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Complete Solution for Stresses in Terms of Stress Functions Part II: Modification of Variational Principles

I. Kozák, Gy. Szeidl

In the second part of the paper the authors consider the variational problem from which the general and complete solution of equilibrium equations can be obtained as Euler equations. By deducing the dual pairs of the strain boundary conditions the static-kinematic analogy has been made complete.

1 Introduction

1.1 The book by Abovski, Andreev and Deruga (1978), which we have also cited in the first part of the paper, presents variational principles from which the solutions of equilibrium equations in terms of stress functions are obtained as Euler equations. Contrary to the papers by Tonti (1967) and Stippes (1966) there is a step ahead in the treatment of the boundary surface but all those terms needed for a complete solution on multiple-bordered regions are missing. The reason for this is that the particular solutions of the equilibrium equations are assumed to be known in advance therefore the difference between homogeneous and particular solutions, i.e., self-equilibrated stresses, are given by the Euler equations mentioned above. It is a further problem that the contradiction between the number of side conditions (six compatibility differential equations on the volume V) and the number of necessary stress functions (although three stress functions are sufficient to describe any stress condition the resulting Euler equations involve six stress functions) is also not resolved.

1.2 It is well known that the mathematical structure of the compatibility equations and the stress representations found by Beltrami are the same. This similarity is often called as static-kinematic analogy. It is obvious that the fulfillment of strain boundary conditions is the way to cause no incompatibility on S_u. Recalling that compatibility and equilbrium are dual concepts one can raise the question: under what conditions are there no stresses due to stress functions on S_t? In other words, is there a possibility to extend the static-kinematic analogy to boundary conditions?

1.3 In view of the foregoing the aims in the second part of the paper are as follows:

- With regard to the previous ideas (completeness, number of necessary stress functions, transformations of integrals on the boundary etc.) to modify and supplement the corresponding variational principles.
- If possible to extend the static-kinematic analogy to the boundary conditions on S_t.

1.4 In section 2 we focus on the free variational problem and briefly show what equations follow from the stationary condition. Section 3 is devoted to a modification of the principle of minimum potential energy and it is proved that the dual counterparts of the strain boundary conditions are also stationary conditions. Conclusions are presented in section 4 which is a short summary of the results. The last section is again an Appendix, i.e., a collection of some longer transformations.

2 Free Variational Problem

2.5 Notations and notational conventions are the same as in the first part. When citing equations of the first part the equation number is followed by a comma and the roman number I.

2.6 There arises the question in connection with equation (3.31,I) obtained from the general primal form of the principle of virtual work whether it is possible or not to establish a free variational problem where

— vanishing of variations with respect to strain fields e_{kl} of the corresponding functional ensures the fulfillment of field equations (3.32,I) on the volume V of body and that of boundary conditions (3.33,I) on the part S_t of boundary

— furthermore vanishing of variations with respect to the displacements u_k yields the fulfillment of boundary conditions (3.34a-b,I), consequently the fulfillment of stress boundary conditions on S_t.

The functional sought can be derived from the functional of the total potential energy by applying the method of Lagrange multipliers. The domain of the functional involves

the strain fields

$$e_{kl}(x) \qquad x \in V$$

the displacements

$$u_k(\xi) \qquad \xi \in S_t$$

and

the stress functions

$$H_{kl}(x) \qquad x \in V$$

as well as

$$\tilde{H}_{\kappa\lambda}(\xi) \quad \text{and} \quad \tilde{H}_{\kappa\lambda;3}(\xi) \qquad \xi \in S_t$$

In the latter case, as we have assumed so far, the stress functions meet the preconditions
$H_{AB}(X) \equiv 0 \quad x \in V$ and $\tilde{H}_{k3}(\xi) \equiv 0 \quad \xi \in S_t$.

NOTE 1: These preconditions are based on those results presented in Section 2 of part I. We remind the reader that there are three independent compatibility differential equations $\eta^{RS} = 0$ and because of that three multipliers H_{RS} are needed to maintain the equilibrium on V.

2.7 Equations of linear elasticity in terms of the variables mentioned above consist of the field equations

$$C^{plrs} e_{rs} = \epsilon^{pyk} \epsilon^{ldr} H_{yd;kr} + g^{pq} B^{l}_{.;q} + g^{lq} B^{p}_{.;q} - g^{pl} B^{k}_{.;k} \qquad x \in V \tag{2.1}$$

$$\epsilon^{Rkm} \epsilon^{Slp} e_{kl;mp} = 0 \qquad x \in V \tag{2.2}$$

and boundary conditions

$$\tilde{H}_{\kappa\lambda} - H_{\kappa\lambda} = 0 \qquad \tilde{H}_{\kappa\lambda;3} - H_{\kappa\lambda;3} = 0 \qquad \xi \in S_t \tag{2.3}$$

$$e_{\lambda\kappa} - u_{(\lambda;\kappa)} = 0 \qquad \xi \in S_t \tag{2.4a}$$

$$(e_{3\kappa} - u_{3|\kappa})_{\|\lambda} + b^{\alpha}_{\lambda}(e_{\alpha\kappa} - u_{\alpha|\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda 3;\kappa}) = 0 \qquad \xi \in S_t \tag{2.4b}$$

$$e_{\lambda\kappa} - \hat{u}_{(\lambda;\kappa)} = 0 \qquad \xi \in S_u \tag{2.5a}$$

$$(e_{3\kappa} - \hat{u}_{3|\kappa})_{\|\lambda} + b^{\alpha}_{\lambda}(e_{\alpha\kappa} - \hat{u}_{\alpha|\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda 3;\kappa}) = 0 \qquad \xi \in S_u \tag{2.5b}$$

$$\hat{t}^{\rho} = \epsilon^{3\lambda\vartheta} \epsilon^{\rho dp} \tilde{H}_{\lambda d;p\vartheta} + a^{3q} B^{\rho}_{.;q} + a^{\rho q} B^{3}_{.;q} - a^{3\rho} B^{k}_{.;k} \qquad \xi \in S_t \tag{2.6a}$$

$$\hat{t}^{3} = \epsilon^{3\lambda\vartheta} \epsilon^{3\kappa\rho} \tilde{H}_{\lambda\kappa;\rho\vartheta} + a^{3q} B^{3}_{.;q} + a^{3q} B^{3}_{.;q} - a^{33} B^{k}_{.;k} \qquad \xi \in S_t \tag{2.6b}$$

associated with a continuity condition

$$\hat{u}_l - u_l = 0 \qquad \xi \in g \tag{2.7}$$

Really, simultaneous fulfilment of equations (2.2), (2.4a,b), (2.5a,b) and (2.7) ensures that the strains e_{kl} are kinematically admissible. Recalling the assertion from the beginning of paragraph 3.10 in part I we can conclude with regard to continuity conditions (2.7) that the integration of conditions (2.4a) and (2.4b) yields the actual displacement $u_k(\xi)$ on S_t. If in addition to this, field equation (2.1) is also satisfied then the equilibrium on V is maintained while simultaneous fulfilment of (2.6a) and (2.6b) is equivalent to that of stress boundary conditions.

NOTE 2: Here and in the sequel, with regard to its simplicity, we confine ourselves to Schaefer's solution. However, the line of thought presented herein can be applied with ease to Gurtin's solution.

2.8 Now let

$$\Pi_2 = \Pi_2(e_{kl}, u_l, H_{RS}, \tilde{H}_{\kappa\lambda}, \tilde{H}_{\kappa\lambda;3}) = \Pi_2^{V} + \Pi_2^{S_t} + \Pi_2^{Su} + \Pi_2^{G} \tag{2.8}$$

be the functional sought in which

$$\Pi_2^V = \int_V [\frac{1}{2} e_{pl} C^{plrs} e_{rs} - (g^{pq} B^l_{.;q} + g^{lq} B^p_{.;q} - g^{pl} B^k_{.;k}) e_{pl}] \, dV + \int_V \epsilon^{krm} \epsilon^{lsp} e_{rs;mp} H_{lk} \, dV \tag{2.9a}$$

$$\begin{aligned} \Pi_2^{S_t} = & -\int_{S_t} [\hat{t}^l - n_3(a^{3q} B^l_{.;q} + a^{lq} B^3_{.;q} - a^{3l} B^k_{.;k})] \, u_l \, dA \\ & - \int_{S_t} n_3 \epsilon^{\kappa\eta 3} \epsilon^{\lambda\vartheta 3} \{ (e_{\lambda\kappa} - u_{(\lambda|\kappa)}) \tilde{H}_{\eta\vartheta;3} \\ & + [(e_{3\kappa} - u_{3|\kappa})_{\|\lambda} + b^\alpha_\lambda (e_{\alpha\kappa} - u_{\alpha|\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda 3;\kappa}) - b^\beta_\beta (e_{\lambda\kappa} - u_{(\lambda|\kappa)})] \tilde{H}_{\eta\vartheta} + \\ & [e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta} - (u_{\lambda|\kappa})_{\|\vartheta} - u_{3|\lambda} b_{\vartheta\kappa}] \tilde{H}_{\eta 3} - b_{\eta\vartheta} (e_{\lambda\kappa} - u_{(\lambda|\kappa)}) \tilde{H}_{33} \} \, dA \end{aligned} \tag{2.9b}$$

$$\begin{aligned} \Pi_2^{S_u} = & \int_{S_u} n_3(a^{3q} B^l_{.;q} + a^{lq} B^3_{.;q} - a^{3l} B^k_{.;k}) \, \hat{u}_l \, dA \\ & - \int_{S_u} n_3 \epsilon^{\kappa\eta 3} \epsilon^{\lambda\vartheta 3} \{ (e_{\lambda\kappa} - \hat{u}_{(\lambda|\kappa)}) H_{\eta\vartheta;3} \\ & + [(e_{3\kappa} - \hat{u}_{3|\kappa})_{\|\lambda} + b^\alpha_\lambda (e_{\alpha\kappa} - \hat{u}_{\alpha|\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda 3;\kappa}) - b^\beta_\beta (e_{\lambda\kappa} - \hat{u}_{(\lambda|\kappa)})] H_{\eta\vartheta} \\ & + [e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta} - (\hat{u}_{\lambda|\kappa})_{\|\vartheta} - \hat{u}_{3|\lambda} b_{\vartheta\kappa}] H_{\eta 3} - b_{\eta\vartheta} (e_{\lambda\kappa} - \hat{u}_{(\lambda|\kappa)}) H_{33} \} \, dA \end{aligned} \tag{2.9c}$$

$$\Pi_2^G = -\oint_g n_3 \epsilon^{\kappa\eta 3} \tau^\vartheta [(u_{\vartheta|\kappa} - \hat{u}_{\vartheta|\kappa}) \tilde{H}_{\eta 3} - (u_{3|\kappa} - \hat{u}_{3|\kappa}) \tilde{H}_{\eta\vartheta}] \, ds + \oint_g \tau^\eta \epsilon^{ldp} \tilde{H}_{\eta d;p} (u_l - \hat{u}_l) \, ds \tag{2.10}$$

Observe that the functional contains all the stress functions including those regarded to be zero. When investigating what stationary conditions follow from equation $\delta\Pi_2 = 0$ as a variational principle we shall take into consideration, as we did earlier, that $H_{kl}(x)$ and $H_{kl}(\xi)$ are of special structure – see NOTE 8 in part I and paragraph 2.2 .

2.9 Vanishing of variation

$$\delta\Pi_2 = \delta_e \Pi_2 + \delta_u \Pi_2 + \delta_H \Pi_2 + \delta_{\tilde{H}} \Pi_2 = 0 \tag{2.11}$$

as a variational principle ensures the fulfilment not only of field equations (2.1) and (2.2) but also of the boundary conditions (2.3), (2.4a,b), (2.5a,b), (2.6a,b) and continuity condition (2.7).

In what follows we briefly outline the proof of the above assertion. Because of the independence of variations taken with respect to distinct variables stationary condition (2.11) is equivalent to the equations

$$\delta_e \Pi_2 = \delta_e \Pi_2^V + \delta_e \Pi_2^{S_t} + \delta_e \Pi_2^{Su} = 0 \tag{2.12a}$$

$$\delta_u \Pi_2 = \delta_u \Pi_2^{St} + \delta_u \Pi_2^G = 0 \tag{2.12b}$$

$$\delta_H \Pi_2 = \delta_H \Pi_2^V + \delta_H \Pi_2^{Su} = 0 \tag{2.12c}$$

and

$$\delta_{\tilde{H}} \Pi_2 = \delta_{\tilde{H}} \Pi_2^{S_t} + \delta_{\tilde{H}} \Pi_2^G = 0 \tag{2.12d}$$

2.10 Equation (2.12a) can be transformed into a suitable form if utilizing (3.26a-b,I) we substitute

— (3.30,I) into the expression $\delta_e \Pi_2^V$ replacing first e_{rs}, $e_{\rho\vartheta;3}$ and $e_{\rho\vartheta}$ by their variations δe_{rs}, $\delta e_{\rho\vartheta;3}$ and $\delta e_{\rho\vartheta}$

— the opposite of (3.29,I) for $\delta_e \Pi_2^{S_t}$ replacing first S, $\tilde{H}_{\kappa\lambda}$ and $\tilde{H}_{\kappa\lambda;3}$ by S_t, $\delta\tilde{H}_{\kappa\lambda}$ and $\delta\tilde{H}_{\kappa\lambda;3}$

and

— the opposite of (3.29,I) for $\delta_e \Pi_2^{S_u}$ replacing first S, $\tilde{H}_{\kappa\lambda}$ and $\tilde{H}_{\kappa\lambda;3}$ by S_u, $\delta H_{\kappa\lambda}$ and $\delta H_{\kappa\lambda;3}$.

Upon a subsequent rearrangement we have

$$\begin{aligned} \delta_e \Pi_2 = & \int_V [C^{plrs} e_{rs} - (\epsilon^{pyk} \epsilon^{ldr} H_{yd;kr} + g^{pq} B^l_{.;q} + g^{lq} B^p_{.;q} - g^{pl} B^k_{.;k})] \, \delta e_{lp} \, dV \\ & + \int_{S_t} n_3 \epsilon^{\kappa\rho 3} \epsilon^{\lambda\vartheta 3} [-(H_{\lambda\kappa} - \tilde{H}_{\lambda\kappa}) \delta e_{\rho\vartheta;3} + (H_{\lambda\kappa;3} - \tilde{H}_{\lambda\kappa;3}) \delta e_{\rho\vartheta}] \, dA = 0 \end{aligned} \tag{2.13}$$

Since the variations δe_{lp} , $\delta e_{\rho\vartheta;3}$ and $\delta e_{\rho\vartheta}$ are arbitrary, this equation can only be satisfied when field equation (2.1) and boundary conditions (2.3-a,b) are also fulfilled.

2.11 Observing that $\delta_u\Pi_2^{S_t}$ is the opposite of I_{1U}^S if in the latter S, u_λ and u_3 are respectively replaced by S_u, δu_λ and δu_3, then utilizing (A.50) — in which S_o, g_o and u_3 are also to be replaced by S_u, g and δu_3 — and (2.10) we obtain from (2.12b)

$$\delta_u\Pi_2 = -\int_{S_t} [\hat{t}^l - n_3(\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa} + a^{3q}B^l_{.;q} + a^{lq}B^3_{.;q} - a^{3l}B^k_{.;k})]\,\delta u_l\,dA \,=\, 0 \tag{2.14}$$

Since in (2.14) no condition for δu_l is set down it is arbitrary. Consequently, fulfillment of equation (2.14), or what is the same thing, fulfillment of stationary condition (2.12b) yields the boundary conditions (2.6a) and (2.6b).

2.12 As regards equation (2.12c) one should remember that condition (3.19,I) is not independent of (3.8a-b,I). In the light of this circumstance it can easily be shown that the fulfillment of stationary condition (2.12c) is equivalent to that of field equation (2.2) – $H_{AB}\,\equiv\,0$ on V, consequently, we assume that $\delta H_{AB}\,\equiv\,0$ — and boundary conditions (2.5a,b) even if $\delta H_{\eta 3}$ and δH_{33} are different from zero, otherwise arbitrary on S_u.

2.13 Making use of the independence of variations $\delta_{\tilde{H}}\Pi_2^{S_t}$ and $\delta_{\tilde{H}}\Pi_2^G$ we can replace (2.12d) by the following two conditions

$$\delta_{\tilde{H}}\Pi_2^{S_t} = 0 \qquad \text{and} \qquad \delta_{\tilde{H}}\Pi_2^G = 0 \tag{2.15}$$

Since $\tilde{H}_{\kappa 3} = 0$ we can cancel those terms in (2.15-a) — see (2.9b) — which involve $\tilde{H}_{\kappa 3}$ and $\tilde{H}_{33}$. In this way we obtain from (2.15-a) with regard to the arbitrariness of $\delta\tilde{H}_{\eta\vartheta}$ and $\delta\tilde{H}_{\eta\vartheta;3}$ that the boundary conditions (2.4a,b) also hold.

2.14 Before investigating what equations follow from the stationary condition (2.15-b) we define two vector fields $\delta\tilde{r}^l(\xi)$ and $\delta\tilde{w}_l(\xi)$ on the curve g separating boundary parts S_u and S_t in order to simplify the necessary transformations. Let

$$\frac{d\,\delta\tilde{r}^l}{ds} = -\tau^\eta\epsilon^{ldp}\delta\tilde{H}_{\eta d;p} \qquad\qquad \xi\in g \tag{2.16}$$

This equation always has a solution for the unknown vector field.

$$\delta\tilde{r}^l\mathbf{a}_l = -\int_{s_o}^{s} \tau^\eta\epsilon^{ldp}\delta\tilde{H}_{\eta d;p}\mathbf{a}_l\,ds$$

In addition to this

$$\frac{d\,\delta\tilde{r}^1}{ds} = -\tau^\eta\epsilon^{123}(\delta\tilde{H}_{\eta 3;2} + \delta\tilde{H}_{\eta 2;3}) \qquad\qquad \frac{d\,\delta\tilde{r}^2}{ds} = \tau^\eta\epsilon^{123}(\delta\tilde{H}_{\eta 3;1} + \delta\tilde{H}_{\eta 1;3}) \qquad\qquad \xi\in g \tag{2.17}$$

$$\frac{d\,\delta\tilde{r}^3}{ds} = -\tau^\eta\epsilon^{3\delta\pi}\delta\tilde{H}_{\eta\delta;\pi} \qquad\qquad \xi\in g \tag{2.18}$$

where with regard to (3.35a-b,I)

$$\delta\tilde{H}_{\lambda 3;\kappa} = \delta\tilde{H}_{\lambda 3|\kappa} = \delta\tilde{H}_{\lambda 3,\kappa} - \Gamma^r_{\lambda\kappa}\delta\tilde{H}_{r3} - \Gamma^r_{3\kappa}\delta\tilde{H}_{\lambda r} \qquad\qquad \xi\in S \tag{2.19a}$$

$$\delta\tilde{H}_{\lambda\kappa;\rho} = \delta\tilde{H}_{\lambda\kappa|\rho} = \delta\tilde{H}_{\lambda\kappa\|\rho} - b_{\lambda\rho}\delta\tilde{H}_{3\kappa} - b_{\kappa\rho}\delta\tilde{H}_{\lambda 3} \qquad\qquad \xi\in S \tag{2.19b}$$

Further let

$$\frac{d\,\delta\tilde{w}_\eta}{ds} = \tau^\vartheta(\epsilon_{\vartheta\eta 3}\delta\tilde{r}^3 + \delta\tilde{H}_{\eta\vartheta}) \qquad\qquad \xi\in g \tag{2.20}$$

It is clear that the latter equation also has a solution for the vector field $\delta\tilde{w}_\eta$.

NOTE 3: In view of (2.17), (2.20) and (2.19a,b) we may write

$$\delta\tilde{r}^1 = \delta\tilde{r}^1(\delta\tilde{H}_{\eta 2;3},\dots) \qquad \delta\tilde{r}^2 = \delta\tilde{r}^2(\delta\tilde{H}_{\eta 1;3},\dots) \qquad\qquad \xi\in g$$

and

$$\delta\tilde{w}_1 = \delta\tilde{w}_1(\delta\tilde{H}_{1\vartheta},\dots) \qquad \delta\tilde{w}_2 = \delta\tilde{w}_2(\delta\tilde{H}_{2\vartheta},\dots) \qquad\qquad \xi\in g$$

where the variations of $\tilde{H}_{\eta 2;3}$, $\tilde{H}_{\eta 1;3}$, $\tilde{H}_{1\vartheta}$ and $\tilde{H}_{2\vartheta}$ are independent of each other and arbitrary. Consequently, we may assume without any loss of generality that $\delta\tilde{r}^\lambda$ and $\delta\tilde{w}_\eta$ are independent and arbitrary on g. Later on it will also turn out that $\delta\tilde{r}^3$ plays no role in the final form of stationary condition $\delta_{\tilde{H}}\Pi_2^G = 0$.

2.15 Now we can turn our attention to the stationary condition (2.15-b). Using (2.10), substituting (2.16) and (2.20) for

$$\tau^{\eta}\epsilon^{ldp}\delta\tilde{H}_{\eta d;p} \quad \text{and} \quad \tau^{\vartheta}\delta\tilde{H}_{\eta\vartheta}$$

and entering into no details — these are presented in paragraph 5.1 — we have

$$\delta_{\tilde{H}}\Pi_2^G = -\oint_g [\epsilon^{\kappa\eta 3}\frac{d}{ds}(u_{3|\kappa} - \hat{u}_{3|\kappa})]\delta\tilde{w}_\eta ds + \oint_g [\frac{d}{ds}(u_\kappa - \hat{u}_\kappa)]\delta\tilde{r}^\kappa ds = 0 \tag{2.21}$$

if we also bear in mind that because of the assumption $\tilde{H}_{\eta 3} = 0$ the corresponding term has been cancelled in (2.10).

It is obvious that the vanishing of $\delta_{\tilde{H}}\Pi_2^G$ for arbitrary $\delta\tilde{w}_\eta$ and $\delta\tilde{r}^\kappa$ is equivalent to the fulfilment of equations

$$\frac{d}{ds}(u_{3|\kappa} - \hat{u}_{3|\kappa}) = 0 \quad \text{and} \quad \frac{d}{ds}(u_\kappa - \hat{u}_\kappa) = 0 \qquad \xi \in g \tag{2.22}$$

If the latter two equations hold then continuity condition (2.7) can always be satisfied by means of a proper choice in respect of the initial values.

NOTE 4: When applying direct methods there is no need to utilize the line of thought presented in paragraphs 2.8 and 2.9 in order that one can prove the fulfilment of continuity condition (2.7).

NOTE 5: Functional defined by the equations (2.8), (2.9a,b,c) and (2.10) corresponds to the last functional published on p.224 in Abovski et al. (1978). There are, however, some significant differences detailed as follows:

1. The functional presented in this paper does not imply any contradiction concerning the number of compatibility differential equations and that of stress functions. Both are three and not six as it is the case in Abovski et al. (1978).
2. The present formulation allows us to divide the boundary into parts S_u and S_t on which various boundary conditions can be imposed.
3. The domain of functional Π_2 involves stress functions defined on S_t and this is the circumstance which enables us to handle boundary conditions of various types.
4. It is also worthy of mention that the continuity of displacements on curve g is not a precondition but it follows from the stationarity of functional Π_2.

3 Static-Kinematic Analogy

3.1 If preconditions are set down on some variables then functional Π_2 can assume a much simpler form. If the strains are kinematically admissible then (2.2) and (2.4a,b) hold and both the displacements and their derivatives on S are continuous along the curve g. If in addition to this stress functions $\tilde{H}_{\kappa\lambda}$ and $\tilde{H}_{\kappa\lambda;3}$ satisfying stress boundary conditions (2.6a,b) are known then functional Π_2 — see equations (2.8) to (2.10) — reduces to functional

$$\Pi_1(e_{kl}, u_l) = \Pi_1^V(e_{kl}) + \Pi_1^{S_t}(u_l) + C_1^{S_u} \tag{3.1}$$

where

$$\Pi_1^V(e_{kl}) = \int_V [\frac{1}{2}e_{pl}C^{plrs}e_{rs} - (g^{pq}B^l_{.;q} + g^{lq}B^p_{.;q} - g^{pl}B^k_{.;k})e_{pl}]\, dV \tag{3.2a}$$

$$\Pi_1^{S_t}(u_l) = -\int_{St} n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa}u_l dA \tag{3.2b}$$

and

$$C_1^{S_u} = \int_{S_u} n_3(a^{3q}B^l_{.;q} + a^{lq}B^3_{.;q} - a^{3l}B^k_{.;k})]\, \hat{u}_l\, dA \tag{3.2c}$$

NOTE 6: The same functional can be obtained from that of the total potential energy

$$\Pi(e_{kl}, u_l) = \frac{1}{2}\int_V e_{pl}C^{plrs}e_{rs}\, dV - \int_V b^l u_l\, dV - \int_{S_t} \hat{t}^l u_l\, dA$$

if one substitutes I^B_{V1} (equations (3.2-a,I) and (3.3,I)) and (2.6a,b) for the second volume integral and $\hat{t}^l$, respectively, keeping in mind that $u_l(\xi) = \hat{u}_l(\xi)$ on S_u.

3.2 Functional (3.1) can be transformed further performing partial integrations in $\Pi_1^{S_t}$ in order that Π_1 should depend on e_{kl} only. As regards the details we refer to paragraph 5.2 . Finally one has

$$\Pi_1(e_{kl}) = \Pi_1^V(e_{kl}) + \Pi_1^{S_t 1}(e_{kl}) + \Pi_1^G + C_1^G + C_1^{S_u} \tag{3.3}$$

in which

$$\Pi_1^{S_t 1}(e_{kl}) = \int_{S_t} n_3 \epsilon^{\kappa\eta 3} \epsilon^{ldp} (-\tilde{H}_{\eta d;p} e_{l\kappa} + \tilde{H}_{\eta d} e_{\kappa l;p})\, dA \tag{3.4a}$$

$$\Pi_1^G(e_{kl}) = \oint_g \tau^\eta \epsilon^{\kappa\eta 3} (\tilde{H}_{\eta\vartheta} e_{3\kappa} - \tilde{H}_{\eta 3} e_{\vartheta\kappa})\, ds \tag{3.4b}$$

and

$$C_1^G = -\oint_g \tau^\eta \epsilon^{ldp} \hat{u}_l \tilde{H}_{\eta d;p}\, ds - \oint_g \tau^\eta \epsilon^{\kappa\eta 3} (\tilde{H}_{\eta\vartheta} \hat{u}_{3;\kappa} - \tilde{H}_{\eta 3} \hat{u}_{\vartheta;\kappa})\, ds \tag{3.4c}$$

3.3 Functional (3.3) is subjected to subsidiary conditions which ensure that the strains e_{kl} are kinematically admissible. In contrast to the foregoing one has to choose those conditions of single-valuedness being given in terms of strains e_{kl}. Consequently, for strains to be kinematically admissible it is necessary that the field equation (3.7-a,I), the kinematic boundary condition (2.5a,b) and the boundary condition of compatibility

$$n_3 \eta^{3b} = \epsilon^{3\eta\kappa} \epsilon^{ldp} e_{\eta d;p\kappa} = 0 \qquad \xi \in S_t \tag{3.5}$$

should be fulfilled. Let

$$\check{H}_{RS}(x) = \check{H}_{SR}(x) \qquad x \in V$$

$$\overset{*}{H}_{kl}(\xi) = \overset{*}{H}_{lk}(\xi) \quad \text{and} \quad H_{\eta\vartheta;3} = \overset{*}{H}_{\vartheta\eta;3} \qquad \xi \in S_u$$

and

$$w_b(\xi) \qquad \xi \in S_t$$

be undetermined Lagrange multipliers.

In accordance with all that has been said when seeking what equations can be obtained from the stationarity of functional Π_1 one should supplement the functional by the sum of integrals

$$\Pi_S = \Pi_S^V + \Pi_S^{S_t} + \Pi_S^{S_u} = 0 \tag{3.6}$$

where

$$\Pi_S^V = -I_1^V(\check{H}_{RS}) \tag{3.7a}$$

$$\Pi_S^{S_u} = -I_1^S(S_u, \hat{u}_l, \overset{*}{H}_{kl}, \overset{*}{H}_{\eta\vartheta;3}) \tag{3.7b}$$

and

$$\Pi_S^{S_t} = -\int_{S_t} \epsilon^{3\eta\kappa} \epsilon^{ldp} e_{\eta d;p\kappa} w_l\, dA \tag{3.7c}$$

As regards the notations the parameters that have been changed are marked in equations (3.7a) and (3.7b) — see (3.21a,I) and (3.21b,I) for details.

It is also worthy of mention that integrals (3.7a) and (3.7b) are considered under the same assumptions as I_1^V and I_1^S were earlier, including the structure of multipliers as well as the not independent condition (3.19,I).

NOTE 7: For strains e_{kl} to be kinematically admissible it is also necessary that some further conditions, referred to as continuity conditions, should be satisfied on curve g. Joint fulfilment of the former and the latter conditions — which are presented in the next paragraph — is not only necessary but also sufficient for strains e_{kl} to be kinematically admissible.

3.4 It follows from the kinematic equations that

$$\tau^\vartheta e_{\eta\vartheta} = \tau^\vartheta \hat{u}_{(\eta;\vartheta)} \qquad \xi \in g \tag{3.8a}$$

must hold. Using (A.25) it is immediately verified that

$$\frac{d\,\omega^3}{ds} = \tau^\eta \omega^3_{;\eta} = \tau^\eta \epsilon^{3\lambda\vartheta} \frac{1}{2} \hat{u}_{\vartheta;\lambda\eta} = \tau^\eta \epsilon^{3\lambda\vartheta} e_{\vartheta\eta;\lambda} \qquad \xi \in g \tag{3.8b}$$

This equation expresses that the rigid body rotation ω^3 should be the same in both sides of curve g. As regards the other two components of rigid body rotation one obtains from (A.26) and (A.27)

$$\omega^r_{.;p} = \epsilon^{rql}(\frac{1}{2}u_{l;qp} + \frac{1}{2}u_{q;lp} - \frac{1}{2}u_{q;lp}) = \epsilon^{rql}(e_{lq;p} - e_{q;lp}) \qquad x \in V$$

Changing r and p to ϑ and η and decomposing the sums one may write after some manipulation that

$$\frac{d\omega^\vartheta}{ds} = \tau^\eta \omega^\vartheta_{.;\eta} = \tau^\eta \epsilon^{\vartheta 3\lambda}(e_{\lambda 3;\eta} - \hat{u}_{3;\lambda\eta}) = \tau^\eta \epsilon^{\vartheta 3\lambda}(e_{\lambda\eta;3} - e_{3\eta;\lambda}) \qquad \xi \in g \tag{3.8c}$$

The above line of thought implies the assumption that the displacements and their covariant derivatives taken on the surface are continuous when one goes through curve g. Since neither $\hat{u}_k$ nor $\hat{u}_{k;\lambda\vartheta}$ can be varied freely

$$\delta\hat{u}_k = 0 \qquad \text{and} \qquad \delta\hat{u}_{k;\lambda\vartheta} = 0 \qquad \xi \in Su$$

from which in comparison with (3.8a,b,c) it follows immediately that the variations δe_{kl} on g are subject to the conditions

$$\tau^\vartheta \delta e_{\eta\vartheta} = 0, \qquad \tau^\eta \epsilon^{3\lambda\vartheta} \delta e_{\vartheta\eta;\lambda} = 0 \qquad \xi \in g \tag{3.9a}$$

and

$$\tau^\eta \epsilon^{\vartheta 3\lambda} \delta e_{\lambda 3;\eta} = \tau^\eta \epsilon^{\vartheta 3\lambda}(\delta e_{\lambda\eta;3} - \delta e_{3\eta;\lambda}) \qquad \xi \in g \tag{3.9b}$$

Since

$$e_{\lambda 3} = \frac{1}{2}(u_{\lambda;3} + u_{3;\lambda}) = \frac{1}{2}(u_{\lambda;3} + \hat{u}_{3;\lambda}) \qquad \xi \in g \tag{3.10}$$

it is easily seen that $e_{\kappa 3}$ can be varied freely on g.

Consequently, when varying the sum $\Pi_1 + \Pi_S$ with respect to strains e_{kl} in order to find what equations follow from the stationarity condition one should keep in mind that e_{kl} can be varied freely everywhere on V and S except the curve g on which the variations δe_{kl} are to meet the preconditions (3.9a) and (3.9b).

3.5 Now we shall consider what equations can be obtained from the stationary condition

$$\delta_e\Pi_1 + \delta_e\Pi_S = I^V_\Pi + I^{S_u}_\Pi + I^{S_t}_\Pi + I^G_\Pi = 0 \tag{3.11}$$

in which I^V_Π, $I^{S_t}_\Pi$, $I^{S_u}_\Pi$ and I^G_Π denote respectively the integrals taken on V, S_t, S_u and g when the transformations aimed to bring $\delta_e\Pi_1 + \delta_e\Pi_S$ into a suitable form have been completed. At present they are not known. It is, however, obvious that each of the integrals I^V_Π, $I^{S_u}_\Pi$, $I^{S_t}_\Pi$ and I^G_Π must vanish separately since the domains are different. In what follows we shall utilize this circumstance without referring to it again.

3.6 Recalling (3.2a), (3.7a), (3.21a,I) and repeating the line of thought leading from (A.58) to (A.59a,b) and (A.61) we have

$$\delta_e\Pi^V_1 + \delta_e\Pi^V_S = \delta_e\Pi^V_1 + I^V_1(\delta e_{kl}, \check{H}_{RS}) = I^V_\Pi + I^{S_u}_4 + I^{S_t}_4 \tag{3.12}$$

where

$$I^V_\Pi \;=\; \int_V [C^{plrs}e_{rs} - (\epsilon^{pyk}\epsilon^{ldr}\check{H}_{yd;kr} + g^{pq}B^l_{.;q} + g^{lq}B^p_{.;q} - g^{pl}B^k_{.;k})]\,\delta e_{lp}\,dV \;=\; 0 \tag{3.13}$$

and

$$I^{S_u}_4 + I^{S_t}_4 = -\int_{S_u+S_t} n_3\epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}(\check{H}_{\lambda\kappa}\delta e_{\rho\vartheta;3} - \check{H}_{\lambda\kappa;3}\delta e_{\rho\vartheta})\,dA \tag{3.14}$$

3.7 It is obvious that the integral $I^{S_u}_\Pi$ consists of two parts.

$$I^{S_u}_\Pi = I^{S_u}_4 + \delta_e\Pi^{S_u}_S \tag{3.15}$$

As regards the variation $\delta\Pi^{S_u}_S$ let us consider the equations (3.7b), (3.21b,I), (3.26a,I) and (3.29,I) from which follows that

$$\delta\Pi^{S_u}_S = -I^S_1(S_u, \delta e_{kl}, \overset{*}{H}_{kl})|_{\hat{u}=0} \;=\; -I^S_{1E}(S_u, \delta e_{kl}, \overset{*}{H}_{kl})$$

since the variation is taken with respect to e_{kl}. After performing the necessary letter changes one can substitute (3.29,I) for $\delta_e \Pi_S^{S_u}$ in (3.15). Keeping (3.14) also in mind we obtain

$$\begin{aligned} I_\Pi^{S_u} &= I_4^{S_u} + \delta_e \Pi_S^{S_u} \\ &= \int_{S_u} n_3 \epsilon^{\kappa\rho 3} \epsilon^{\lambda\vartheta 3} [(\overset{*}{H}_{\lambda\kappa} - \check{H}_{\lambda\kappa}) \delta e_{\rho\vartheta;3} - (\overset{*}{H}_{\lambda\kappa;3} - \check{H}_{\lambda\kappa;3}) \delta e_{\rho\vartheta}] \, dA = 0 \end{aligned} \tag{3.16}$$

3.8 Now we concentrate on the last two integrals $I_\Pi^{S_u}$ and I_Π^G whose sum will be separated into two groups depending on whether they include $\tilde{H}_{kl}$, $\check{H}_{kl}$ or v_l:

$$I_\Pi^{S_t} + I_\Pi^G = (I_{\Pi H}^{S_t} + I_{\Pi H}^G) + (I_{\Pi w}^{S_t} + I_{\Pi w}^G) \tag{3.17}$$

To begin with we shall consider those integrals containing $\tilde{H}_{kl}$, $\check{H}_{kl}$. It is clear on the basis of (3.3), (3.4a), (3.4b), (3.6), (3.7a), (3.11) and (3.12) that

$$I_{\Pi H}^{S_t} + I_{\Pi H}^G = \delta_e \Pi_1^{S_t 1} + I_4^{S_t} + \delta_e \Pi_1^G (\tilde{H}_{kl}) \tag{3.18}$$

Comparison of (3.4a) to (A.59b) yields

$$\delta_e \Pi_1^{S_t 1} = I_{2E}^S (S_t, \delta e_{kl}, \tilde{H}_{kl}) \tag{3.19a}$$

Next integral $I_4^{S_t}$ will be considered. An appropriate result can be achieved in three steps.

1. We notice that the surface integral in (A.60) is equal to $I_4^{S_t}$ provided that the following replacements are made.

$$S_o \longrightarrow S_t \qquad H \longrightarrow \check{H} \qquad e \longrightarrow \delta e$$

2. Comparing (3.4b) to (A.60) we also notice, that the line integral in (A.60) coincides with $\delta_e \Pi_1^G$ if further letter replacements are made.

$$g_o \longrightarrow g \qquad H \longrightarrow \check{H} \qquad e \longrightarrow \delta e$$

3. Then we solve the equation resulting for $I_4^{S_t}$.

Finally we have

$$I_4^{S_t} = -I_{2E}^S (S_t, \delta e_{kl}, \check{H}_{kl}) - \Pi_1^G (\delta e_{kl}, \check{H}_{kl}) \tag{3.19b}$$

Upon substitution of equations (3.19a) and (3.19b) into (3.18) we obtain

$$I_{\Pi H}^{S_t} + I_{\Pi H}^G = I_{2E}^S (S_t, \delta e_{kl}, \tilde{H}_{kl} - \check{H}_{kl}) + \Pi_1^G (\delta e_{kl}, \tilde{H}_{kl} - \check{H}_{kl}) \tag{3.20}$$

since the integrals are linear in H_{kl}. Let

$$\bar{H}_{kl} = \tilde{H}_{kl} - \check{H}_{kl} \tag{3.21}$$

With (A.60), (3.4b) and (3.21) it follows from (3.20) that

$$\begin{aligned} I_{\Pi H}^{S_t} + I_{\Pi H}^G = & \int_{S_t} n_3 \epsilon^{\kappa\rho 3} \epsilon^{\lambda\vartheta 3} [\bar{H}_{\lambda\kappa} \delta e_{\rho\vartheta;3} - \bar{H}_{\lambda\kappa} \delta e_{\rho 3;\vartheta} - \bar{H}_{3\kappa} \delta e_{\rho\vartheta;\lambda} \\ & - \bar{H}_{\lambda\kappa;3} \delta e_{\rho\vartheta} + \bar{H}_{\lambda\kappa|\vartheta} \delta e_{\rho 3} + \bar{H}_{3\kappa|\lambda} \delta e_{\rho\vartheta}] \, dA \\ & - \oint_{g_o} n_3 \epsilon^{\kappa\eta 3} (\tau^\vartheta \bar{H}_{\eta\vartheta} \delta e_{3\kappa} - \tau^\lambda \delta e_{\lambda\kappa} \bar{H}_{\eta 3}) \, ds \end{aligned} \tag{3.22}$$

If in (3.22) we substitute

$$\tilde{H}_{kl} \quad \text{for} \quad \delta e_{kl}$$

and

$$e_{kl} \quad \text{for} \quad \bar{H}_{kl}$$

we arrive at (A.54). It immediately follows from this that the point of departure of those transformations leading to (A.54), i.e., equation (3.26a,I), is the final form of (3.22) provided that e_{kl} and H_{kl} are respectively replaced by $\bar{H}_{kl}$ and δe_{kl}. In this way we have

$$\begin{aligned} I_{\Pi H}^{S_t} + I_{\Pi H}^G = & \int_S n_3 \epsilon^{\kappa\eta 3} \epsilon^{\lambda\vartheta 3} \{ \bar{H}_{\lambda\kappa} \delta e_{\eta\vartheta;3} + (\bar{H}_{\kappa\lambda\|\vartheta} + \bar{H}_{\lambda\kappa\|\vartheta}) \delta e_{\eta 3} \\ & + (\bar{H}_{3\kappa\|\lambda} + b_\lambda^\alpha \bar{H}_{\alpha\kappa} - \bar{H}_{\kappa\lambda;3} + \bar{H}_{\lambda 3|\kappa} - b_\beta^\beta \bar{H}_{\lambda\kappa}) \delta e_{\eta\vartheta} + b_{\eta\vartheta} \bar{H}_{\lambda\kappa} \delta e_{33} \} \, dA \end{aligned} \tag{3.23a}$$

which means that

$$I^G_{\Pi H} = 0 \tag{3.23b}$$

3.9 The last integral to be considered is the one which involves the multiplier w_l. It is clear from (3.6), (3.7a,b,c) and the resolution (3.17) that

$$I^{S_t}_{\Pi w} + I^G_{\Pi w} = \delta_e \Pi^{S_t}_S = \Pi^{S_t}_S(\delta e_{kl}) \tag{3.24}$$

In the sequel it is our aim to make use of equations (A.51) and (3.25,I) in order to avoid carrying out long formal transformations. Comparing (3.17) and the surface integral in (A.51) it becomes clear that after substituting respectively

$$S_t\,,g\,,\delta e_{kl} \text{ and } v_l \text{ for } S_o\,,g_o\,,\tilde{H}_{kl} \text{ and } w_l$$

in (A.51) and (3.25,I) one obtains an equation with the unknown $\Pi^{S_t}_S(\delta e_{kl})$ or since the two expressions are of the same value of I^S_{1U}. Consequently, one may write by separating surface and line integrals

$$\begin{aligned} I^{S_t}_{\Pi w} = &-\int_{S_t} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3} w_{(\lambda|\kappa)}\delta e_{\eta\vartheta;3}dA \\ &+\int_{S_t} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[-w_{\lambda|\kappa\|\vartheta}\delta e_{\eta 3} - w_{(\lambda|\kappa)} b_{\eta\vartheta}\delta e_{33} + (b^\beta_\beta w_{(\lambda|\kappa)} - b^\alpha_\lambda w_{\alpha|\kappa})\delta e_{\eta\vartheta}]\,dA \\ &+\int_{S_t} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[-w_{3|\kappa\|\lambda}\delta e_{\eta\vartheta} - w_{3|\lambda} b_{\vartheta\kappa}\delta e_{3\eta} - w_{3|\kappa} b_{\vartheta\lambda}\delta e_{\eta 3}]dA \end{aligned} \tag{3.25}$$

and

$$I^G_{\Pi w} = \oint_g \tau^\eta \epsilon^{ldp}\delta e_{\eta d;p} w_l\,ds - \oint_g n_3\epsilon^{\kappa\eta 3}\tau^\vartheta(w_{\vartheta|\kappa}\delta e_{\eta 3} - w_{3|\kappa}\delta e_{\eta\vartheta})\,ds\,. \tag{3.26}$$

With respect to (3.17), (3.23a,b) and (3.25) we have

$$\begin{aligned} I^{S_t}_{\Pi} = \int_{S_t} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{&(\bar{H}_{\lambda\kappa} - w_{(\lambda|\kappa)})\delta e_{\eta\vartheta;3} \\ &+[(\bar{H}_{3\kappa} - w_{3|\kappa})_{\|\lambda} + b^\alpha_\lambda(\bar{H}_{\alpha\kappa} - w_{\alpha|\kappa}) - (\bar{H}_{\kappa\lambda;3} - \bar{H}_{\lambda 3;\kappa}) - b^\beta_\beta(\bar{H}_{\lambda\kappa} - w_{(\lambda|\kappa)})]\delta e_{\eta\vartheta} \\ &+[\bar{H}_{\kappa\lambda\|\vartheta} + \bar{H}_{\lambda\kappa\|\vartheta} - (w_{\lambda|\kappa})_{\|\vartheta} - w_{3|\lambda} b_{\vartheta\kappa}]\delta e_{\eta 3} - b_{\eta\vartheta}(\bar{H}_{\lambda\kappa} - w_{(\lambda|\kappa)})\delta e_{33}\}\,dA = 0 \end{aligned} \tag{3.27}$$

Making use of (3.17), (3.23a,b) and (3.26) it follows that

$$I^G_\Pi = I^G_{\Pi w}$$

Decomposing the sum involving ϵ in the first line integral we obtain from (3.26)

$$\begin{aligned} I^G_\Pi = I^G_{\Pi w} = &-\oint_g \tau^\eta \epsilon^{3\lambda\vartheta}\delta e_{\vartheta\eta;\lambda} w_3\,ds - \oint_g \tau^\eta \epsilon^{\vartheta 3\lambda}(\delta e_{\lambda\eta;3} - \delta e_{3\eta;\lambda}) w_\vartheta\,ds \\ &+\oint_g \tau^\eta \epsilon^{\vartheta 3\lambda} w_{\eta|\vartheta}\delta e_{\lambda 3}\,ds + \oint_g \tau^\vartheta \epsilon^{3\kappa\eta} w_{3|\kappa}\delta e_{\eta\vartheta}\,ds \;=\; 0 \end{aligned}$$

Substituting (3.9a) and (3.9b) then performing partial integration with respect to s we get

$$I^G_\Pi = \oint_g \tau^\vartheta \epsilon^{\eta 3\lambda} 2w_{(\eta|\vartheta)}\delta e_{\lambda 3}\,ds \;=\; 0 \tag{3.28}$$

Since in equations (3.13), (3.14), (3.27) and (3.28) no restrictions for

$$\begin{aligned} &\delta e_{lp} && x \in V \\ &\delta e_{\rho\vartheta|3},\ \delta e_{\rho\vartheta} && \xi \in S_u \\ &\delta e_{\eta\vartheta|3},\ \delta e_{\eta\vartheta},\ \delta e_{\eta 3},\ \delta e_{33} && \xi \in S_t \end{aligned}$$

and

$$\delta e_{3\lambda} \qquad \xi \in g$$

are set down they are arbitrary. Consequently, the vanishing of integrals I^V_Π, $I^{S_u}_\Pi$, $I^{S_t}_\Pi$ and I^G_Π yields

— the field equation

$$C^{plrs}e_{rs} = \epsilon^{pyk}\epsilon^{ldr}\check{H}_{yd;kr} + g^{pq}B^{l}_{\ .;q} + g^{lq}B^{p}_{\ .;q} - g^{pl}B^{k}_{\ .;k} \qquad x \in V \tag{3.29}$$

as the Euler equation of the problem

— the boundary conditions

$$\overset{*}{\check{H}}_{\lambda\kappa} = \check{H}_{\lambda\kappa} \quad \text{and} \quad \overset{*}{\check{H}}_{\lambda\kappa;3} = \check{H}_{\lambda\kappa;3} \qquad \xi \in S_u \tag{3.30}$$

$$\tilde{H}_{\lambda\kappa} - \check{H}_{\lambda\kappa} = \bar{H}_{\lambda\kappa} = w_{(\lambda|\kappa)} \qquad \xi \in S_t \tag{3.31a}$$

$$(\bar{H}_{3\kappa} - w_{3|\kappa})_{\|\lambda} + b^{\alpha}_{\lambda}(\bar{H}_{\alpha\kappa} - w_{\alpha|\kappa}) - (\bar{H}_{\kappa\lambda;3} - \bar{H}_{\lambda3;\kappa}) = 0 \qquad \xi \in S_t \tag{3.31b}$$

$$\bar{H}_{\kappa\lambda\|\vartheta} + \bar{H}_{\lambda\kappa\|\vartheta} - w_{\lambda|\kappa\|\vartheta} - w_{3|\lambda}b_{\vartheta\kappa} = 0 \qquad \xi \in S_t \tag{3.31c}$$

and

— the continuity condition

$$\tau^{\vartheta}w_{(\eta|\vartheta)} = 0 \qquad \xi \in g \tag{3.32}$$

The following notes are aimed at interpreting equations (3.29) to (3.32) obtained from the extremum condition (3.11).

NOTE 8: Equation (3.29) is the general and complete stress function solution of equilibrium equations set up in this form by Schaefer (1953). In other words the general and complete solution of equilibrium equations can really be derived from the extremum of the total potential energy provided that the subsidiary conditions are appropriately chosen.

NOTE 9: According to equation (3.30) multipliers defined on S_u coincide with those defined on V. Consequently, the stress function solution is valid on S_u.

NOTE 10: Equations (3.31a,b,c) are the *dual counterparts* of kinematic boundary conditions (3.8a-b,I) and supplementary condition (3.19) since [the former] (the latter) conditions can immediately be obtained from the [the latter] (the former) ones if we substitute [*e* for *H* and *u* for *w*] (*H* for *e* and *w* for *u*). Since (3.19,I) is not independent of (3.8a) nor is (3.31c) of (3.31a). Consequently, (3.31a) and (3.31b) are the substantial boundary conditions.

NOTE 11: It follows from NOTE 8 and equations (3.33,I) that stress functions on V and S_t may differ from each other in the symmetric part of the gradient of a vector field

$$\tilde{H}_{kl}(\xi) - H_{kl}(\xi) = \tilde{w}_{(k;l)}(\xi) \qquad \xi \in S \tag{3.33}$$

In the light of this circumstance there arises the question whether boundary conditions (3.31a) and (3.31b) contradict equation (3.33) or not. In what follows we shall prove that there is no formal contradiction between (3.31a), (3.31b) and (3.33). Our point of departure is the equation

$$\tilde{H}_{kl}(\xi) - \check{H}_{kl}(\xi) = \bar{H}_{kl}(\xi) = w_{(k;l)}(\xi) \qquad \xi \in S_t \tag{3.34}$$

which is obviously equivalent to (3.33). The latter equation implies (3.31a). However, in contrast to (3.34) no derivatives taken along the normal to S_t appear in $w_{(\lambda|\kappa)}$. Proof of the second part of our statement requires some preparations.

Let r^l be the axial vector of $w_{k;l}$. As it is well known

$$r^l = \frac{1}{2}\epsilon^{lpq}w_{q;p} \qquad \text{and} \qquad w_{[l;p]} = -\epsilon_{lps}r^s \qquad \xi \in S_t \tag{3.35}$$

In view of (A.2b) and (3.34) one may write

$$w_{[l;p];\lambda} = \frac{1}{2}(w_{l;p\lambda} - w_{p;l\lambda}) = \frac{1}{2}(w_{l;p\lambda} + w_{\lambda;lp} - w_{\lambda;lp} - w_{p;l\lambda}) \qquad \xi \in S_t$$

or

$$w_{[l;p];\lambda} = \bar{H}_{l\lambda;p} - \bar{H}_{\lambda p;l} \qquad \xi \in S_t$$

After exchanging the left and right sides let us add (3.34) to the latter equation. With respect to (3.35b) we have

$$w_{l;p\lambda} = w_{(l;p);\lambda} - \epsilon_{lps}r^{s}_{\ .;\lambda} = \bar{H}_{l\lambda;p} - \bar{H}_{\lambda p;l} + \bar{H}_{lp;\lambda}\,. \qquad \xi \in S_t \tag{3.36}$$

Now we shall prove that the latter equation, which is a consequence of (3.34), implies (3.31b).

Because of the indices in (3.31a,b) we shall confine ourselves to those equations obtained by setting l and p to 3 and κ:

$$w_{3;\kappa\lambda} = w_{(3;\kappa);\lambda} - \epsilon_{3\kappa\sigma} r^{\sigma}_{.;\lambda} = \bar{H}_{3\lambda;\kappa} - \bar{H}_{\lambda\kappa;3} + \bar{H}_{3\kappa;\lambda} \qquad \xi \in S_t \tag{3.37}$$

It can be shown readily by using (A.9) that

$$\bar{H}_{3\kappa;\lambda} = \bar{H}_{3\kappa\|\lambda} + b^{\alpha}_{\lambda}\bar{H}_{\alpha\kappa} - b_{\kappa\lambda}\bar{H}_{33} \qquad \xi \in S_t$$

Substitution of the latter equation into the right hand side of (3.37) yields

$$\bar{H}_{3\kappa\|\lambda} + b^{\alpha}_{\lambda}\bar{H}_{\alpha\kappa} - (\bar{H}_{\kappa\lambda;3} - \bar{H}_{\lambda3;\kappa}) - b_{\kappa\lambda}\bar{H}_{33} = w_{3;\kappa\lambda} = w_{(3;\kappa);\lambda} - \epsilon_{3\kappa\sigma} r^{\sigma}_{.;\lambda} \qquad \xi \in S_t \tag{3.38}$$

Using again (A.9) to transform the right hand side of (3.38) we have

$$w_{3;\kappa\lambda} = w_{3;\kappa\|\lambda} + b^{\alpha}_{\lambda} w_{\alpha|\kappa} - b_{\kappa\lambda} w_{3;3} \qquad \xi \in S_t$$

With this equation it follows from (3.38) that

$$(\bar{H}_{3\kappa} - w_{3|\kappa})_{\|\lambda} + b^{\alpha}_{\lambda}(\bar{H}_{\alpha\kappa} - w_{\alpha|\kappa}) - (\bar{H}_{\kappa\lambda;3} - \bar{H}_{\lambda3;\kappa}) - b_{\kappa\lambda}(\bar{H}_{33} - w_{3;3}) = 0 \qquad \xi \in S_t \tag{3.39}$$

If $\bar{H}_{33} - w_{3;3} = 0$ equation (3.39) reduces to (3.31b). In this case no derivative of w_l taken along the normal to the surface appear in (3.39). In other words the principle of minimum potential energy ensures the fulfilment of that part of equation (3.36) which does not involve the derivative of w_l along the normal to S_t.

NOTE 12: With regard to (3.30a) and (3.31a,b) condition (3.33) is a continuity condition of the form

$$\tau^{\vartheta}\tilde{H}_{\lambda\vartheta} = \tau^{\vartheta}\overset{*}{H}_{\lambda\vartheta} \qquad \xi \in g$$

for those multipliers defined on S_t and S_u respectively.

4 Concluding Remarks

4.1 The most important functionals of Lagrange's type have been presented in the second part of the article. As a result of our modification the corresponding variational principles imply no contradiction concerning the number of compatibility equations and that of stress functions in terms of which one obtains the general and complete stress function solution of equilibrium equations from the stationary condition.

4.2 The variational formulation presented ensures more freedom in respect of the boundary conditions (both strain and traction boundary conditions can be imposed on distinct parts of S).

4.3 The static–kinematic analogy has been supplemented by appropriate boundary conditions. Each of the strain boundary conditions and the supplementary identity on S_u has its dual counterpart on S_t and vice versa.

5 Appendix

5.1 Transformation of integral $\delta_{\tilde{H}}\Pi_2^G$ of equations (2.10) and (2.15b)

Cancelling the term that involves $\delta\tilde{H}_{\eta3}$ and substituting $\tau^{\eta}\epsilon^{ldp}\delta\tilde{H}_{\eta d;p}$ and $\tau^{\vartheta}\delta\tilde{H}_{\eta\vartheta}$ taken from (2.16) and (2.20) into (2.15b) we obtain

$$\begin{aligned}\delta_{\tilde{H}}\Pi_2^G &= \oint_g n_3\epsilon^{\kappa\eta3}\tau^{\vartheta}(u_{3|\kappa} - \hat{u}_{3|\kappa})\delta\tilde{H}_{\eta\vartheta}\,ds + \oint_g n_3\tau^{\eta}\epsilon^{ldp}\delta\tilde{H}_{\eta d;p}(u_l - \hat{u}_l)\,ds \\ &= \oint_g \epsilon^{\kappa\eta3}(\frac{d\,\delta\tilde{w}_{\eta}}{ds} + \tau^{\vartheta}\epsilon_{\eta\vartheta3}\delta\tilde{r}^3)(u_{3|\kappa} - \hat{u}_{3|\kappa})\,ds - \oint_g (u_l - \hat{u}_l)\frac{d\,\delta\tilde{r}^l}{ds}\,ds = 0\end{aligned}$$

since $n_3 = 1$.

Making use of the equation $\epsilon^{\kappa\eta 3}\epsilon_{\eta\vartheta 3} = -\delta^{\kappa}_{\vartheta}$ and performing partial integrations we arrive at (2.21).

5.2 Transformation of integral (3.2b)

Applying the rule (A.20) of partial integration and observing that

$$\epsilon^{\kappa\eta 3}\epsilon^{ldp}u_{\kappa;lp} \equiv 0 \qquad \xi \in S_t$$

one obtains

$$\Pi_1^{S_t}(e_{kl}) = I_1^{S_t} + I_1^G = -\int_{S_t} n_3\epsilon^{\kappa\eta 3}\epsilon^{ldp}(-\tilde{H}_{\eta d;p}u_{l;\kappa} + \tilde{H}_{\eta d}u_{\kappa;lp})dA - \oint_g \tau^{\eta}\epsilon^{ldp}\tilde{H}_{\eta d;p}\hat{u}_l ds \tag{B.1}$$

where $I_1^{S_t}$ and I_1^G stand for the surface and line integral, respectively. It follows from the decomposition theorem (A.2) and the kinematic equation (2.6,I) that

$$u_{l;\kappa} = e_{l\kappa} + u_{[l;\kappa]} \qquad \text{and} \qquad u_{\kappa;l} = e_{\kappa l} + u_{[\kappa;l]} \qquad \xi \in S_t \tag{B.2}$$

Upon substitution of (B.2) into the surface integral $I_1^{S_t}$ and comparing the result with (3.4a,b,c) one has

$$I_1^{S_t} = \Pi_1^{S_t 1} + I_2^{S_t} \tag{B.3}$$

where

$$\begin{aligned} I_2^{S_t} &= -\int_{S_t} n_3\epsilon^{\kappa\eta 3}\epsilon^{ldp}(-\tilde{H}_{\eta d;p}u_{[l;\kappa]} + \tilde{H}_{\eta d}u_{[\kappa;p]})\,dA \\ &= -\int_{S_t} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}(-\tilde{H}_{\eta\vartheta;3}u_{[\lambda;\kappa]} + \tilde{H}_{\eta 3;\vartheta}u_{[\kappa;p]} + \tilde{H}_{\eta\vartheta;\lambda}u_{[3;\kappa]} \\ &\qquad + \tilde{H}_{\eta\vartheta}u_{[\kappa;\lambda];3} - \tilde{H}_{\eta 3}u_{[\kappa;\lambda];\vartheta} + \tilde{H}_{\eta\vartheta}u_{[\kappa;3];\lambda})\,dA \end{aligned}$$

Since

$$\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\tilde{H}_{\eta\vartheta}u_{[\kappa;\lambda];3} \equiv 0 \qquad \xi \in S_t \tag{B.4-a}$$

and

$$\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\tilde{H}_{\eta\vartheta;3}u_{[\lambda;\kappa]} \equiv 0 \qquad \xi \in S_t \tag{B.4-b}$$

with the rule of partial integrations one obtains

$$\begin{aligned} I_2^{S_t} = I_3^{S_t} + I_1^G = -\int_{S_t} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[-\tilde{H}_{\eta 3}(u_{[\lambda;\kappa]} + u_{[\kappa;\lambda]})_{;\vartheta} - \tilde{H}_{\eta\vartheta}(u_{[3;\kappa]} + u_{[\kappa;3]})_{;\lambda}]\,dA \\ - \oint_g \tau^{\eta}\epsilon^{\kappa\eta 3}(\tilde{H}_{\eta\vartheta}u_{[3;\kappa]} - \tilde{H}_{\eta 3}u_{[\vartheta;\kappa]})\,ds \end{aligned}$$

in which the surface integral vanishes. In view of equation (2.6,I) and continuity condition (2.22) the assumption is made that

$$u_{[3;\kappa]} = e_{\kappa 3} - \hat{u}_{3;\kappa} \qquad \text{and} \qquad u_{[\vartheta;\kappa]} = e_{\vartheta\kappa} - \hat{u}_{\vartheta;\kappa} \qquad \xi \in g \tag{B.5}$$

Substitution of (B.5) and a subsequent comparison of the result with (3.4a,b,c), (B.1) and (B.2) yields

$$\Pi_1^{S_t} = \Pi_1^{S_t 1} + \Pi_1^G + C_1^G$$

which proves the correctness of transformation mentioned in paragraph 3.2 .

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Address: Professor Dr. Imre Kozák, Associate Professor Dr. György Szeidl, Department of Mechanics, University of Miskolc, H–3515 Miskolc–Egyetemváros