} \left(u^o_{k;l} + u^o_{l;k} + g^{omn} u^o_{n;k} u^o_{n;l}\right) \qquad E_{kl} = \frac{1}{2} \left(u_{k;l} + u_{l;k} - g^{mn} u_{n;k} u_{n;l}\right)$$

$$\mathbf{E}^o = \mathbf{F}^T \cdot \mathbf{E} \cdot \mathbf{F} \qquad \mathbf{E} = \left(\mathbf{F}^{-1}\right)^T \cdot \mathbf{E}^o \cdot \mathbf{F}^{-1}$$

Együttmozgó KR-ben

$$E^o_{KL} = \frac{1}{2}\left(g_{KL} - g^o_{KL}\right) = E_{KL}$$

Alakváltozási mértékek

Vonalelemarány

$$\lambda_s = \frac{ds}{ds^o} = \sqrt{\mathbf{e}^o \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{e}^o} = \sqrt{\mathbf{e}^o \cdot \mathbf{C}^o \cdot \mathbf{e}^o} = \sqrt{1 + 2\mathbf{e}^o \cdot \mathbf{E}^o \cdot \mathbf{e}^o}$$

$$\lambda_s = \frac{ds}{ds^o} = \frac{1}{\sqrt{\mathbf{e} \cdot \left(\mathbf{F}^{-1}\right)^T \cdot \mathbf{F}^{-1} \cdot \mathbf{e}}} = \frac{1}{\sqrt{\mathbf{e} \cdot \mathbf{B}^{-1} \cdot \mathbf{e}}} = \frac{1}{\sqrt{1 - 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}}}$$

Fajlagos nyúlások

$$\varepsilon^o_s = \frac{ds - ds^o}{ds^o} = \lambda_s - 1 = \sqrt{1 + 2\mathbf{e}^o \cdot \mathbf{E}^o \cdot \mathbf{e}^o} - 1 = \frac{1}{\sqrt{1 - 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}}} - 1$$

$$\varepsilon_s = \frac{ds - ds^o}{ds} = 1 - \frac{1}{\lambda_s} = \frac{\varepsilon^o_s}{\lambda_s} = 1 - \frac{1}{\sqrt{1 + 2\mathbf{e}^o \cdot \mathbf{E}^o \cdot \mathbf{e}^o}} = 1 - \sqrt{1 - 2\mathbf{e} \cdot \mathbf{E} \cdot \mathbf{e}}$$

Szögváltozások

$$\cos\alpha = \frac{\cos\alpha^0 + 2\mathbf{e}^o_1 \cdot \mathbf{E}^o \cdot \mathbf{e}^o_2}{\left(1 + \varepsilon^o_{s1}\right)\left(1 + \varepsilon^o_{s2}\right)} = \frac{\cos\alpha^o}{\left(1 + \varepsilon^o_{s1}\right)\left(1 + \varepsilon^o_{s2}\right)} + 2\mathbf{e}_1 \cdot \mathbf{E} \cdot \mathbf{e}_2$$

$$\text{ha} \qquad \alpha_o = \frac{\pi}{2} \qquad \alpha = \alpha_0 - \gamma_{12} = \frac{\pi}{2} - \gamma_{12}$$

$$\frac{1}{2}\sin\gamma_{12} = \frac{\mathbf{e}^o_1 \cdot \mathbf{E}^o \cdot \mathbf{e}^o_2}{\left(1 + \varepsilon^o_{s1}\right)\left(1 + \varepsilon^o_{s2}\right)} = \mathbf{e}_1 \cdot \mathbf{E} \cdot \mathbf{e}_2$$

Térfogatelemek, térfogatelemarány

$$dV^0 = \varepsilon^o_{klm}\, dx^{ok}_{\underline{1}} dx^{ol}_{\underline{2}} dx^{om}_{\underline{3}} \qquad dV = \varepsilon_{pqs}\, dx^p_{\underline{1}} dx^q_{\underline{2}} dx^s_{\underline{3}}$$

$$dV = \sqrt{\frac{g}{g^o}} J_{x,x^o} dV^o$$

$$\lambda_V = \frac{dV}{dV^o} = \sqrt{\frac{g}{g^o}} J_{x,x^o}$$

kartéziuszi KR-ben

$$dV = J_{y,y^o} dV^0 \qquad \lambda_V = J_{y,y^o}$$

Felületelemek

$$dA^o_k = \varepsilon^o_{klm}\, dx^{ol}_{\underline{1}} dx^{om}_{\underline{2}} \qquad dA_p = \varepsilon_{pqs}\, dx^q_{\underline{1}} dx^s_{\underline{2}}$$

$$dA_p = \sqrt{\frac{g}{g^o}} J_{x,x^o}\, \overset{o}{F}{}^m_p\, dA^o_m = \lambda_V \overset{o}{F}{}^m_p\, dA^o_m \qquad d\mathbf{A} = \sqrt{\frac{g}{g^o}} J_{x,x^o} \left(\mathbf{F}^{-1}\right)^T \cdot d\mathbf{A}^o = \lambda_V \left(\mathbf{F}^{-1}\right)^T \cdot d\mathbf{A}^o$$

kartéziuszi KR-ben

$$d\mathbf{A} = J_{y,y^o} \left(\mathbf{F}^{-1}\right)^T \cdot d\mathbf{A}^o$$

Materiális idő szerinti deriváltak

Ívelemvektor deriváltja és az Euler sebességgradiens felbontása

$$v = \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial \mathbf{r}}{\partial x^p}\frac{\partial x^p}{\partial t} = \mathbf{g}_p v^p = \mathbf{G}^o_k v^{oK}$$

$$(d\mathbf{r})^{\cdot} = \mathbf{L}\cdot d\mathbf{r} \qquad \mathbf{L} = \mathbf{v}\otimes\nabla$$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \qquad \mathbf{D} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right) \qquad \mathbf{W} = \frac{1}{2}\left(\mathbf{L}-\mathbf{L}^T\right)$$

$$\left(dx^k\right)^{\cdot} = l^k{}_s dx^s = v^k{}_{;s} dx^s$$

$$l_{pq} = d_{pq} + w_{pq} \qquad d_{pq} = \frac{1}{2}\left(v_{p;q}+v_{q;p}\right) \qquad w_{pq} = \frac{1}{2}\left(v_{p;q}-v_{q;p}\right)$$

Euler féle tenzormező

$$\mathbf{H}(x^1,x^2,x^3;t) = h_{pq}\mathbf{g}^p\otimes\mathbf{g}^q$$

$$(\mathbf{H})^{\cdot} = \left.\frac{d\mathbf{H}}{dt}\right|_{(\overset{o}{x})} = \frac{\partial \mathbf{H}}{\partial t} + (\mathbf{H}\otimes\nabla)\cdot\mathbf{v}$$

$$(h_{pq})^{\cdot} = \frac{\partial h_{pq}}{\partial t} + h_{pq;r}v^r$$

Lagrange féle tenzormező

$$\mathbf{H}^o(x^{o1},x^{o2},x^{o3};t) = h^o_{pq}\mathbf{g}^{op}\otimes\mathbf{g}^{oq}$$

$$(\mathbf{H}^o)^{\cdot} = \frac{d\mathbf{H}}{dt} = \frac{\partial \mathbf{H}}{\partial t}$$

$$\left(h^0_{pq}\right)^{\cdot} = \frac{dh^o_{pq}}{dt} = \frac{\partial h^o_{pq}}{\partial t}$$

Alakváltozási gradiens

$$(\mathbf{F})^{\cdot} = \mathbf{L}\cdot\mathbf{F} \qquad \left(\mathbf{F}^T\right)^{\cdot} = \mathbf{F}^T\cdot\mathbf{L}^T$$

$$\left(F^k{}_{\overset{o}{m}}\right) = L^k_s F^s{}_{\overset{o}{m}} = v^k_{;s}F^s{}_{\overset{o}{m}} \qquad \left(F_{\overset{o}{m}}{}^k\right) = F^s{}_{\overset{o}{m}}L^k_s = F^s{}_{\overset{o}{m}}v^k_{;s}$$

Inverz alakváltozási gradiens

$$(\mathbf{F}^{-1})^{\cdot} = -\mathbf{F}^{-1}\cdot\mathbf{L} \qquad \left[(\mathbf{F}^{-1})^T\right]^{\cdot} = -\mathbf{L}^T\cdot(\mathbf{F}^{-1})^T$$

$$\left[(F^{-1})^{\overset{o}{r}}{}_k\right]^{\cdot} = -\left(F^{-1}\right)^{\overset{o}{r}}{}_s L^s{}_k = -\left(F^{-1}\right)^{\overset{o}{r}}{}_s v^s{}_{;k} \qquad \left[(F^{-1})_k{}^{\overset{o}{r}}\right]^{\cdot} = -L_k{}^s\left(F^{-1}\right)_s{}^{\overset{o}{r}} = \left(F^{-1}\right)_s{}^{\overset{o}{r}} v^s{}_{;k}$$

Cauchy féle alakváltozási tenzor

$$(\mathbf{B}^{-1})^{\cdot} = \left[(\mathbf{F}^{-1})^T\cdot\mathbf{F}\right]^{\cdot} = -\mathbf{L}^T\cdot\mathbf{B}^{-1} - \mathbf{B}^{-1}\cdot\mathbf{L}$$

Jobboldali Cauchy-Green féle alakváltozási tenzor

$$(\mathbf{C}^o)^{\cdot} = \left(\mathbf{F}^T\cdot\mathbf{F}\right)^{\cdot} = 2\mathbf{F}^T\cdot\mathbf{D}\cdot\mathbf{F}$$

Euler féle alakváltozási tenzor

$$(\mathbf{E})^{\cdot} = \mathbf{D} - \mathbf{E}\cdot\mathbf{L} - \mathbf{L}^T\cdot\mathbf{E}$$

$$= \mathbf{D} - \mathbf{E}\cdot\mathbf{D} - \mathbf{D}\cdot\mathbf{E} + \boldsymbol{\omega}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\omega}$$

Lagrange féle alakváltozási tenzor

$$(\mathbf{E}^o)^{\cdot} = \mathbf{F}^T\cdot\mathbf{D}\cdot\mathbf{F}$$

Jauman féle objektív derivált

$$\mathbf{H}^{\nabla} \overset{df}{=} (\mathbf{H})^{\cdot} - (\mathbf{W}\cdot\mathbf{H} - \mathbf{H}\cdot\mathbf{W}) = (\mathbf{H})^{\cdot} - (\boldsymbol{\omega}\times\mathbf{H} - \mathbf{H}\times\boldsymbol{\omega})$$

Euler féle alakváltozási tenzorra

$$\mathbf{E}^{\nabla} = (\mathbf{E})^{\cdot} - (\mathbf{W}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{W}) = \mathbf{D} - \mathbf{E}\cdot\mathbf{D} - \mathbf{D}\cdot\mathbf{E}$$

Néhány további nevezetes materiális idő szerinti derivált

$$(\lambda_s)^{\cdot} = \left(\frac{ds}{ds^o}\right)^{\cdot} = \lambda_s \mathbf{e}\cdot\mathbf{D}\cdot\mathbf{e}$$

$$(e_s^o)^{\cdot} = (1+e_s^o)\,\mathbf{e}\cdot\mathbf{D}\cdot\mathbf{e} \qquad (e_s)^{\cdot} = (1-e_s)\,\mathbf{e}\cdot\mathbf{D}\cdot\mathbf{e}$$

$$(\lambda_A)^{\cdot} = \left(\frac{dA}{dA^o}\right)^{\cdot} = \lambda_A\,(D_I - \mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})$$

$$(\lambda_V)^{\cdot} = \left(\frac{dV}{dV^o}\right)^{\cdot} = \lambda_V D_I$$

$$(\varepsilon_V^o)^{\cdot} = \left(\frac{dV - dV^o}{dV^o}\right)^{\cdot} = (1+\varepsilon_V^o)\,D_I \qquad (\varepsilon_V)^{\cdot} = \left(\frac{dV - dV^o}{dV}\right) = (1-\varepsilon_V)\,D_I$$

$$-(\alpha)^{\cdot} = (\gamma_{12})^{\cdot} = 2\mathbf{e}_1\cdot\mathbf{D}\cdot\mathbf{e}_2 \qquad \mathbf{e}_1\perp\mathbf{e}_2$$

A kontinuummechanika alaptörvényei

A kontinuum belső erőrendszere

Cauchy féle feszültségi tenzor

$$\mathbf{T} = t^{kl}\mathbf{g}_k\otimes\mathbf{g}_l$$

*Elemi erő a d**A** felületelemen, feszültségvektor*

$$d\mathbf{F} = \mathbf{T}\cdot d\mathbf{A} = \mathbf{g}_k\,t^{kl}dA_l = \mathbf{g}_k\,t^{kl}n_l\,dA$$

I. Piola Kirchoff féle feszültségi tenzor

$$s^{k\overset{o}{r}} = \sqrt{\frac{g}{g^o}}J_{x,x^o}t^{kl}\left(F^{-1}\right)_l^{\overset{o}{r}}$$

$$\mathbf{S} = \sqrt{\frac{g}{g^o}}J_{x,x^o}\,\mathbf{T}\cdot\left(\mathbf{F}^{-1}\right)^T$$

II. Piola Kirchoff féle feszültségi tenzor

$$t^{opr} = \sqrt{\frac{g}{g^o}}J_{x,x^o}\left(F^{-1}\right)_k^{\overset{o}{p}}t^{kl}\left(F^{-1}\right)_l^{\overset{o}{r}}$$

$$\mathbf{T}^o = \sqrt{\frac{g}{g^o}}J_{x,x^o}\,\left(\mathbf{F}^{-1}\right)\cdot\mathbf{T}\cdot\left(\mathbf{F}^{-1}\right)^T$$

Elemi erő a pillanatnyi és kezdeti konfiguráció felületelemén

$$d\mathbf{F} = \mathbf{S}\cdot d\mathbf{A}^o \qquad d\mathbf{F}^o = \mathbf{T}^o\cdot d\mathbf{A}^o$$

$$d\mathbf{F} = \mathbf{F}\cdot d\mathbf{F}^o$$

(itt **F** az alakváltozási gradiens)

Kartéziuszi KR-ben

$$\mathbf{S} = J_{y,y^o}\mathbf{T}\cdot\left(\mathbf{F}^{-1}\right)^T \qquad \mathbf{T}^o = J_{y,y^o}\left(\mathbf{F}^{-1}\right)\cdot\mathbf{T}\cdot\left(\mathbf{F}^{-1}\right) = \left(\mathbf{F}^{-1}\right)\cdot\mathbf{S}$$

Az alaptörvények

A tömegmegmaradás elve, kontinuitási egyenlet

$$m^{\cdot} = \left(\int_V \rho\, dV\right)^{\cdot} = 0$$

$$\rho^{\cdot} + \rho\, v^k_{\ ;k} = 0 \qquad \rho^{\cdot} + \rho\, \mathbf{v} \cdot \boldsymbol{\nabla} = 0$$

A dinamika alaptétele, a mozgásegyenletek

$$\mathbf{a}\rho dV \text{ a V} - \text{n} \overset{m}{=} \begin{cases} \mathbf{q}dV \text{ a V} - \text{n} \\ \mathbf{T}\cdot d\mathbf{A} \text{ az A} - \text{n} \end{cases}$$

Pillanatnyi konfiguráció

$$\rho a^k = t^{kl}_{\ ;l} + q^k \qquad \rho\mathbf{a} = \mathbf{T} \cdot \nabla + \mathbf{q}$$

$$\varepsilon_{klr} t^{kl} = 0 \qquad \mathbf{T} = \mathbf{T}^T$$

Kezdeti konfiguráció

$$\rho^o a^{ok} = \left[\left(\delta^k_s + u^{ok}_{\ \ ;s}\right) t^{osl}\right]_{;l} + q^{ok} \qquad \rho^o\mathbf{a}^o = [(\mathbf{I} + \mathbf{u}^o \otimes \nabla^o) \cdot \mathbf{T}^o] \cdot \nabla^o + \mathbf{q}^o$$

$$\varepsilon_{klr}\left(\delta^k_s + u^{ok}_{\ \ ;s}\right) t^{osl} = 0 \qquad (\mathbf{I} + \mathbf{u}^o \otimes \nabla^o) \cdot \mathbf{T}^o = [(\mathbf{I} + \mathbf{u}^o \otimes \nabla^o) \cdot \mathbf{T}^o]^T$$

A mechanikai energia tétel

$$T^{\cdot} = \frac{1}{2}\int_V v^2 \rho dV \qquad P_B = \int_V t^{kl} d_{kl}\, dV = \int_V \mathbf{T} \cdot\cdot\, \mathbf{D}\, dV$$

$$P_K = \int_V v^k q_k dV + \int_A v_k t^{kl} n_l dA = \int_V \mathbf{v} \cdot \mathbf{q} dV + \int_A \mathbf{v} \cdot \mathbf{T} \cdot d\mathbf{A}$$

$$T^{\cdot} = P_K + P_B$$

$$t^{kl} d_{kl} dV = t^{ors}\left(E^o_{rs}\right)^{\cdot} dV^o$$

A termodinamika I. főtétele. Az energia egyenlet

$$E_B = \int_V e_B \rho dV \qquad P_Q = \int_V r\rho dV - \int_A h^k n_k\, dA = \int_V r\rho dV - \int_A \mathbf{h}\cdot d\mathbf{A}$$

$$(T + E_B)^{\cdot} = P_K + P_Q$$

$$\rho\,(e_B)^{\cdot} = t^{kl} d_{kl} + r\rho - h^k_{\ ;k} \qquad \rho\,(e_B)^{\cdot} = \mathbf{T} \cdot\cdot\, \mathbf{D} + r\rho - \mathbf{h} \cdot \nabla$$

$$\Phi_M = t^{kl} d_{kl} \qquad \Phi_Q = r\rho - h^k_{\ ;k}$$

$$\rho\,(e_B)^{\cdot} = \Phi_M + \Phi_Q$$

A termodinamika II. főtétele Az entrópia tétel

$$\rho\Theta s^{\cdot} > \rho r - h^{k}_{;k} + \frac{h^k(\Theta_{;k})}{\Theta} = \Phi_Q + \frac{h^k(\Theta_{;k})}{\Theta}$$

$$\rho\Theta s^{\cdot} - \rho\,(e_B)^{\cdot} + t^{kl}d_{kl} - \frac{h^k(\Theta_{;k})}{\Theta} > 0 \qquad \Phi_D - \frac{h^k(\Theta_{;k})}{\Theta} > 0$$

$$\Phi_D = t^{kl}d_{kl} - \rho\left[(e_B)^{\cdot} - \Theta s^{\cdot}\right]$$

Az egyensúlyi egyenletek általános és teljes megoldása

$$t^{kl} = \varepsilon^{kpq}\varepsilon^{lrs}\mathcal{H}_{pr;qs} + g^{ls}\chi^{k}_{\ ;s} + g^{ks}\chi^{l}_{\ ;s} - \chi^{s}_{\ ;s}g^{kl} \qquad \mathcal{H}_{pr} = \mathcal{H}_{rp}$$

$$\mathbf{T} = -\nabla\times\mathbf{H}\times\nabla + \nabla\otimes\boldsymbol{\chi} + \boldsymbol{\chi}\otimes\nabla - (\boldsymbol{\chi}\cdot\nabla)\,\mathbf{I} \qquad \mathbf{H} = \mathbf{H}^T$$

$$\triangle\chi^k = q^k \qquad \triangle\,\boldsymbol{\chi} = \mathbf{q}$$

A feszültségekre vonatkozó homogén megoldás szerkezete

$$\begin{aligned}
{}'t_{11} &= {}'\mathcal{H}_{22,33} + {}'\mathcal{H}_{33,22} - {}'\mathcal{H}_{23,23}\\
{}'t_{22} &= {}'\mathcal{H}_{33,11} + {}'\mathcal{H}_{11,33} - {}'\mathcal{H}_{31,31}\\
{}'t_{33} &= {}'\mathcal{H}_{11,22} + {}'\mathcal{H}_{22,11} - {}'\mathcal{H}_{12,12}
\end{aligned}$$

$$\begin{aligned}
{}'t_{12} &= \frac{1}{2}\left({}'\mathcal{H}_{13,2} + {}'\mathcal{H}_{23,1} - {}'\mathcal{H}_{12,3}\right)_{,3} - {}'\mathcal{H}_{33,12}\\
{}'t_{23} &= \frac{1}{2}\left({}'\mathcal{H}_{21,3} + {}'\mathcal{H}_{31,2} - {}'\mathcal{H}_{23,1}\right)_{,1} - {}'\mathcal{H}_{11,23}\\
{}'t_{31} &= \frac{1}{2}\left({}'\mathcal{H}_{32,1} + {}'\mathcal{H}_{12,3} - {}'\mathcal{H}_{31,2}\right)_{,2} - {}'\mathcal{H}_{22,31}
\end{aligned}$$

A $\left\{\begin{array}{c} H_{kl}\ \textit{és}\ H_{kl} + \frac{1}{2}(w_{k;l} + w_{l;k}) \\ \mathbf{H}\ \textit{és}\ \mathbf{H} + \frac{1}{2}(\mathbf{w}\otimes\nabla + \nabla\otimes\mathbf{w}) \end{array}\right\}$ feszültségfüggvényekből azonos a feszültségmező

Maxwell féle előállítás

$$[\mathbf{H}] = \begin{bmatrix} {}'\mathcal{H}_{11} & 0 & 0 \\ 0 & {}'\mathcal{H}_{22} & 0 \\ 0 & 0 & {}'\mathcal{H}_{33} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{xx} & 0 & 0 \\ 0 & \mathcal{H}_{yy} & 0 \\ 0 & 0 & \mathcal{H}_{zz} \end{bmatrix}$$

Morera féle előállítás

$$[\mathbf{H}] = \begin{bmatrix} 0 & {}'\mathcal{H}_{12} & {}'\mathcal{H}_{13} \\ {}'\mathcal{H}_{21} & 0 & {}'\mathcal{H}_{23} \\ {}'\mathcal{H}_{31} & {}'\mathcal{H}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{H}_{xy} & \mathcal{H}_{xz} \\ \mathcal{H}_{yx} & 0 & \mathcal{H}_{yz} \\ \mathcal{H}_{zx} & \mathcal{H}_{zy} & 0 \end{bmatrix}$$

A virtuális teljesítmény elv és a kiegészítő virtuális teljesítmény elv

$$\int_V t^{kl}\hat{d}_{kl}\,dV = \int_V \overset{(a)}{q}{}^k\hat{v}_k\,dV + \int_{A_v}\tilde{v}_k t^{kl} n_l\,dA + \int_{A_p}\tilde{p}^k\hat{v}_k\,dA$$
$$\int_V \mathbf{T}\cdot\cdot\hat{\mathbf{D}}\,dV = \int_V \overset{(a)}{\mathbf{q}}\cdot\hat{\mathbf{v}}\,dV + \int_{A_v}\tilde{\mathbf{v}}\cdot\mathbf{T}\cdot\mathbf{n}\,dA + \int_{A_p}\tilde{\mathbf{p}}\cdot\hat{\mathbf{v}}\,dA$$

$$\int_V \bar{t}^{kl} d_{kl}\,dV = \int_V \overset{(a)}{q}{}^k v_k\,dV + \int_{A_v}\tilde{v}_k \bar{t}^{kl} n_l\,dA + \int_{A_p}\tilde{p}^k v_k\,dA$$
$$\int_V \bar{\mathbf{T}}\cdot\cdot\mathbf{D}\,dV = \int_V \overset{(a)}{\mathbf{q}}\cdot\mathbf{v}\,dV + \int_{A_v}\tilde{\mathbf{v}}\cdot\bar{\mathbf{T}}\cdot\mathbf{n}\,dA + \int_{A_p}\tilde{\mathbf{p}}\cdot\mathbf{v}\,dA$$

vagy

$$\int_V t^{kl}\delta d_{kl}\,dV = \int_V \overset{(a)}{q}{}^k\delta v_k\,dV + \int_{A_p}\tilde{p}^k\delta v_k\,dA$$
$$\int_V \mathbf{T}\cdot\cdot\delta\mathbf{D}\,dV = \int_V \overset{(a)}{\mathbf{q}}\cdot\delta\mathbf{v}\,dV + \int_{A_p}\tilde{\mathbf{p}}\cdot\delta\mathbf{v}\,dA$$

$$\int_V d_{kl}\delta t^{kl}\,dV = \int_{A_v}\tilde{v}_k\delta t^{kl} n_l\,dA$$
$$\int_V \mathbf{D}\cdot\cdot\delta\mathbf{T}\,dV = \int_{A_v}\tilde{\mathbf{v}}\cdot\delta\mathbf{T}\cdot\mathbf{n}\,dA$$

A virtuális munka elv és a kiegészítő virtuális munka elv

$$\int_V t^{kl}\delta\overset{L}{E}_{kl}\,dV = \int_V \overset{(a)}{q}{}^k\delta u_k\,dV + \int_{A_p}\tilde{p}^k\delta u_k\,dA$$
$$\int_V \mathbf{T}\cdot\cdot\delta\overset{L}{\mathbf{E}}\,dV = \int_V \overset{(a)}{\mathbf{q}}\cdot\delta\mathbf{u}\,dV + \int_{A_p}\tilde{\mathbf{p}}\cdot\delta\mathbf{u}\,dA$$

$$\int_V \varepsilon_{kl}\delta\sigma^{kl}\,dV = \int_{A_u}\tilde{u}_k\delta\sigma^{kl} n_l\,dA$$
$$\int_V \boldsymbol{\varepsilon}\cdot\cdot\delta\boldsymbol{\sigma}\,dV = \int_{A_u}\tilde{\mathbf{u}}\cdot\delta\mathbf{T}\cdot\mathbf{n}\,dA$$

Egy iterációs eljárás főbb lépései

$${}^{(k+1)}\mathbf{u}^o = {}^{(k)}\mathbf{u}^o + \Delta^{(k)}\mathbf{u}^o$$

$${}^{(k+1)}\mathbf{E}^o = {}^{(k)}\mathbf{E}^o + \Delta^{(k)}\mathbf{E}^o$$

$${}^{(k)}\mathbf{E}^o = \frac{1}{2}\left[{}^{(k)}\mathbf{u}^o\otimes\nabla^o + \nabla^o\otimes{}^{(k)}\mathbf{u}^o + \left(\nabla^o\otimes{}^{(k)}\mathbf{u}^o\right)\cdot\left({}^{(k)}\mathbf{u}^o\otimes\nabla^o\right)\right]$$

$$\Delta^{(k)}\overset{L}{\mathbf{E}}{}^o = \frac{1}{2}\left[\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o + \nabla^o\otimes\left(\Delta^{(k)}\mathbf{u}^o\right)\right] +$$

$$+\frac{1}{2}\left\{\left(\nabla^o\otimes{}^{(k)}\mathbf{u}^o\right)\cdot\left[\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o\right] + \left[\nabla^o\otimes\left(\Delta^{(k)}\mathbf{u}^o\right)\right]\cdot\left({}^{(k)}\mathbf{u}^o\otimes\nabla^o\right)\right\}$$

$$\Delta^{(k)}\overset{N}{\mathbf{E}}{}^o = \frac{1}{2}\left[\nabla^o\otimes\left(\Delta^{(k)}\mathbf{u}^o\right)\right]\cdot\left[\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o\right]$$

$${}^{(k+1)}\mathbf{T}^o = {}^{(k)}\mathbf{T}^o + \Delta^{(k)}\mathbf{T}^o$$

$$\delta\mathbf{u}^o = \delta\left(\Delta^{(k)}\mathbf{u}^o\right)$$

$$\delta^{(k)}\mathbf{E}^o = \delta\Delta^{(k)}\overset{L}{\mathbf{E}}{}^o + \delta\Delta^{(k)}\overset{N}{\mathbf{E}}{}^o$$

$$\delta\Delta^{(k)}\overset{L}{\mathbf{E}}{}^o = \frac{1}{2}\left[\delta\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o + \nabla^o\otimes\delta\left(\Delta^{(k)}\mathbf{u}^o\right)\right]$$

$$+\frac{1}{2}\left\{\left(\nabla^o\otimes{}^{(k)}\mathbf{u}^o\right)\cdot\left[\delta\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o\right] + \left[\nabla^o\otimes\delta\left(\Delta^{(k)}\mathbf{u}^o\right)\right]\cdot\left({}^{(k)}\mathbf{u}^o\otimes\nabla^o\right)\right\}$$

$$\delta\Delta^{(k)}\overset{N}{\mathbf{E}}{}^o = \frac{1}{2}\left\{\left[\nabla^o\otimes\delta\left(\Delta^{(k)}\mathbf{u}^o\right)\right]\cdot\left[\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o\right] + \left[\nabla^o\otimes\left(\Delta^{(k)}\mathbf{u}^o\right)\right]\cdot\left[\delta\left(\Delta^{(k)}\mathbf{u}^o\right)\otimes\nabla^o\right]\right\}$$

A virtuális munka elv

$$\int_{V^o}\left[{}^{(k)}\mathbf{T}^o + \Delta^{(k)}\mathbf{T}^o\right]\cdot\cdot\,\delta^{(k)}\mathbf{E}^o\,dV^o = \int_{V^o}\left[\lambda_V + \Delta^{(k)}\lambda_V\right]\mathbf{q}\cdot\delta\mathbf{u}^o\,dV^o$$

$$+\int_{A_p^o}\left[\lambda_A + \Delta^{(k)}\lambda_A\right]\tilde{\mathbf{p}}\cdot\delta\mathbf{u}^o\,dA^o$$

Linearizált alak

$$\int_{V^o}\Delta^{(k)}\mathbf{T}^o\cdot\cdot\,\delta\left(\Delta^{(k)}\overset{L}{\mathbf{E}}{}^o\right)dV^o + \int_{V^o}{}^{(k)}\mathbf{T}^o\cdot\cdot\,\delta\left(\Delta^{(k)}\overset{N}{\mathbf{E}}{}^o\right)dV^o =$$

$$= -\int_{V^o}^{(k)}\mathbf{T}^o\cdot\cdot\,\delta\left(\Delta^{(k)}\overset{L}{\mathbf{E}}{}^o\right)dV^o$$

$$\int_{V^o}{}^{(k)}\lambda_V\mathbf{q}\cdot\delta\left(\Delta^{(k)}\mathbf{u}^o\right)dV^o + \int_{A_p^o}{}^{(k)}\lambda_A\tilde{\mathbf{p}}\cdot\delta\left(\Delta^{(k)}\mathbf{u}^o\right)dA^o$$

Anyagegyenletek

Hőrugalmas test

$$f(\Theta, E^o_{kl}) = e_B - s\Theta$$

$$\Phi_D = t^{kl}d_{kl} - \rho\left[f^{\cdot} - s\Theta^{\cdot}\right] = \left(t^{okl} - \rho^o\frac{\partial f}{\partial E^o_{kl}}\right)\left(E^o_{kl}\right)^{\cdot} - \rho^o\left(s + \frac{\partial f}{\partial\Theta}\right)\Theta^{\cdot} = 0$$

$$t^{okl} = \rho^o\frac{\partial f}{\partial E^o_{kl}} \qquad t^{kl} = \rho\frac{\partial f}{\partial E_{kl}} \qquad s = -\frac{\partial f}{\partial\Theta}$$

Hővezetési egyenlet

$$h^k_{;k} - \rho^o\Theta\left[\frac{\partial^2 f}{\partial\Theta\partial E^o_{kl}}\left(E^o_{kl}\right)^{\cdot} + \frac{\partial^2 f}{\partial\Theta^2}\Theta^{\cdot}\right] - \rho r = 0$$

Linearizált egyenletek

$$\rho^o f(\Theta_o,0)=0 \qquad \rho^o\frac{\partial f(\Theta_o,0)}{\partial\varepsilon_{kl}}=0 \qquad \rho^o\frac{\partial f(\Theta_o,0)}{\partial\Theta}=0$$

$$\rho^o\frac{\partial^2 f(\Theta_o,0)}{\partial\varepsilon_{kl}\partial\varepsilon_{pq}}=C^{klpq} \qquad \rho^o\frac{\partial^2 f(\Theta_o,0)}{\partial\varepsilon_{kl}\partial\Theta}=-\beta^{kl} \qquad \rho^o\frac{\partial^2 f}{\partial\Theta^2}=c$$

$$\rho^o f(\Theta_o,\varepsilon_{kl})=\frac{1}{2}\left[\varepsilon_{kl}C^{klpq}\varepsilon_{pq}-2\beta^{kl}\varepsilon_{kl}(\Theta-\Theta_o)-c(\Theta-\Theta_o)^2\right]$$

$$\sigma^{kl}=C^{klpq}\varepsilon_{pq}-\beta^{kl}(\Theta-\Theta_o)$$
$$\varepsilon_{pq}=C^{-1}_{pqkl}\sigma^{kl}+\alpha_{pq}(\Theta-\Theta_o)$$

$$C^{klpq}C^{-1}_{pqrs}=\delta^k_r\delta^l_s \qquad \alpha_{pq}=C^{-1}_{pqrs}\beta^{rs}$$

$$h^k=-\kappa^{kl}\Theta_{,l}$$
$$\left(\kappa^{kl}\Theta_{,l}\right)_{;k}-\Theta\beta^{kl}(\varepsilon_{kl})^{\cdot}-c\Theta\Theta^{\cdot}=-\rho^o r$$
$$\left(\kappa^{kl}\vartheta_{,l}\right)_{;k}-\Theta_o\beta^{kl}(\varepsilon_{kl})^{\cdot}-c\Theta_o\vartheta^{\cdot}=-\rho^o r$$

Egyenletek homogén és izotróp testre

$$\kappa^{kl}=\kappa g^{kl} \qquad \beta^{kl}=\beta g^{kl} \qquad \alpha_{kl}=\beta g_{kl}$$

$$C^{klpq}=\lambda g^{kl}g^{pq}+\mu\left(g^{kp}g^{lq}+g^{kq}g^{lp}\right)=\frac{2\mu\nu}{1-2\nu}g^{kl}g^{pq}+\mu\left(g^{kp}g^{lq}+g^{kq}g^{lp}\right)$$

$$C^{-1}_{pqkl}=-\frac{\lambda}{2\mu(2\mu+3\lambda)}g^{kl}g^{pq}+\frac{1}{2\mu}\left(g^{kp}g^{lq}+g^{kq}g^{lp}\right)$$
$$=-\frac{\nu}{2\mu(1+\nu)}g^{kl}g^{pq}+\frac{1}{2\mu}\left(g^{kp}g^{lq}+g^{kq}g^{lp}\right)$$

$$\sigma_{kl}=2\mu\varepsilon_{kl}+\lambda\varepsilon_I g_{kl}-\beta g_{kl}(\Theta-\Theta_o)=2\mu\left(\varepsilon_{kl}+\frac{\nu}{1-2\nu}\varepsilon_I g_{kl}\right)-\beta g_{kl}\vartheta$$

$$\varepsilon_{kl}=\frac{1}{2\mu}\left(\sigma_{kl}-\frac{\lambda}{2\mu+3\lambda}\sigma_I g_{kl}\right)+\alpha g_{kl}\vartheta=\frac{1}{2\mu}\left(\sigma_{kl}-\frac{\nu}{1+\nu}\sigma_I g_{kl}\right)+\alpha g_{kl}\vartheta$$

$$\kappa\triangle\vartheta-\Theta_o\beta(\varepsilon_I)^{\cdot}-c\Theta_o\vartheta^{\cdot}=-\rho^o r$$

Rugalmas test

$$E^o_I=g^{kl}E^o_{kl} \qquad E^o_{II}=\frac{1}{2}\left[(E^o_I)^2-E^{op}_qE^{oq}_p\right]$$
$$E^o_{III}=\frac{1}{6}\left[-2(E^o_I)^3+6E^o_IE^o_{II}-E^{om}_sE^{or}_mE^{os}_r\right]$$

$$e_B = e_B(E_I^o, E_{II}^o, E_{III}^o)$$
$$t^{okl} = \rho^o \frac{\partial e_B}{\partial E_{kl}^o} \qquad t^{kl} = \rho \frac{\partial e_B}{\partial E_{kl}}$$

$$t^{okl} = a_o^o g^{okl} + a_1^o E^{okl} + a_2^o E^{ok}{}_s E^{osl}$$
$$t^{kl} = a_o g^{kl} + a_1 E^{kl} + a_2 E^k{}_s E^{sl}$$

$$a_o^o = \rho^o \left[\frac{\partial e_B}{\partial E_I^o} + E_I^o \frac{\partial e_B}{\partial E_{II}^o} + E_{II}^o \frac{\partial e_B}{\partial E_{III}^o} \right]$$
$$a_1^o = -\rho^o \left[\frac{\partial e_B}{\partial E_{II}^o} + E_I^o \frac{\partial e_B}{\partial E_{III}^o} \right] \qquad a_2^o = \rho^o \frac{\partial e_B}{\partial E_{III}^o}$$

Képlékeny test

Képlékenységi feltétel

$$f(\sigma^{kl}) \leq 0 \qquad \frac{\partial f}{\partial \sigma^k_l} \left(\sigma^k_l\right)^{\cdot} < 0 \qquad \text{rugalmas alakváltozás}$$
$$f(\sigma^{kl}) = 0 \qquad \frac{\partial f}{\partial \sigma^k_l} \left(\sigma^k_l\right)^{\cdot} \geq 0 \qquad \text{képlékeny alakváltozás}$$

Mises féle képlékenységi feltétel összenyomhatatlan testre

$$\begin{aligned} (\boldsymbol{\sigma}')_{II} + (\tau_F)^2 &= 0 \\ \sqrt{\frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} &= \tau_F \end{aligned}$$

Lévy-Mises féle anyagtörvény

$$\left(\sigma^{kl}\right)' = \frac{\tau_F}{\sqrt{-D_{II}}} d_{kl} \qquad D_I = 0$$

Prandtl-Reuss egyenlet

$$\left[(\varepsilon_{kl})'\right]^{\cdot} = \frac{1}{2G}\left[\left(\sigma^{kl}\right)'\right]^{\cdot} + \lambda \left(\sigma^{kl}\right)'$$

Lineáris rugalmasságtan

A primál rendszer egyenletei és peremfeltételei

Primál értelmező egyenlet (kinematikai egyenlet)

$$\varepsilon_{kl} = \frac{1}{2}(u_{k;l} + u_{l;k}) \qquad \boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u}) \qquad x \in V$$

Primál konstitutív egyenlet

$$\sigma^{kl} = C^{klrs} \varepsilon_{rs} \qquad \boldsymbol{\sigma} = {}^{(4)}\mathbf{C} \cdot\cdot \boldsymbol{\varepsilon} \qquad x \in V$$

Primál egyensúlyi egyenlet

$$\sigma^{kl}{}_{;l} + q^k = 0 \qquad \boldsymbol{\sigma} \cdot \nabla + \mathbf{q} = 0 \qquad x \in V$$

Peremfeltételek

$$u_k = \tilde{u}_k \qquad \mathbf{u} = \tilde{\mathbf{u}} \qquad x \in A_u$$
$$\sigma^{kl} n_l = \tilde{p}^k \qquad \boldsymbol{\sigma} \cdot \mathbf{n} = \tilde{\mathbf{p}} \qquad x \in A_p$$

Fajlagos alakváltozási energia

$$u = u(\varepsilon_{kl}) = \frac{1}{2}\sigma^{kl}\varepsilon_{kl} = \frac{1}{2}\varepsilon_{rs}C^{rskl}\varepsilon_{kl}$$

$$u = u(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot\,^{(4)}\mathbf{C}\cdot\cdot\,\boldsymbol{\varepsilon}$$

$$\sigma^{kl} = \frac{\partial u}{\partial \varepsilon_{kl}} \qquad \boldsymbol{\sigma} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}}$$

Fajlagos kiegészítő alakváltozási energia

$$e = e(\sigma^{kl}) = \sigma^{kl}\varepsilon_{kl} - u = \frac{1}{2}\sigma^{kl}C^{-1}_{klrs}\sigma^{rs}$$

$$e = e(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\varepsilon} - u = \frac{1}{2}\boldsymbol{\sigma}\cdot\cdot\,^{(4)}\mathbf{C}^{-1}\cdot\cdot\,\boldsymbol{\sigma}$$

$$\varepsilon_{kl} = \frac{\partial e}{\partial \sigma^{kl}} \qquad \boldsymbol{\varepsilon} = \frac{\partial e}{\partial \boldsymbol{\sigma}}$$

Izotróp, lineárisan rugalmas test

$$\sigma^k{}_l = 2\mu\varepsilon^k{}_l + \lambda\varepsilon_I\delta^k_l \qquad \boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\varepsilon_I\mathbf{I}$$

$$\varepsilon^k{}_l = \frac{1}{2\mu}\left(\sigma^k{}_l - \frac{\lambda}{2\mu+3\lambda}\sigma_I\delta^k_l\right) \qquad \boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu+3\lambda}\sigma_I\mathbf{I}\right)$$

$$\sigma_I = (2\mu+3\lambda)\,\varepsilon_I$$

$$\sigma^k{}_l = 2G\left(\varepsilon^k{}_l + \frac{\nu}{1-2\nu}\varepsilon_I\delta^k_l\right) \qquad \boldsymbol{\sigma} = 2G\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\varepsilon_I\mathbf{I}\right)$$

$$\varepsilon^k{}_l = \frac{1}{2\mu}\left(\sigma^k{}_l - \frac{\nu}{1+\nu}\sigma_I\delta^k_l\right) \qquad \boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\nu}{1+\nu}\sigma_I\mathbf{I}\right)$$

$$\sigma_I = 2G\frac{1+\nu}{1-2\nu}\varepsilon_I = 3K\varepsilon_I$$

Alakváltozási és feszültségi deviátor

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{\sigma_I}{3}\mathbf{I} \qquad \boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \frac{\varepsilon_I}{3}\mathbf{I}$$

$$\boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon}' \qquad \sigma_I = 2G\frac{1+\nu}{1-2\nu}\varepsilon_I$$

Alakváltozási energia térfogatváltozási és izotróp része

$$u = \frac{1}{2}\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\sigma}' + \frac{\sigma_I}{3}\mathbf{I}\right)\cdot\cdot\left(\boldsymbol{\varepsilon}' + \frac{\varepsilon_I}{3}\mathbf{I}\right) = u_V + u_T$$

$$u_V = \frac{\sigma_I\varepsilon_I}{6} = \frac{K}{2}\left(\varepsilon_I\right)^2 = \frac{1}{18K}\left(\sigma_I\right)^2$$

$$u_T = \frac{1}{2}\boldsymbol{\sigma}'\cdot\cdot\,\boldsymbol{\varepsilon}' = \frac{1}{4G}\boldsymbol{\sigma}'\cdot\cdot\,\boldsymbol{\sigma}' =$$

$$= \frac{1}{12G}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)\right] =$$

$$= \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

A mechanikai energia tétel

$$U=\int_V u dV=\frac{1}{2}\int_V \sigma^{kl}\varepsilon_{kl} dV=\frac{1}{2}\int_V \boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\varepsilon}\, dV=-W_B$$
$$W_K=\frac{1}{2}\int_V u_k\, q^k dV+\frac{1}{2}\int_A u_k\,\sigma^{kl} n_l dA=\frac{1}{2}\int_V \mathbf{u}\cdot\mathbf{q}\, dV+\frac{1}{2}\int_A \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\, dA$$
$$U=W_K$$

A Navier féle egyenlet

$$\Delta u_k+\frac{1}{1-2\nu}u^l{}_{;lk}+\frac{q_k}{G}=0$$
$$\Delta\mathbf{u}+\frac{1}{1-2\nu}\mathbf{u}\cdot(\nabla\otimes\nabla)+\frac{\mathbf{q}}{G}=0$$

A Lagrange féle variációs elv

Teljes potenciális energia funkcionál

$$\Pi(\hat{u}_k)=\frac{1}{2}\int_V \hat{\sigma}^{kl}\hat{\varepsilon}_{kl} dV-\int_V \hat{u}_k\, q^k dV-\int_{A_p}\hat{u}_k\tilde{p}^k dA$$
$$\Pi(\hat{\mathbf{u}})=\frac{1}{2}\int_V \hat{\boldsymbol{\sigma}}\cdot\cdot\,\hat{\boldsymbol{\varepsilon}}\, dV-\int_V \hat{\mathbf{u}}\cdot\mathbf{q}\, dV-\int_{A_p}\hat{\mathbf{u}}\cdot\tilde{\mathbf{p}}\, dA$$

Mellékfeltételek

$$\hat{\varepsilon}_{kl}=\tfrac{1}{2}(\hat{u}_{k;l}+\hat{u}_{l;k})\qquad \hat{\boldsymbol{\varepsilon}}=\tfrac{1}{2}(\hat{\mathbf{u}}\otimes\nabla+\nabla\otimes\hat{\mathbf{u}})\qquad x\in V$$
$$\hat{\sigma}^{kl}=C^{klrs}\hat{\varepsilon}_{rs}\qquad \hat{\boldsymbol{\sigma}}={}^{(4)}\mathbf{C}\cdot\cdot\,\hat{\boldsymbol{\varepsilon}}\qquad x\in V$$
$$\hat{u}_k=\tilde{u}_k\qquad \hat{\mathbf{u}}=\tilde{\mathbf{u}}\qquad x\in A_u$$

Első és második variáció

$$\delta\Pi(\hat{u}_k)=\int_V \hat{\sigma}^{kl}\delta\hat{\varepsilon}_{kl} dV-\int_V \delta\hat{u}_k\, q^k dV-\int_{A_p}\delta\hat{u}_k\tilde{p}^k dA$$
$$\delta\Pi(\hat{\mathbf{u}})=\int_V \hat{\boldsymbol{\sigma}}\cdot\cdot\,\delta\hat{\boldsymbol{\varepsilon}}\, dV-\int_V \delta\hat{\mathbf{u}}\cdot\mathbf{q}\, dV-\int_{A_p}\delta\hat{\mathbf{u}}\cdot\tilde{\mathbf{p}}\, dA$$
$$\delta^2\Pi(\hat{u}_k)=\frac{1}{2}\int_V \delta\hat{\sigma}^{kl}\delta\hat{\varepsilon}_{kl} dV=\int_V \delta\hat{\boldsymbol{\sigma}}\cdot\cdot\,\delta\hat{\boldsymbol{\varepsilon}}\, dV>0$$

A duál rendszer egyenletei és peremfeltételei

Duál értelmező egyenlet (duál kinematikai egyenlet)

$$\sigma^{kl}-\overset{h}{\sigma}{}^{kl}=\varepsilon^{kpq}\varepsilon^{lrs}\mathcal{H}_{pr;qs}\qquad \mathcal{H}_{pr}=\mathcal{H}_{rp}\qquad x\in V$$
$$\boldsymbol{\sigma}-\overset{h}{\boldsymbol{\sigma}}=-\nabla\times\mathbf{H}\times\nabla\qquad \mathbf{H}=\mathbf{H}^T\qquad x\in V$$
$$\mathcal{H}_{XY}\neq 0\qquad \mathcal{H}_{AB}=0\qquad \{XY\}\cup\{AB\}=\{pr\}$$

Duál konstitutív egyenlet

$$\varepsilon_{kl}=C^{-1}_{klrs}\sigma^{rs}\qquad \boldsymbol{\varepsilon}={}^{(4)}\mathbf{C}^{-1}\cdot\cdot\,\boldsymbol{\sigma}\qquad x\in V$$

Duál egyensúlyi egyenlet (kompatibilitási egyenlet)

$$\eta^{XY} = \varepsilon^{Xpq}\varepsilon^{Yrs}\varepsilon_{pr;qs} = 0 \qquad x \in V$$

$$\boldsymbol{\eta} = -\nabla \times \boldsymbol{\varepsilon} \times \nabla = 0 \qquad \text{(három alkalmas egyenlet)} \qquad x \in V$$

Peremfeltételek

$$\eta^{ab}n_b = 0 \qquad \boldsymbol{\eta}\cdot\mathbf{n} = 0 \qquad x \in A_p$$

$$\sigma^{kl}n_l = \tilde{p}^k \qquad \boldsymbol{\sigma}\cdot\mathbf{n} = \tilde{\mathbf{p}} \qquad x \in A_p$$

$$\varepsilon_{\kappa\lambda} = \tfrac{1}{2}\left(\tilde{u}_{\kappa;\lambda} + \tilde{u}_{\lambda;\kappa}\right) \qquad x \in A_u$$

$$\varepsilon_{\kappa\lambda;3} - \varepsilon_{\kappa 3;\lambda} = \left(\varepsilon_{3\kappa} - \tilde{u}_{3;\kappa}\right)_{,\lambda} - \Gamma^{\nu}_{\kappa\lambda}\left(\varepsilon_{3\kappa} - \tilde{u}_{3;\kappa}\right) + b^{\nu}_{\lambda}\left(\varepsilon_{\nu\kappa} - \tilde{u}_{\nu;\kappa}\right) \qquad x \in A_u$$

Beltrami-Mitchell féle kompatibilitási egyenletek

$$\Delta\sigma_{kl} + \frac{1}{1+\nu}\left(\sigma_I\right)_{;kl} + \left(q_{k;l} + q_{l;k}\right) + \frac{\nu}{1-\nu}q^{s}_{\ ;s}g_{kl} = 0$$
$$\Delta\boldsymbol{\sigma} + \frac{1}{1+\nu}\sigma_I\left(\nabla\otimes\nabla\right) + \left(\mathbf{q}\otimes\nabla + \nabla\otimes\mathbf{q}\right) + \frac{\nu}{1-\nu}\mathbf{q}\cdot\nabla\,\mathbf{I} = 0$$

A Castigliano féle variációs elv

A teljes kiegészitő energia funkcionál

$$K\left(\bar{\sigma}^{kl}\right) = \frac{1}{2}\int_V \bar{\sigma}^{kl}\bar{\varepsilon}_{kl}dV - \int_{A_u}\tilde{u}_k\bar{\sigma}^{kl}n_l\,dA$$
$$K\left(\bar{\boldsymbol{\sigma}}\right) = \frac{1}{2}\int_V \bar{\boldsymbol{\sigma}}\cdot\cdot\,\bar{\boldsymbol{\varepsilon}}\,dV - \int_{A_p}\tilde{\mathbf{u}}\cdot\bar{\boldsymbol{\sigma}}\cdot\mathbf{n}\,dA$$

Mellékfeltételek

$$\bar{\sigma}^{kl}_{\ ;l} + q^k = 0 \qquad \bar{\boldsymbol{\sigma}}\cdot\nabla + \mathbf{q} = 0 \qquad x \in V$$

$$\bar{\varepsilon}_{kl} = C^{-1}_{klrs}\bar{\sigma}^{rs} \qquad \bar{\boldsymbol{\varepsilon}} = {}^{(4)}\mathbf{C}^{-1}\cdot\cdot\,\bar{\boldsymbol{\sigma}} \qquad x \in V$$

$$\bar{\sigma}^{kl}n_l = \tilde{p}^k \qquad \bar{\boldsymbol{\sigma}}\cdot\mathbf{n} = \tilde{\mathbf{p}} \qquad x \in A_p$$

Első és második variáció

$$\delta K\left(\bar{\sigma}^{kl}\right) = \int_V \delta\bar{\sigma}^{kl}\bar{\varepsilon}_{kl}dV - \int_{A_u}\tilde{u}_k\delta\bar{\sigma}^{kl}n_l\,dA$$
$$\delta K\left(\bar{\boldsymbol{\sigma}}\right) = \int_V \delta\bar{\boldsymbol{\sigma}}\cdot\cdot\,\bar{\boldsymbol{\varepsilon}}\,dV - \int_{A_p}\tilde{\mathbf{u}}\cdot\delta\bar{\boldsymbol{\sigma}}\cdot\mathbf{n}\,dA$$
$$\delta^2 K\left(\bar{\boldsymbol{\sigma}}\right) = \frac{1}{2}\int_V \delta\bar{\sigma}^{kl}\delta\bar{\varepsilon}_{kl}dV = \frac{1}{2}\int_V \delta\bar{\boldsymbol{\sigma}}\cdot\cdot\,\delta\bar{\boldsymbol{\varepsilon}}\,dV$$

Reissner féle funkcionál

$$R\left(\sigma^{kl},u_r\right)=\int_V\left[\sigma^{kl}u_{(k;l)}-q^ku_k-\frac{1}{2}\sigma^{kl}C^{-1}_{klrs}\sigma^{rs}\right]dV$$
$$+\int_{A_u}\left(\tilde{u}_k-u_k\right)\sigma^{kl}n_l\,dA-\int_{A_p}u_k\tilde{p}^kdA$$

$$R\left(\boldsymbol{\sigma},\mathbf{u}\right)=\int_V\left[\boldsymbol{\sigma}\cdot\cdot\frac{1}{2}\left(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}\right)-\mathbf{q}\cdot\mathbf{u}-\frac{1}{2}\boldsymbol{\sigma}\cdot\cdot^{(4)}\mathbf{C}^{-1}\cdot\cdot\boldsymbol{\sigma}\right]dV$$
$$+\int_{A_u}\left(\tilde{\mathbf{u}}-\mathbf{u}\right)\delta\boldsymbol{\sigma}\cdot\mathbf{n}\,dA-\int_{A_p}\mathbf{u}\cdot\tilde{\mathbf{p}}\,dA$$

$$\delta_\sigma R\left(\boldsymbol{\sigma},\mathbf{u}\right)=0\quad\Longleftrightarrow\quad\begin{cases}{}^{(4)}\mathbf{C}^{-1}\cdot\cdot\boldsymbol{\sigma}-\frac{1}{2}\left(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}\right)=0 & x\in V\\ \tilde{\mathbf{u}}-\mathbf{u}=0 & x\in A_u\end{cases}$$

$$\delta_u R\left(\boldsymbol{\sigma},\mathbf{u}\right)=0\quad\Longleftrightarrow\quad\begin{cases}\boldsymbol{\sigma}\cdot\nabla+\mathbf{q}=0 & x\in V\\ \boldsymbol{\sigma}\cdot\mathbf{n}-\tilde{\mathbf{p}}=0 & x\in A_u\end{cases}$$

Hu-Washizu féle funkcionál

$$H\left(u_k,\varepsilon_{kl},\sigma^{kl}\right)=\int_V\left[\frac{1}{2}\varepsilon_{kl}C^{klrs}\varepsilon_{rs}-\sigma^{kl}\left(\varepsilon_{kl}-u_{(k;l)}\right)-q^ku_k\right]dV$$
$$-\int_{A_u}\left(u_k-\tilde{u}_k\right)\sigma^{kl}n_l\,dA-\int_{A_p}u_k\tilde{p}^kdA$$

$$H\left(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}\right)=\int_V\left[\frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot^{(4)}\mathbf{C}\cdot\cdot\boldsymbol{\varepsilon}-\boldsymbol{\sigma}\cdot\cdot\left(\boldsymbol{\varepsilon}-\frac{1}{2}\left(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}\right)\right)-\mathbf{q}\cdot\mathbf{u}\right]dV$$
$$-\int_{A_u}\left(\mathbf{u}-\tilde{\mathbf{u}}\right)\delta\boldsymbol{\sigma}\cdot\mathbf{n}\,dA-\int_{A_p}\mathbf{u}\cdot\tilde{\mathbf{p}}\,dA$$

$$\delta_u H\left(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}\right)=0\quad\Longleftrightarrow\quad\begin{cases}\boldsymbol{\sigma}\cdot\nabla+\mathbf{q}=0 & x\in V\\ \boldsymbol{\sigma}\cdot\mathbf{n}-\tilde{\mathbf{p}}=0 & x\in A_u\end{cases}$$

$$\delta_\varepsilon H\left(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}\right)=0\quad\Longleftrightarrow\quad\boldsymbol{\sigma}={}^{(4)}\mathbf{C}\cdot\cdot\boldsymbol{\varepsilon}\qquad x\in V$$

$$\delta_\sigma H\left(\boldsymbol{\sigma},\mathbf{u}\right)=0\quad\Longleftrightarrow\quad\begin{cases}\boldsymbol{\varepsilon}-\frac{1}{2}\left(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}\right)=0 & x\in V\\ \tilde{\mathbf{u}}-\mathbf{u}=0 & x\in A_u\end{cases}$$