1. GYAKORLAT

Sebességgradiens számítása

Sebességgradiens jelölése: $\underline{\underline{L}}$

$$\frac{d}{dt}(\overrightarrow{dr}) = (\overrightarrow{dr}) = \underline{\underline{L}}(\overrightarrow{dr})$$

$$\underline{\underline{L}} = \overrightarrow{v} \circ \nabla = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

1.1. Írjuk fel az xyz derékszögű Descartes koordinátarendszerben a sebességgradiens mátrixát

$$\underline{\underline{L}} = \overrightarrow{v} \circ \nabla = v^p \overrightarrow{g}_p \circ \frac{\partial}{\partial x^k} \overrightarrow{g}^k = \frac{\partial v^p}{\partial x^k} \overrightarrow{g}_p \circ \overrightarrow{g}^k + v^p \frac{\partial \overrightarrow{g}_p}{\partial x^k} \circ \overrightarrow{g}^k = (v^p{}_{,k} + v^m \Gamma^p_{mk}) \overrightarrow{g}_p \circ \overrightarrow{g}^k$$

helykoordináták	sebességkoordináták	bázisvektorok
$x_1 = x^1 = x$	$v_1 = v^1 = v_x$	$\overrightarrow{g}_1 = \overrightarrow{g}^1 = \overrightarrow{e}_x$
$x_2 = x^2 = x$	$v_2 = v^2 = v_y$	$\overrightarrow{g}_2 = \overrightarrow{g}^2 = \overrightarrow{e}_y$
$x_3 = x^3 = x$	$v_3 = v^3 = v_z$	$\overrightarrow{g}_3 = \overrightarrow{g}^3 = \overrightarrow{e}_z$

A metrikus tenzor determinánsa:

$$|g_{kl}| = g = 1$$

A Christofel szimbólumokról pedig tudjuk, hogy

$$\Gamma^m_{pk} = 0$$

Ekkor a sebességgradiens mátrix a következő formában írható fel:

$$[l^k_l] = \begin{bmatrix} v^1{}_{;1} & v^1{}_{;2} & v^1{}_{;3} \\ v^2{}_{;1} & v^2{}_{;2} & v^2{}_{;3} \\ v^3{}_{;1} & v^3{}_{;2} & v^3{}_{;3} \end{bmatrix} = \begin{bmatrix} v^1{}_{,1} & v^1{}_{,2} & v^1{}_{,3} \\ v^2{}_{,1} & v^2{}_{,2} & v^2{}_{,3} \\ v^3{}_{,1} & v^3{}_{,2} & v^3{}_{,3} \end{bmatrix}$$

Descartes rendszerben a pontosvessző helyett vesszőt írhatunk, mivel a Christoffel szimbólumok zérus értékűek. Továbbá mivel a metrikus tenzor egységtenzor, így írhatjuk, hogy

$$[l_l{}^k] = [l^k_l] = [l^{kl}] = [l_{kl}]$$

1.2. Írjuk fel az R, φ, z hengerkoordináta rendszerben a sebességgradienst

A bázisvektorok

$$\overrightarrow{e}_z = \text{áll.} \qquad \overrightarrow{e}_\varphi = \overrightarrow{e}_\varphi(\varphi) \rightarrow \frac{\partial \overrightarrow{e}_\varphi}{\partial \varphi} = -\overrightarrow{e}_R \qquad \overrightarrow{e}_R = \overrightarrow{e}_R(\varphi) \rightarrow \frac{\partial \overrightarrow{e}_R}{\partial \varphi} = \overrightarrow{e}_\varphi$$

Sebességvektor:

$$\vec{v} = v_R \vec{e}_R + v_\varphi \vec{e}_\varphi + v_z \vec{e}_z$$

$$\nabla = \frac{\partial}{\partial R}\vec{e}_R + \frac{1}{R}\frac{\partial}{\partial \varphi}\vec{e}_\varphi + \frac{\partial}{\partial z}\vec{e}_z$$

Szimbolikus jelöléssel:

$$\underline{\underline{L}} = \vec{v} \circ \nabla = \begin{bmatrix} v_R \\ v_\varphi \\ v_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial R} & \frac{1}{R}\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R}(\frac{\partial v_R}{\partial \varphi} - v_\varphi) & \frac{\partial v_R}{\partial z} \\ \frac{\partial v_\varphi}{\partial R} & \frac{1}{R}(\frac{\partial v_\varphi}{\partial \varphi} + v_R) & \frac{\partial v_\varphi}{\partial z} \\ \frac{\partial v_z}{\partial R} & \frac{1}{R}\frac{\partial v_z}{\partial \varphi} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

helykoordináták	bázisvektorok
$R = x_1 = x^1$	$\vec{g}_1 = \vec{g}^1 = \vec{e}_R$
$\varphi = x_2 = x^2$	$\vec{g}_2 = \vec{g}^2 = \vec{e}_\varphi$
$z = x_3 = x^3$	$\vec{g}_3 = \vec{g}^3 = \vec{e}_z$

A nem zérus Christofel szimbólumok:

$$\Gamma^1_{22} = -R = -x^1 \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{R} = \frac{1}{x^1}$$

A metrikus tenzor:

$$|g_{kl}| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A v^k-hoz tartozó fizikai koordináták:

$$v^{\langle K\rangle} = v^K\sqrt{g_{KK}} \Rightarrow \begin{array}{l} v^{\langle 1\rangle} = v^1\sqrt{g_{11}} = v^1 \cdot 1 = v^1 = v_1 = v_R \\ v^{\langle 2\rangle} = v^2\sqrt{g_{22}} = v^2 \cdot R = v^2 \cdot R = v_2 \cdot \frac{1}{R} = v_\varphi \\ v^{\langle 3\rangle} = v^3\sqrt{g_{33}} = v^3 \cdot 1 = v^3 = v_3 = v_z \end{array}$$

Vagyis a v^k illetve v_k fizikai koordinátái a következőek:

$$\begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} = \begin{bmatrix} v_R & \frac{v_\varphi}{R} & v_z \end{bmatrix}$$
$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_R & Rv_\varphi & v_z \end{bmatrix}$$

Ezáltal a sebbességgradiens koordinátái:

$$[l^p_k] \Rightarrow \qquad \begin{array}{l} v^1{}_{;1} = v^1{}_{,1} + \Gamma^1_{m1}v^m = v^1{}_{,1} \\ v^1{}_{;2} = v^1{}_{,2} + \Gamma^1_{m2}v^m = v^1{}_{,2} - Rv^2 \\ v^1{}_{;3} = v^1{}_{,3} + \Gamma^1_{m3}v^m = v^1{}_{,3} \\ v^2{}_{;1} = v^2{}_{,1} + \Gamma^2_{m1}v^m = v^2{}_{,1} + \frac{1}{R}v^2 \\ v^2{}_{;2} = v^2{}_{,2} + \Gamma^2_{m2}v^m = v^2{}_{,2} + \frac{1}{R}v^1 \\ v^2{}_{;3} = v^2{}_{,3} + \Gamma^2_{m3}v^m = v^2{}_{,3} \\ v^3{}_{;1} = v^3{}_{,1} + \Gamma^3_{m1}v^m = v^3{}_{,1} \\ v^3{}_{;2} = v^3{}_{,2} + \Gamma^3_{m2}v^m = v^3{}_{,2} \\ v^3{}_{;3} = v^3{}_{,3} + \Gamma^3_{m3}v^m = v^3{}_{,3} \end{array}$$

Az $l^k_{\ l}$-hez tartozó fizikai koordináták:

$$l^{\langle K\rangle}_{\ \langle L\rangle} = l^K_{\ L}\sqrt{\frac{g_{KK}}{g_{LL}}} \Rightarrow \qquad l^{\langle 1\rangle}_{\ \langle 1\rangle} = l_{RR} = l^1_{\ 1}\cdot 1 = v^1_{\ ,1} = \frac{\partial v_R}{\partial R}$$

$$l^{\langle 1\rangle}_{\ \langle 2\rangle} = l_{R\varphi} = l^1_{\ 2}\cdot\sqrt{\frac{1}{R^2}} = l^1_{\ 2}\cdot\frac{1}{R} = \frac{1}{R}(\frac{\partial v_R}{\partial\varphi} - v_\varphi)$$

$$l^{\langle 1\rangle}_{\ \langle 3\rangle} = l_{Rz} = l^1_{\ 3}\cdot 1 = \frac{\partial v_R}{\partial z}$$

$$l^{\langle 2\rangle}_{\ \langle 1\rangle} = l_{\varphi R} = l^2_{\ 1}\cdot R = (\frac{\partial}{\partial R}(\frac{v_\varphi}{R}) + \frac{1}{R}\frac{v_\varphi}{R})\cdot R = \frac{\partial v_\varphi}{\partial R}$$

$$l^{\langle 2\rangle}_{\ \langle 2\rangle} = l_{\varphi\varphi} = l^2_{\ 2}\cdot 1 = \frac{\partial}{\partial\varphi}\frac{v_\varphi}{R} + \frac{1}{R}v_R = \frac{1}{R}(\frac{\partial v_\varphi}{\partial\varphi} + v_R)$$

$$l^{\langle 2\rangle}_{\ \langle 3\rangle} = l_{\varphi z} = l^2_{\ 3}\cdot R = \frac{\partial}{\partial z}\frac{v_\varphi}{R}R = \frac{\partial v_\varphi}{\partial z}$$

$$l^{\langle 3\rangle}_{\ \langle 1\rangle} = l_{zR} = l^3_{\ 1}\cdot 1 = \frac{\partial v_z}{\partial R}$$

$$l^{\langle 3\rangle}_{\ \langle 2\rangle} = l_{z\varphi} = l^3_{\ 2}\cdot\frac{1}{R} = \frac{1}{R}\frac{\partial v_z}{\partial\varphi}$$

$$l^{\langle 3\rangle}_{\ \langle 3\rangle} = l_{zz} = l^3_{\ 3}\cdot 1 = \frac{\partial v_z}{\partial z}$$

Az $\vec{g}_1$, $\vec{g}_2$, $\vec{g}_3$ bázisokkal a sebességgradiens mátrixa:

$$[l_{kp}] = \begin{bmatrix} l_{RR} & Rl_{R\varphi} & l_{Rz} \\ \frac{1}{R}l_{\varphi R} & l_{\varphi\varphi} & \frac{1}{R}l_{\varphi z} \\ l_{zR} & Rl_{z\varphi} & l_{zz} \end{bmatrix}$$

1.3. Példa a sebességradiens számítására

Ismertek a sebességvektor koordinátái:

$$v_1 = v_x = \frac{3x}{1+t} \qquad v_2 = v_y = \frac{y}{1+t} \qquad v_3 = v_z = \frac{5z^2}{1+t}$$

melyből a sebességgradiens mátrixa a következő képpen alakul

$$\underline{\underline{L}} = \vec{v}\circ\nabla = \frac{1}{1+t}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10z \end{bmatrix}$$

Alakváltozási sebesség mátrixa:

$$\underline{\underline{D}} = \frac{1}{2}(\underline{\underline{L}} + \underline{\underline{L}}^T) = \underline{\underline{L}}$$

illetve az örvénytenzor

$$\underline{\underline{W}} = \frac{1}{2}(\underline{\underline{L}} - \underline{\underline{L}}^T) = \underline{\underline{0}}$$

2. GYAKORLAT

ALAKVÁLTOZÁSI JELLEMZŐK ELŐÁLLÍTÁSA

2.1. Derékszögű Descartes féle koordinátarendszerben

t^0 : kezdeti (azonosító) vagy vonatkoztató rendszer (konfiguráció)

t : pillanatnyi vonatkoztatási rendszer (konfiguráció)

a bázisvektorok:

$\vec{e}^{\,0}_x, \vec{e}^{\,0}_y, \vec{e}^{\,0}_z$: ortonormált bázisvektorok az azonosító konfigurációban

$\vec{e}_x, \vec{e}_y, \vec{e}_z$: ortonormált bázisvektorok a pillanatnyi konfigurációban

$$P^0 : \text{helyvektora:} \quad \vec{r}^{\,0} = \vec{r}^{\,0}(x^0, y^0, z^0)$$

$$P : \text{helyvektora:} \quad \vec{r} = \vec{r}(x, y, z)$$

a mozgástörvények:

$$\vec{r} = \vec{r}(x^0, y^0, z^0; t)$$

$$x = x(x^0, y^0, z^0; t) \qquad y = y(x^0, y^0, z^0; t) \qquad z = z(x^0, y^0, z^0; t)$$

és

$$\vec{r}^{\,0} = \vec{r}^{\,0}(x, y, z; t)$$

$$x^0 = x^0(x, y, z; t) \qquad y^0 = y^0(x, y, z; t) \qquad z^0 = z^0(x, y, z; t)$$

Az elmozdulásmező:

$$\vec{u}^{\,0} = u^0_x \vec{e}^{\,0}_x + u^0_y \vec{e}^{\,0}_y + u^0_z \vec{e}^{\,0}_z$$

$$\vec{r} = \vec{r}^{\,0} + \vec{u}^{\,0}$$

továbbá

$$\underline{\underline{F}} = \vec{r} \circ \nabla^0 = \left(\vec{r}^{\,0} + \vec{u}^{\,0}\right) \circ \nabla^0 = \vec{r}^{\,0} \circ \nabla^0 + \vec{u}^{\,0} \circ \nabla^0 = \vec{u}^{\,0} \circ \nabla^0 + \underline{\underline{I}}$$

mivel:

$$\vec{r}^{\,0} = x^0 \vec{e}^{\,0}_x + y^0 \vec{e}^{\,0}_y + z^0 \vec{e}^{\,0}_z$$

$$\vec{r}^{\,0} \circ \nabla^0 = \underline{\underline{I}}$$

$$\vec{dr} = \underline{\underline{F}} \cdot \vec{dr}^{\,0}$$

$$\nabla^0 = \frac{\partial}{\partial x^0} \vec{e}^{\,0}_x + \frac{\partial}{\partial y^0} \vec{e}^{\,0}_y + \frac{\partial}{\partial z^0} \vec{e}^{\,0}_z$$

$$[\underline{\underline{F}}] = \left(u_x^0 \overrightarrow{e}_x^0 + u_y^0 \overrightarrow{e}_y^0 + u_z^0 \overrightarrow{e}_z^0\right) \circ \left(\frac{\partial}{\partial x^0} \overrightarrow{e}_x^0 + \frac{\partial}{\partial y^0} \overrightarrow{e}_y^0 + \frac{\partial}{\partial z^0} \overrightarrow{e}_z^0\right) + \left[\overrightarrow{I}\right] =$$

$$= \begin{bmatrix} 1 + \frac{\partial u_x^0}{\partial x^0} & \frac{\partial u_x^0}{\partial y^0} & \frac{\partial u_x^0}{\partial z^0} \\ \frac{\partial u_y^0}{\partial x^0} & 1 + \frac{\partial u_y^0}{\partial y^0} & \frac{\partial u_y^0}{\partial z^0} \\ \frac{\partial u_z^0}{\partial x^0} & \frac{\partial u_z^0}{\partial y^0} & 1 + \frac{\partial u_z^0}{\partial z^0} \end{bmatrix}$$

A jobboldali Cauchy-Green-féle alakváltozási tenzor

$$\underline{\underline{C}}^0 = \underline{\underline{F}}^T \cdot \underline{\underline{F}}$$

A mátrix koordinátái pedig a következőképpen írhatóak fel, mivel szimmetrikus

$$c_{xx}^0 = \left(1 + \frac{\partial u_x^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial x^0}\right)^2$$

$$c_{yy}^0 = \left(\frac{\partial u_x^0}{\partial y^0}\right)^2 + \left(1 + \frac{\partial u_y^0}{\partial y^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial y^0}\right)^2$$

$$c_{zz}^0 = \left(\frac{\partial u_x^0}{\partial z^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial z^0}\right)^2 + \left(1 + \frac{\partial u_z^0}{\partial z^0}\right)^2$$

$$c_{xy}^0 = c_{yx}^0 = \left(1 + \frac{\partial u_x^0}{\partial x^0}\right)\frac{\partial u_x^0}{\partial y^0} + \left(1 + \frac{\partial u_y^0}{\partial y^0}\right)\frac{\partial u_y^0}{\partial x^0} + \frac{\partial u_z^0}{\partial y^0}\frac{\partial u_z^0}{\partial x^0}$$

$$c_{xz}^0 = c_{zx}^0 = \left(1 + \frac{\partial u_x^0}{\partial x^0}\right)\frac{\partial u_x^0}{\partial z^0} + \frac{\partial u_y^0}{\partial z^0}\frac{\partial u_y^0}{\partial x^0} + \left(1 + \frac{\partial u_z^0}{\partial z^0}\right)\frac{\partial u_z^0}{\partial x^0}$$

$$c_{zy}^0 = c_{yz}^0 = \frac{\partial u_x^0}{\partial z^0}\frac{\partial u_x^0}{\partial y^0} + \left(1 + \frac{\partial u_y^0}{\partial y^0}\right)\frac{\partial u_y^0}{\partial z^0} + \left(1 + \frac{\partial u_z^0}{\partial z^0}\right)\frac{\partial u_z^0}{\partial y^0}$$

Green-Lagrange-féle alakváltozási tenzor

$$\underline{\underline{E}}^0 = \frac{1}{2}\left(\underline{\underline{C}}^0 - \underline{\underline{I}}^0\right)$$

egy elem részletesen kiírva

$$c_{xx}^0 = \left(1 + \frac{\partial u_x^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial x^0}\right)^2 = 1 + 2\frac{\partial u_x^0}{\partial x^0} + \left(\frac{\partial u_x^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial x^0}\right)^2$$

ebből

$$E_{xx}^0 = \frac{\partial u_x^0}{\partial x^0} + \frac{1}{2}\left[\left(\frac{\partial u_x^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial x^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial x^0}\right)^2\right]$$

ehhez hasonlóan a főátlóban lévő elemek

$$E_{yy}^0 = \frac{\partial u_y^0}{\partial y^0} + \frac{1}{2}\left[\left(\frac{\partial u_x^0}{\partial y^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial y^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial y^0}\right)^2\right]$$

$$E_{zz}^0 = \frac{\partial u_z^0}{\partial z^0} + \frac{1}{2}\left[\left(\frac{\partial u_x^0}{\partial z^0}\right)^2 + \left(\frac{\partial u_y^0}{\partial z^0}\right)^2 + \left(\frac{\partial u_z^0}{\partial z^0}\right)^2\right]$$

a többi elemre pedig az alábbi öszefüggések adódnak

$$E^0_{xy} = E^0_{yx} = \frac{1}{2}C^0_{xy} = \frac{1}{2}C^0_{yx} = \frac{1}{2}\left(\frac{\partial u^0_x}{\partial y^0} + \frac{\partial u^0_y}{\partial x^0}\right) + \frac{1}{2}\left(\frac{\partial u^0_x}{\partial x^0}\frac{\partial u^0_x}{\partial y^0} + \frac{\partial u^0_y}{\partial y^0}\frac{\partial u^0_y}{\partial x^0} + \frac{\partial u^0_z}{\partial y^0}\frac{\partial u^0_z}{\partial x^0}\right)$$

$$E^0_{xz} = E^0_{zx} = \frac{1}{2}C^0_{xz} = \frac{1}{2}C^0_{zx} = \frac{1}{2}\left(\frac{\partial u^o_x}{\partial z^0} + \frac{\partial u^o_z}{\partial x^0}\right) + \frac{1}{2}\left(\frac{\partial u^0_x}{\partial x^0}\frac{\partial u^0_x}{\partial z^0} + \frac{\partial u^0_y}{\partial z^0}\frac{\partial u^0_y}{\partial x^0} + \frac{\partial u^0_z}{\partial z^0}\frac{\partial u^0_z}{\partial x^0}\right)$$

$$E^0_{yz} = E^0_{zy} = \frac{1}{2}C^0_{yz} = \frac{1}{2}C^0_{zy} = \frac{1}{2}\left(\frac{\partial u^0_y}{\partial z^0} + \frac{\partial u^0_z}{\partial y^0}\right) + \frac{1}{2}\left(\frac{\partial u^0_x}{\partial y^0}\frac{\partial u^0_x}{\partial z^0} + \frac{\partial u^0_y}{\partial z^0}\frac{\partial u^0_y}{\partial y^0} + \frac{\partial u^0_z}{\partial z^0}\frac{\partial u^0_z}{\partial y^0}\right)$$

2.2. Példa: egytengelyű húzás (a húzás tengelye az x tengely)

$$\vec{u}^0 = \varepsilon x^0 \vec{e}^{\,0}_x - \nu\varepsilon y^0 \vec{e}^{\,0}_y - \nu\varepsilon z^0 \vec{e}_z$$

$$u^0_y = -\nu\varepsilon y^0 \qquad u^0_z = -\nu\varepsilon z^0 \qquad u^0_x = \varepsilon x^0$$

Határozzuk meg az alakváltozási gradiens, a jobboldali Cauchy-Green, és a Green-Lagrange féle alakváltozási tenzorokat!

$$\underline{\underline{F}} =?,\ \underline{\underline{C}}^0 =?,\ \underline{\underline{E}}^0 =?$$

Az alakváltozási gradiens:

$$\left[\underline{\underline{F}}\right] = \left[\underline{\underline{I}} + \vec{u}^0 \circ \nabla^0\right] = \begin{bmatrix} 1+\varepsilon & 0 & 0 \\ 0 & 1-\nu\varepsilon & 0 \\ 0 & 0 & 1-\nu\varepsilon \end{bmatrix}$$

A jobboldi Cauchy-Green alakváltozási tenzor:

$$\left[\underline{\underline{C}}^0\right] = \left[\underline{\underline{F}}^T \cdot \underline{\underline{F}}\right] = \begin{bmatrix} (1+\varepsilon)^2 & 0 & 0 \\ 0 & (1-\nu\varepsilon)^2 & 0 \\ 0 & 0 & (1-\nu\varepsilon)^2 \end{bmatrix}$$

mivel

$$(1+\varepsilon)^2 = 1 + 2\varepsilon + \varepsilon^2 \qquad (1-\nu\varepsilon)^2 = 1 - 2\nu\varepsilon + \nu^2\varepsilon^2$$

így a Green-Lagrange féle alakváltozási tenzor:

$$\left[\underline{\underline{E}}^0\right] = \left[\frac{1}{2}\left(\underline{\underline{C}}^0 - \underline{\underline{I}}^0\right)\right] = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & -\nu\varepsilon & 0 \\ 0 & 0 & -\nu\varepsilon \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \nu^2\varepsilon^2 & 0 \\ 0 & 0 & \nu^2\varepsilon^2 \end{bmatrix}$$

Kis alakváltozásokra a lineáris közelítést használjuk, nagy alakváltozásoknál a nemlináris tag is domináns!

$$\vec{G}_1 = \underline{\underline{F}} \cdot \vec{e}^{\,0}_x = (1+\varepsilon)\,\vec{e}^{\,0}_x$$
$$\vec{G}_2 = \underline{\underline{F}} \cdot \vec{e}^{\,0}_y = (1-\nu\varepsilon)\,\vec{e}^{\,0}_y$$
$$\vec{G}_3 = \underline{\underline{F}} \cdot \vec{e}^{\,0}_z = (1-\nu\varepsilon)\,\vec{e}^{\,0}_z$$

$$\left[\underline{\underline{C}}^0\right] = [G_{KL}] = \begin{bmatrix} \vec{G}_1 \cdot \vec{G}_1 & 0 & 0 \\ 0 & \vec{G}_2 \cdot \vec{G}_2 & 0 \\ 0 & 0 & \vec{G}_3 \cdot \vec{G}_3 \end{bmatrix}$$

mint az látható.

2.3. Alakváltozási jellemzők hengerkoordináta rendszerben

t^0: a kezdeti konfiguráció

	DDKR			HKR		
koordináták:	x^0	y^0	z^0	R^0	φ^0	z^0
bázisvektorok:	$\vec{e}^{\,0}_x$	$\vec{e}^{\,0}_y$	$\vec{e}^{\,0}_z$	$\vec{e}^{\,0}_R$	$\vec{e}^{\,0}_\varphi$	$\vec{e}^{\,0}_z$

t^0: a pillanatnyi konfiguráció

	DDKR			HKR		
koordináták:	x	y	z	R	φ	z
bázisvektorok:	$\vec{e}_x$	$\vec{e}_y$	$\vec{e}_z$	$\vec{e}_R$	$\vec{e}_\varphi$	$\vec{e}_z$

Bázisvektorok:

$$\vec{e}^{\,0}_x = \vec{e}_x \qquad \vec{e}^{\,0}_y = \vec{e}_y \qquad \vec{e}^{\,0}_z = \vec{e}_z$$

$$\vec{e}^{\,0}_R = \vec{g}^{\,0}_1 \qquad \vec{e}^{\,0}_\varphi = \frac{1}{R}\vec{g}^{\,0}_2 \qquad \vec{e}^{\,0}_z = \vec{g}^{\,0}_3$$

$$\vec{e}_R = \vec{g}_1 \qquad \vec{e}_\varphi = \frac{1}{R}\vec{g}_2 \qquad \vec{e}_z = \vec{g}_3$$

Alakváltozási gradiens

$$\underline{\underline{F}} = \vec{I} + \vec{u}^{\,0} \circ \nabla^0$$

Az elmozdulásmező:

$$\vec{u}^{\,0} = u^0_R \vec{e}^{\,0}_R + u^0_\varphi \vec{e}^{\,0}_\varphi + u^0_z \vec{e}^{\,0}_z$$

A deriválási operátor:

$$\nabla^0 = \frac{\partial}{\partial R^0}\vec{e}^{\,0}_R + \frac{1}{R^0}\frac{\partial}{\partial \varphi^0}\vec{e}^{\,0}_\varphi + \frac{\partial}{\partial z^0}\vec{e}^{\,0}_z$$

A hengerkoordinátarendszerbeli egységvektok deriváltjai:

$$\frac{\partial \vec{e}^{\,0}_R}{\partial \varphi} = \vec{e}^{\,0}_\varphi \qquad \frac{\partial \vec{e}^{\,0}_\varphi}{\partial \varphi} = -\vec{e}^{\,0}_R$$

Ezek után az alakváltozási gradiens mátrixa:

$$\underline{\underline{F}} = \begin{bmatrix} \frac{\partial u^0_R}{\partial R^0} + 1 & \frac{1}{R^0}\left(\frac{\partial u^0_R}{\partial \varphi^0} - u^0_\varphi\right) & \frac{\partial u^0_R}{\partial z^0} \\ \frac{\partial u^0_\varphi}{\partial R^0} & \frac{1}{R^0}\left(\frac{\partial u^0_\varphi}{\partial \varphi^0} + u^0_R\right) + 1 & \frac{\partial u^0_\varphi}{\partial z^0} \\ \frac{\partial u^0_z}{\partial R^0} & \frac{1}{R^0}\left(\frac{\partial u^0_z}{\partial \varphi^0}\right) & \frac{\partial u^0_z}{\partial z^0} + 1 \end{bmatrix}$$

A jobboldali Couchy-Green féle alakváltozási tenzor

$$\underline{\underline{C}}^0 = \underline{\underline{F}}^T \cdot \underline{\underline{F}}$$

melyből a tenzor koordinátái a következőképpen írhatók

$$C_{xx} = (\frac{\partial u_R^0}{\partial R^0} + 1)^2 + (\frac{\partial u_\varphi^0}{\partial R^0})^2 + (\frac{\partial u_z^0}{\partial R^0})^2$$

$$C_{yy} = (\frac{1}{R^0}(\frac{\partial u_R^0}{\partial \varphi^0} - u_\varphi^0))^2 + (\frac{1}{R^0}(\frac{\partial u_\varphi^0}{\partial \varphi^0} + u_R^0) + 1)^2 + (\frac{1}{R^0}(\frac{\partial u_z^0}{\partial \varphi^0}))^2$$

$$C_{zz} = (\frac{\partial u_R^0}{\partial z^0})^2 + (\frac{\partial u_\varphi^0}{\partial z^0})^2 + (\frac{\partial u_z^0}{\partial z^0})^2$$

$$C_{xy} = C_{yx} = (\frac{\partial u_R^0}{\partial R^0} + 1)(\frac{1}{R^0}(\frac{\partial u_R^0}{\partial \varphi^0} - u_\varphi^0)) + (\frac{\partial u_\varphi^0}{\partial R^0})(\frac{1}{R^0}(\frac{\partial u_\varphi^0}{\partial \varphi^0} + u_R^0) + 1) + (\frac{\partial u_z^0}{\partial R^0})(\frac{1}{R^0}(\frac{\partial u_z^0}{\partial \varphi^0}))$$

$$C_{yz} = C_{zy} = (\frac{1}{R^0}(\frac{\partial u_R^0}{\partial \varphi^0} - u_\varphi^0))(\frac{\partial u_R^0}{\partial z^0}) + (\frac{1}{R^0}(\frac{\partial u_\varphi^0}{\partial \varphi^0} + u_R^0) + 1)(\frac{\partial u_\varphi^0}{\partial z^0}) + (\frac{1}{R^0}(\frac{\partial u_z^0}{\partial \varphi^0}))(\frac{\partial u_z^0}{\partial z^0})$$

$$C_{xz} = C_{zx} = (\frac{\partial u_R^0}{\partial R^0} + 1)(\frac{\partial u_R^0}{\partial z^0}) + (\frac{\partial u_\varphi^0}{\partial R^0})(\frac{\partial u_\varphi^0}{\partial z^0}) + (\frac{\partial u_z^0}{\partial R^0})(\frac{\partial u_z^0}{\partial z^0})$$

Green-Lagrange féle alakváltozási tenzor

$$\underline{\underline{E}}^0 = \frac{1}{2}(\underline{\underline{C}}^0 - \underline{\underline{I}})$$

melyből a tenzor koordinátái a következőképpen írhatók

$$E_{xx}^0 = \frac{1}{2}[(\frac{\partial u_R^0}{\partial R^0} + 1)^2 + (\frac{\partial u_\varphi^0}{\partial R^0})^2 + (\frac{\partial u_z^0}{\partial R^0})^2 - 1]$$

$$E_{yy}^0 = \frac{1}{2}[(\frac{1}{R^0}(\frac{\partial u_R^0}{\partial \varphi^0} - u_\varphi^0))^2 + (\frac{1}{R^0}(\frac{\partial u_\varphi^0}{\partial \varphi^0} + u_R^0) + 1)^2 + (\frac{1}{R^0}(\frac{\partial u_z^0}{\partial \varphi^0}))^2 - 1]$$

$$E_{zz}^0 = \frac{1}{2}[(\frac{\partial u_R^0}{\partial z^0})^2 + (\frac{\partial u_\varphi^0}{\partial z^0})^2 + (\frac{\partial u_z^0}{\partial z^0})^2 - 1]$$

$$E_{xy}^0 = E_{yx}^0 = \frac{1}{2}[(\frac{\partial u_R^0}{\partial R^0} + 1)(\frac{1}{R^0}(\frac{\partial u_R^0}{\partial \varphi^0} - u_\varphi^0)) + (\frac{\partial u_\varphi^0}{\partial R^0})(\frac{1}{R^0}(\frac{\partial u_\varphi^0}{\partial \varphi^0} + u_R^0) + 1) + (\frac{\partial u_z^0}{\partial R^0})(\frac{1}{R^0}(\frac{\partial u_z^0}{\partial \varphi^0}))]$$

$$E_{yz}^0 = E_{zy}^0 = \frac{1}{2}[(\frac{1}{R^0}(\frac{\partial u_R^0}{\partial \varphi^0} - u_\varphi^0))(\frac{\partial u_R^0}{\partial z^0}) + (\frac{1}{R^0}(\frac{\partial u_\varphi^0}{\partial \varphi^0} + u_R^0) + 1)(\frac{\partial u_\varphi^0}{\partial z^0}) + (\frac{1}{R^0}(\frac{\partial u_z^0}{\partial \varphi^0}))(\frac{\partial u_z^0}{\partial z^0})]$$

$$E_{xz}^0 = E_{zx}^0 = \frac{1}{2}[(\frac{\partial u_R^0}{\partial R^0} + 1)(\frac{\partial u_R^0}{\partial z^0}) + (\frac{\partial u_\varphi^0}{\partial R^0})(\frac{\partial u_\varphi^0}{\partial z^0}) + (\frac{\partial u_z^0}{\partial R^0})(\frac{\partial u_z^0}{\partial z^0})]$$