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Complete Solution for Stresses in Terms of Stress Functions Part I. Derivation from the Principle of Virtual Work

I. Kozák, Gy. Szeidl

In the first part of the paper it is proved that for solid bodies the general and complete solution of equilibrium equations in terms of stress functions can be derived from the general primal form of the virtual work principle provided that the necessary and sufficient conditions for the strains to be kinematically admissible are known. This result is of methodological significance since the line of thought can be applied to every case for which the sufficient and necessary conditions of kinematical admissibility of strains are clarified.

1 Introduction

1.1 The general solution of the two dimensional equilibrium equations in terms of a stress function was found by Airy (1863). Three dimensional generalizations of Airy's function are the stress function solutions obtained by Maxwell (1870) and Morera (1892) (cf. e.g. Gurtin, 1972) who established two alternative solutions, each involving a triplet of stress functions. Beltrami (1892) observed that these solutions represent two special cases of setting equal to zero three components of the stress function tensor involved in his solution.

Completeness proofs for Beltrami's solution, which were given among others by Ornstein (1954), Günther (1954) and Dorn & Schield (1956), are valid only for those regions whose boundary consists of a single closed surface. This fact was noticed by Rieder (1960) who observed that for regions bordered by more than one closed surface (multiple-bordered region) Beltrami's solution is totally self-equilibrated on each surface therefore it can not be complete. By supplementing Beltrami's solution but independently of each other Schaefer (1953) and Gurtin (1963) found formally different and complete solutions.

According to the papers cited introduction of stress functions took place intuitively. In this respect a step ahead was made by Tonti (1967) who derived the incomplete Beltrami solution from a variational principle. Although the paper by Stippes (1966) derived the complete solution, like Tonti he assumed that there are no body forces. It is a further problem that both Tonti and Stippes used the six Saint Venant compatibility conditions as side conditions although these are not independent of each other. Consequently the solution involves six stress functions (As it is well known any state of stress can be given in terms of three stress functions.). This contradiction is the dual counterpart of the paradox found by Southwell (1938) who gave the statically admissible stresses in terms of three stress functions and got only three (instead of six) compatibility equations from the principle of minimum complementary energy – he did not know that the six compatibility equations are not independent. It is also worthy of mention that the surface integrals obtained during the mathematical transformations are completely left out of consideration. In addition both Stippes and Tonti assumed that there are no body forces.

The book (1978) written by Abovski, Andreev and Deruga provides a detailed description of variational principles in classical elastostatics including those variational principles where the solutions of the equilibrium equations in terms of stress functions appear as Euler equations. In comparison with the papers by Tonti (1967) and Stippes (1966) there is a step ahead in the treatment of the boundary

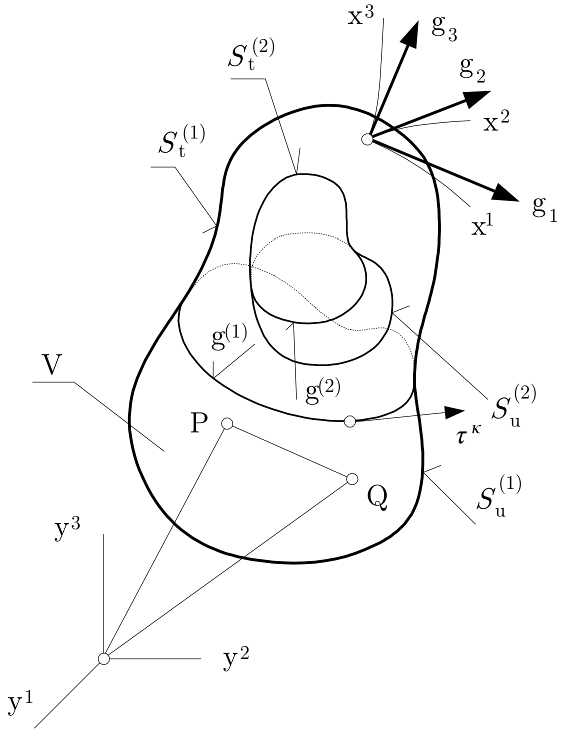

Figure 1. Body with Two Single Closed Surfaces

surface but all the terms needed for a complete solution on multiple-bordered regions are missing. The reason for this is the assumption that the particular solutions of equilibrium equations are assumed to be known therefore the difference between homogeneous and particular solutions, i.e., self-equilibrated stresses are given by the above mentioned Euler equations.

1.2 In a view of the foregoing the aims in the first part of our paper are as follows

- Derivation of the general and complete solution of equilibrium equations in terms of three stress functions from the principle of virtual work and by solving in this way the dual counterpart of the Southwell paradox.
- To point out clearly in connection with the first aim how important a role the side conditions – three independent comptibility conditions on the volume and the so called strain (or kinematic) boundary conditions – play.
- To show how the integrals taken on the boundary can be transformed into a suitable form and what mechanical meaning the resulting boundary integrals and boundary conditions have.

In section 2 we give the notations and notational conventions and collect some preliminary results. Section 3 is devoted to the derivation of the complete solution of equilibrium equations from the principle of virtual work. Section 4 is a summary of the results. The last section is an Appendix, i.e., a collection of some longer transformations.

In the second part of the paper the modification of the corresponding variational principles and the dual pairs of the strain boundary conditions are presented.

2 Preliminaries

2.1 The bounded region of three-dimensional space occupied by the body and the surface of the body are denoted respectively by V and S. For the sake of simplicity we shall assume that the region V is simple-connected. The surface S may, however, consist of not only one but more closed surfaces — a multiply-bordered region — as well. The surface S is divided into parts S_u and S_t whose common bounding curve is denoted by g. The body represented in Fig.1. is bordered by two single closed surfaces.

If the body is bordered by N closed regular surfaces $S^{(i)}$ $(i = 1, \ldots, N; \quad N \geq 2)$ and each surface is divided into two parts $S_u^{(i)}$, $S_t^{(i)}$ separated from each other by a bounding curve $g^{(i)}$ then S_u, S_t and

g are the unions of the subsurfaces $S_u^{(i)}$ and $S_t^{(i)}$ and the bounding curves g_i, respectively. Any of the surfaces $[S_u]$ $\{S_u^{(i)}\}$ or $[S_t]$ $\{S_t^{(i)}\}$ may be an empty set.

Indicial notations and three coordinate systems

- the $(y^1y^2y^3)$ Cartesian
- the $(x^1x^2x^3)$ curvilinear and
- the $(\xi^1\xi^2\xi^3)$ curvilinear, defined on the surface S,

are employed throughout this paper. Scalars and tensors, unless the opposite is stated, are denoted independently of the coordinate system by the same letter. Distinction is aided by the indication of the arguments y, x and ξ being used to denote the totality of the corresponding coordinates.

Volume integrals — except the formula (2.8) — and surface integrals are considered, respectively, in the coordinate systems $(x^1x^2x^3)$ and $(\xi^1\xi^2\xi^3)$. Consequently, in the case of integrals, arguments are omitted.

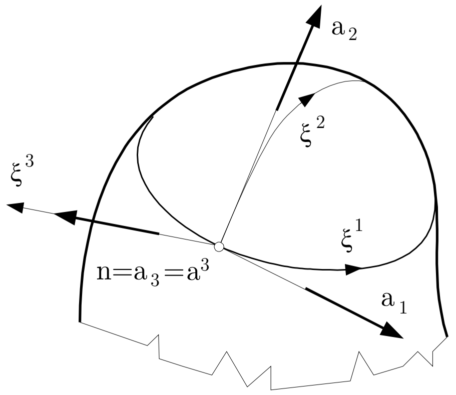

Figure 2. Coordinate System

In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a semicolon denote covariant differentiation with respect to the corresponding subscripts. Latin and Greek indices range over the integers 1, 2, 3 and 1, 2, respectively. ϵ^{klm} and ϵ_{pqr} stand for the permutation tensors; δ_k^l is the *Kronecker* delta. In the Cartesian system $(y^1y^2y^3)$ $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the base vectors while the covariant and contravariant components of tensors are coinciding. In the system of coordinates $(x^1x^2x^3)$ $\mathbf{g}_k$ and $\mathbf{g}^l$ are the covariant and contravariant base vectors. The corresponding metric tensors are denoted by g_{kl} and g^{pq}.

We assume that there exists one-to-one relationship $y^k = y^k(x^1, x^2, x^3)$ between the Cartesian coordinate y^k and the curvilinear coordinates x^1, x^2 and x^3 where y^k is differentiable with respect to x^l as many times as required. Consequently

$$J_{y,x} = \left| \frac{\partial y^k}{\partial x^l} \right| \neq 0.$$

Contravariant and covariant vector fields B^l and C_b are transformed in accordance with the rules

$$C_b(x) = C_p(y)\frac{\partial y^p}{\partial x^b} \qquad B^k(x) = B^p(y)\frac{\partial x^k}{\partial y^p}\,. \tag{2.1}$$

Equations and calculations can be better understood by introducing a suitable surface oriented coordinate system. The equation of a boundary surface is written as $x^k = x^k(\xi^1, \xi^2)$ where ξ^1 and ξ^2 are the surface coordinates. Let ξ^3 be the distance measured on the outward unit normal $\mathbf{n}$ to the surface. On S $\xi^3 = 0$. [Base vectors] {Metric tensors} on S are denoted by [$\mathbf{a}^k$ and $\mathbf{a}_k$] {a_{kl} and a^{kl}}. In the coordinate system $(\xi^1\xi^2\xi^3)$

$$\mathbf{n} = \mathbf{a}_3 = \mathbf{a}^3\,, \qquad n^3 = 1 \qquad \text{and} \qquad n^\eta = 0. \tag{a*}$$

If $\left|\xi^3\right| / (\min\{|R_1|, |R_2|\}) < 1$ in which R_1 and R_2 are the principal radii of curvature on S then the relationship $x^k = x^k(\xi^1, \xi^2, \xi^3)$ is always one-to-one. Under this condition the functional determinant is not vanishing:

$$J_{x,\xi} = \left| \frac{\partial x^k}{\partial \xi^l} \right| \neq 0.$$

Upon change of coordinates (x^1, x^2, x^3) and (ξ^1, ξ^2, ξ^3) a tensor $D^p_{.q}(x)$ of the second order follows the transformation rules

$$D^k_{.l}(\xi) = D^p_{.q}(x)\frac{\partial \xi^k}{\partial x^p}\frac{\partial x^q}{\partial \xi^l} \qquad D^p_{.q}(x) = D^k_{.l}(\xi)\frac{\partial x^p}{\partial \xi^k}\frac{\partial \xi^p}{\partial x^k} \tag{2.2}$$

where

$$\frac{\partial x^k}{\partial \xi^l}\frac{\partial \xi^p}{\partial x^k} = \delta^p_l \ .$$

We shall assume that the vector and tensor fields involved in the investigations are sufficiently smooth.

2.2 Let u_k be the displacement field (or displacements for brevity's sake). Further let e_{kl} be the strain tensor (or strains for brevity's sake). By t^{kl} we denote the stress tensor (or stresses for brevity's sake). Displacements and strains will be assumed to be small.
Boundary conditions — inasmuch as there are any boundary conditions prescribed — have the following forms:
Displacement boundary condition:

$$u_k = \hat{u}_k \ . \qquad \xi \in S_u \tag{2.3}$$

Stress boundary condition:

$$n_k t^{kl} = \hat{t}^l \qquad \xi \in S_t \tag{2.4}$$

where $\hat{u}_k$ and $\hat{t}^l$ are the prescribed displacements and tractions.

Kozák (1980) systematizees [the general primal forms]{the primal forms ordered to prescribed boundary conditions} of the principle of virtual work, the corresponding assertions and, in addition to this, it gives the missing [general dual forms] {dual forms ordered to prescribed boundary conditions} and dual assertions together with their proofs.

The line of thought of the present section is grounded on a well known assertion related to the general primal form of principle of virtual work and on a proper choice of the corresponding subsidiary conditions.

2.3 The strains $e_{kl}(x)$ are said to be [compatible] {kinematically admissible} if the differential equations

$$e_{kl}(x) = (u_{l;k} + u_{k;l})/2 = u_{(l;k)} \qquad x \in V \tag{2.5}$$

have a single-valued solution – irrespective of a rigid body motion – for the displacements $u_l(x)$ $x \in V$ and the solution [does not satisfy other conditions] {satisfies the displacement boundary conditions (2.3)}.

Accordingly, the displacements u_k are [compatible] {kinematically admissible} if they are differentiable at least twice and meet [no other conditions]{the displacement boundary condition (2.3)}.

2.4 Let b^l be the body forces. The stresses $t^{kl}(x)$ $x \in V$ will be referred to as [equilibrated]{statically admissible} if they satisfy the equilibrium equations

$$t^{kl}_{..;k}(x) + b^l = 0 \qquad x \in V \tag{2.6}$$

and [meet no other conditions]{the stress boundary conditions (2.4)}.

For a linearly elastic body the boundary conditions (2.3), (2.4) and field equations (2.5), (2.6) should be supplemented by the stress-strain relations. Assuming anisotropic material the stress strain relations have the form

$$t^{kl} = C^{klpq}\, e_{pq}$$

where C^{klpq} is the tensors of elastic coefficients.

2.5 According to a fundamental result of potential theory (Gurtin, 1972) the body forces b^l always admit the representation

$$b^l = -\Delta B^l = -g^{pq} B^l_{.;pq} \qquad x \in V \tag{2.7}$$

where $B^l(x)$ is obtained from the transformation formulas (2.1) provided that the integral

$$B^l\left[y^r(Q)\right] = \frac{1}{4\pi}\int_V \frac{b^l\left[y^r(P)\right]}{|y^s(P)-y^s(Q)|}dV_P \qquad Q\in V \tag{2.8}$$

have been determined first. With reference to the above result we shall assume that the vector field $B^l(x)$ is known. Repeated application of (2.7) results in that b^l admitting the representation

$$b^l = -\Delta\Delta\Psi^l = -g^{pq}g^{mn}\Psi^l{}_{.;mnpq}. \qquad x\in V \tag{2.9}$$

In what follows we shall assume that the vector field Ψ^l is also known.

3 Derivation of the Stress Function Solution from the Principle of Virtual Work

3.1 Equation

$$\int_V t^{kl}\,e_{kl}\,dV - \int_V b^l u_l\,dV - \int_S n_3 t^{3l} u_l\,dA\;=\;0 \tag{3.1}$$

is the general primal form of principle of virtual work. The above equation is associated with the following direct assertion: *Suppose that the strains $e_{kl}(x)$ are obtained from equation* (2.5). *If equation* (3.1) *holds for any compatible displacements $u_k(x)$ then the stresses $t^{kl}(x)$ are equilibrated.*

By substituting the kinematic equations (2.5) as subsidiary conditions and performing partial integrations the assertion can easily be proved. Really, upon substitution of the integral

$$\int_V t^{kl}u_{(l;k)}dV = \int_S n_3 t^{3l}u_l dA + \int_V t^{kl}_{..;k}u_l dV$$

into (3.1) and a subsequent rearrangement it follows the fulfillment of the equilibrium equations if we take into consideration that the coefficient u_k in the resulting equation

$$\int_V (t^{kl}_{..;k}+b^l)u_l dV = 0$$

is arbitrary in V.

3.2 It can be expected that the above assertion will remain valid when the subsidiary conditions (2.5) are replaced by such side conditions which have a different mathematical form but otherwise are equivalent to (2.5).

3.3 Representations (2.7) and (2.9) enable us to rewrite the volume integral

$$I_V^B = -\int_V b^l u_l\,dV$$

involved in the principle of virtual work:

$$I_{V1}^B = \int_V \Delta B^l\,u_l\,dV \qquad I_{V2}^B = \int_V \Delta\Delta\Psi^l\,u_l\,dV\,. \tag{3.2}$$

Our aim is to transform them into such a form that the strain tensor is involved instead of the displacement field. With (A.28) and (A.30), integral I_{V1}^B changes into

$$I_{V1}^B = -\int_V (g^{pq}B^l{}_{.;q} + g^{lq}B^p{}_{.;q} - g^{pl}B^k{}_{.;k})\,e_{lp}\,dV + \int_S n_3\,(a^{3q}B^l{}_{.;q} + a^{lq}B^3{}_{.;q} - a^{3l}B^k{}_{.;k})\,u_l\,dA\,. \tag{3.3}$$

Upon substitution of (3.3) for the second volume integral in (3.1) we obtain

$$\int_V [t^{pl} - (g^{pq}B^l{}_{.;q} + g^{lq}B^p{}_{.;q} - g^{pl}B^k{}_{.;k})]\,e_{lp}\,dV - \int_S n_3\,[t^{3l} - (a^{3q}B^l{}_{.;q} + a^{lq}B^3{}_{.;q} - a^{3l}B^k{}_{.;k})]\,u_l\,dA\;=0\,. \tag{3.4}$$

With the aid of (A.31) and (A.33) it can easily be shown that

$$\begin{aligned} I^B_{V2} = & - \int_V (g^{pq}\Delta\Psi^l{}_{.;q} + g^{lq}\Delta\Psi^p{}_{.;q} - g^{pq}g^{ml}\Psi^k{}_{.;kmq})\, e_{lp}\, dV \\ & + \int_S n_3\, (a^{3q}\Delta\Psi^l{}_{.;q} + a^{lq}\Delta\Psi^3{}_{.;q} - a^{3q}a^{ml}\Psi^k{}_{.;kmq})\, u_l\, dA\,. \end{aligned} \tag{3.5}$$

Substitution of (3.5) for the second volume integral in (3.1) yields

$$\begin{aligned} & \int_V [t^{pl} - (g^{pq}\Delta\Psi^l{}_{.;q} + g^{lq}\Delta\Psi^p{}_{.;q} - g^{pq}g^{ml}\Psi^k{}_{.;kmq})]\, e_{lp}\, dV \\ & - \int_S n_3\, [t^{3l} - (a^{3q}\Delta\Psi^l{}_{.;q} + a^{lq}\Delta\Psi^3{}_{.;q} - a^{3q}a^{ml}\Psi^k{}_{.;kmq})]\, u_l\, dA = 0\,. \end{aligned} \tag{3.6}$$

Paragraphs 3.4 to 3.6 are devoted to the problem of how to find a proper form of the side conditions.

3.4 Equations (3.4) and (3.6) are the general primal forms of the principle of virtual work provided that the body forces are given in terms of the potential functions $B^l(x)$ and $\Psi^l(x)$, respectively. Observe that in the above forms of principle of virtual work the kinematic variables u_l and e_{kl} appear either on the boundary S only as it is the case for u_l or on the volume V as it is the case for e_{kl}. Keeping this circumstance in mind and recalling all that has been said about the side conditions in paragraph 3.2 one has to raise the following two questions:

(a) Under what conditions are the strains $e_{kl}(x) \quad x \in V$ compatible?

(b) What further conditions should be satisfied if we want the displacements $u_l(x) \quad x \in V$ obtained from the compatible strains e_{kl} to coincide with those appearing in the surface integral in (3.4) or (3.6), i.e., with displacements given on S?

3.5 Solution to problem (a) is presented herein on the basis of papers by Kozák (1980b, c). To begin with, we have to introduce some new notations. The index pairs which range over a subset of the nine possible values will be capitalized. Let α_{ab} be a sufficiently smooth otherwise arbitrary symmetric tensor field in V. Further let $v^l(x)$ be an unknown vector field on V. By ${}_{AB}$ we denote those subsets of the possible values of index pairs ${}_{ab}$ for which the differential equation

$$\frac{1}{2}(v_{A;B} + v_{B;A}) = \alpha_{AB}(x) \qquad x \in V$$

always have a solution for the vector field $v(x)$. It is clear that the index pairs ${}_{AB}$ may have only three different values. Let ${}_{RS}$ be the supplementary subset of index pairs whose union with ${}_{AB}$ is the set of index pairs ${}_{ab}$. Obviously, the index pairs ${}_{RS}$ may have six distinct values. Because of the symmetry, however, the corresponding tensor components α_{RS} represent three distinct functions only.

The tensor of incompatibility η^{ab} is defined by the equation

$$\eta^{ab} = \epsilon^{akm}\epsilon^{blp}e_{kl;mp}\,. \qquad x \in V$$

Returning to question (a) the independent necessary and sufficient conditions for the strains e_{kl} to be compatible in a simply-connected region V are the fulfillment of differential equations of compatibility

$$\eta^{RS} = \epsilon^{Rkm}\epsilon^{Slp}e_{kl;mp} = 0 \qquad x \in V \tag{3.7a}$$

and that of boundary conditions of compatibility

$$n_a\eta^{ab} = n_3\eta^{3b} = n_3\epsilon^{3km}\epsilon^{dlp}e_{kl;mp} = n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}e_{\eta d;p\kappa} = 0\,. \qquad \xi \in S \tag{3.7b}$$

Observe that (3.7a) and (3.7b) are equivalent to three-three scalar equations.

3.6 Referring again to Kozák (1980b, c) solution for problem (b) is provided by the following assertion: Suppose that the strains e_{kl} fulfill the kinematic boundary conditions

$$e_{\lambda\kappa} = u_{(\lambda;\kappa)} = u_{(\lambda|\kappa)} \qquad \xi \in S \tag{3.8a}$$

$$(e_{3\kappa} - u_{3|\kappa})_{\|\lambda} + b^\alpha_\lambda(e_{\alpha\kappa} - u_{\alpha|\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda3;\kappa}) = 0 \qquad \xi \in S \tag{3.8b}$$

where b^α_λ is the tensor of curvature, index pairs in parentheses stand for the symmetric part of a tensor of order two while surface covariant derivative and covariant derivative on surface are respectively denoted by ${}_{\|\lambda}$ and $|\kappa$. (See paragraphs 5.1. to 5.4. for further details.) Then, on one hand,

– the boundary conditions of compatibility (3.7b) are fulfilled

and on the other hand

– the displacement field $u_k(\xi) \quad \xi \in S$ can be determined from $e_{kl}(\xi)$ by integrations.

The papers by Kozák (1980b,c) cited above do not contain the whole proof of the assertion. For this reason a short proof is presented in paragraphs 3.7 and 3.8. During the transformations we shall need the equation

$$e_{\kappa\mu|\lambda} - e_{\lambda\mu|\kappa} = \frac{1}{2}(u_{\kappa|\lambda} - u_{\lambda|\kappa})_{\|\mu} + b_{\kappa\mu}(e_{3\lambda} - u_{3|\lambda}) - b_{\lambda\mu}(e_{\kappa 3} - u_{3|\kappa}) \tag{3.9}$$

whose validity is proved in paragraph 5.13 .

3.7 First we shall consider the boundary conditions of compatibility. What we are going to prove is the identical fulfillment of

$$n_3\eta^{33} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}e_{\kappa\lambda;\pi\mu} = 0 \qquad \xi \in S \tag{3.10}$$

and

$$n_3\eta^{3\beta} = \epsilon^{3\kappa\mu}\epsilon^{\beta\lambda 3}(e_{\kappa\lambda;3\mu} - e_{\kappa 3;\lambda\mu}) = 0 \qquad \xi \in S \tag{3.11}$$

provided that u_l meets the kinematic boundary conditions (3.8a) and (3.8b). With the aid of (A.11) it follows from (3.10) and (3.11) that

$$n_3\eta^{33} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}[(e_{\kappa\lambda|\pi})_{\|\mu} - b_{\lambda\mu}e_{\kappa 3|\pi} - b_{\pi\mu}e_{\kappa\lambda;3}] \;=\; 0 \qquad \xi \in S \tag{3.12}$$

$$n_3\eta^{3\beta} = \epsilon^{3\kappa\mu}\epsilon^{\beta\lambda 3}[(e_{\kappa\lambda;3} - e_{\kappa 3|\lambda})_{\|\mu} + b^{\nu}_{\mu}(e_{\kappa\lambda|\nu} - e_{\kappa\nu|\lambda})] \;=\; 0\,. \qquad \xi \in S \tag{3.13}$$

With regard to the identity

$$-\epsilon^{3\lambda\pi}b_{\lambda\mu}e_{\kappa 3|\pi} \;=\; \epsilon^{3\lambda\pi}b_{\pi\mu}e \tag{3.14}$$

obtained by interchanging the indices λ, π we can substitute the condition (3.8a) into both (3.12) and (3.13). Then substituting condition (3.8b) and performing further transformations we find that the equations (3.11) and (3.12) are really fulfilled identically.

These manipulations are presented in paragraphs 5.14 and 5.15 . However, the crux of the matter is inherent in the circumstance that $u_k = u_k(\xi)$ i.e. u_k is given on S, consequently all its derivatives should be taken on S.

3.8 Now we shall prove that $u_k = u_k(\xi)$ can be determined by direct integrations from $e_{kl}(\xi)$ provided that $e_{kl}(\xi)$ meets the conditions (3.8a) and (3.8b).

Let L be a sufficiently smooth otherwise arbitrary curve on S. Since $d\mathbf{r} = d\xi^{\mu}\mathbf{a}_{\mu}\ \xi \in L$ in view of (A.27) one can write for the rotation tensor that

$$\begin{aligned}\Omega_{kl}\mathbf{a}^k\mathbf{a}^l|_B^P &= \int_L (e_{k\mu;l} - e_{l\mu;k})\mathbf{a}^k\mathbf{a}^l\, d\xi^{\mu} = \\ &= \int_L \{(e_{\kappa\mu;\lambda} - e_{\lambda\mu;\kappa})\mathbf{a}^{\kappa}\mathbf{a}^{\lambda} + (e_{\kappa\mu;3} - e_{3\mu;\kappa})\mathbf{a}^{\kappa}\mathbf{a}^3 + (e_{3\mu;\lambda} - e_{\lambda\mu;3})\mathbf{a}^3\mathbf{a}^{\lambda}\}d\xi^{\mu}\,.\end{aligned} \tag{3.15}$$

Upon substitution of (3.9) and (3.12) we obtain from (3.15) that

$$\begin{aligned}\Omega_{kl}\,\mathbf{a}^k\mathbf{a}^l|_B^P = \int_L \{[\frac{1}{2}(u_{\kappa|\lambda} - u_{\lambda|\kappa})_{\|\mu} + b_{\kappa\mu}(e_{3\lambda} - u_{3|\lambda}) - b_{\lambda\mu}(e_{\kappa 3} - u_{3|\kappa})]\mathbf{a}^{\kappa}\mathbf{a}^{\lambda} \\ +[(e_{3\kappa} - u_{3|\kappa})_{\|\mu} + b^{\nu}_{\mu}(e_{\nu\kappa} - u_{\nu|\kappa})]\mathbf{a}^{\kappa}\mathbf{a}^3 - [(e_{3\lambda} - u_{3|\lambda})_{\|\mu} + b^{\nu}_{\mu}(e_{\nu\lambda} - u_{\nu|\lambda})]\mathbf{a}^3\mathbf{a}^{\lambda}\}d\xi^{\mu}\,.\end{aligned} \tag{3.16}$$

By making use of the derivatives of base vectors (A.7) and taking the kinematic boundary condition (3.8a) into consideration equation (3.16) can be transformed into the form

$$\Omega_{kl}\mathbf{a}^k\mathbf{a}^l|_B^P = \int_L \frac{\partial}{\partial\xi^{\mu}}[\frac{1}{2}(u_{\kappa|\lambda} - u_{\lambda|\kappa})\mathbf{a}^{\kappa}\mathbf{a}^{\lambda} + (e_{\kappa 3} - u_{3|\kappa})\mathbf{a}^{\kappa}\mathbf{a}^3 - (e_{\lambda 3} - u_{3|\lambda})\mathbf{a}^3\mathbf{a}^{\lambda}]d\xi^{\mu}$$

from which performing the integration and omitting the distinguishing letter P we have

$$\Omega_{kl}(\xi)\mathbf{a}^k\mathbf{a}^l = \frac{1}{2}(u_{\kappa|\lambda} - u_{\lambda|\kappa})\mathbf{a}^{\kappa}\mathbf{a}^{\lambda} + (e_{\kappa 3} - u_{3|\kappa})\mathbf{a}^{\kappa}\mathbf{a}^3 - (e_{\lambda 3} - u_{3|\lambda})\mathbf{a}^3\mathbf{a}^{\lambda}\,. \tag{3.17}$$

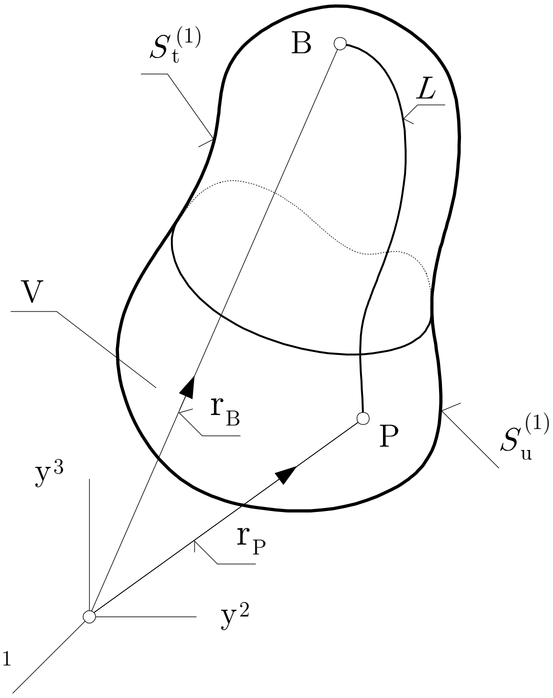

Figure 3. An Arbitrary Curve L on Surface S

The displacement field can also be determined by integration performed along the curve L. In view of the decomposition (A.25) we can write

$$\mathbf{u}|_B^P = \int_L (u_{k;l}\,\mathbf{a}^k\mathbf{a}^l)\cdot d\mathbf{r} = \int_L \mathbf{a}^k\,(e_{k\lambda} + \Omega_{k\lambda})\,d\xi^\lambda\,.$$

Substituting (3.17) for $\Omega_{k\lambda}$ and utilizing the condition (3.8a) we arrive at

$$\begin{aligned}\mathbf{u}|_B^P &= \int_L \{\mathbf{a}^k\,[e_{k\lambda} + \frac{1}{2}(u_{\kappa|\lambda} - u_{\lambda|\kappa})] + \mathbf{a}^3[e_{3\lambda} - (e_{\lambda 3} - u_{3|\lambda})]\,\}d\xi^\lambda \\ &= \int_L \mathbf{a}^k u_{k|\lambda}\,d\xi^\lambda = \int_L \frac{\partial}{\partial\xi^\lambda}(\mathbf{a}^k u_k)d\xi^\lambda\,. \qquad (3.18)\end{aligned}$$

The last formula really proves that the fulfillment of kinematic boundary conditions (3.8a) and (3.8b) enables us to determine the displacement field $u_k(\xi)$ on the surface S.

3.9 In order to cast those integrals involving the side conditions – these are discussed in paragraph 3.10 – into a proper form we shall need the following assertion: *Suppose that the kinematic boundary conditions* (3.8a,b) *hold. Then*

$$e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta} - (u_{\lambda|\kappa})_{\|\vartheta} - u_{3|\lambda}b_{\vartheta\kappa} = 0\,. \qquad \xi\in S \qquad (3.19)$$

In other words the above equation is not an independent condition. The proof is presented in paragraph 5.16.

3.10 On the basis of paragraphs 3.5 to 3.9 we can draw the following inference: Let $e_{kl}(x)$ be a strain field on V. Let further $u_l(\xi)$ be a displacement field on S. If $e_{kl}(x)$ satisfies the differential equations of compatibility (3.7a) as well as the kinematic boundary conditions (3.8a) and (3.8b) then the kinematic equations (2.5) have a solution for $u_l(x)$ and the solution coincides with the displacement field $u_l(\xi)$ given on the surface S. In addition to this, condition (3.19) is an identity.

In other words conditions (3.7a), (3.8a) and (3.8b) are the sought side conditions. For simplicity in the further transformations identity (3.19) will also be taken into consideration when those integrals involving the side conditions are being set up.

Since the conditions mentioned cannot be substituted directly into the general primal forms (3.4) and (3.6) of principle of virtual work Lagrange's method of undetermined multipliers should be used. Let

$$\begin{aligned} H_{lk} &= H_{kl}\,, & x\in V \\ \tilde{H}_{\eta\vartheta} &= \tilde{H}_{\vartheta\eta}\,, & \xi\in S \\ \tilde{H}_{\eta\vartheta;3} &= \tilde{H}_{\vartheta\eta;3}\,, & \xi\in S \\ \tilde{H}_{\eta 3} &= \tilde{H}_{3\eta}\equiv 0 & \xi\in S \end{aligned} \tag{3.20}$$

and

$$\tilde{H}_{33}\equiv 0 \qquad\qquad \xi\in S$$

be the undetermined *Lagrange*'s multipliers. Suppose that the side conditions (3.7a) and (3.8a,b) hold. Then the integrals I_1^V and I_1^S are identically vanishing:

$$I_1^V=\int_V \epsilon^{Rkm}\epsilon^{Slp}e_{kl;mp}H_{RS}\,dV\;\equiv 0 \tag{3.21a}$$

and

$$\begin{aligned} I_1^S=\int_S & n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{(e_{\lambda\kappa}-u_{(\lambda|\kappa)})\tilde{H}_{\eta\vartheta;3} \\ &+[(e_{3\kappa}-u_{3|\kappa})_{\|\lambda}+b^{\alpha}_{\lambda}(e_{\alpha\kappa}-u_{\alpha|\kappa})-(e_{\kappa\lambda;3}-e_{\lambda 3;\kappa})-b^{\beta}_{\beta}(e_{\lambda\kappa}-u_{(\lambda|\kappa)})]\tilde{H}_{\eta\vartheta} \\ &+[e_{\kappa\lambda\|\vartheta}+e_{\lambda\kappa\|\vartheta}-(u_{\lambda|\kappa})_{\|\vartheta}-u_{3|\lambda}b_{\vartheta\kappa}]\tilde{H}_{\eta 3}-b_{\eta\vartheta}(e_{\lambda\kappa}-u_{(\lambda|\kappa)})\tilde{H}_{33}\}\,dA\equiv 0 \end{aligned} \tag{3.21b}$$

Consequently, the sum of the above integrals is also vanishing:

$$I_1^{VS}=I_1^V+I_1^S\equiv 0 \tag{3.22}$$

Since the integral form of the side conditions on S is not obvious NOTEs 1 to 7 are aimed to interpret our choice from which after long and hard transformations and taking into account the other integrals it follows the correctness of the resulting surface integrals.

NOTE 1.: It is temporarily assumed that the Lagrange multipliers $\tilde{H}_{\eta 3}=\tilde{H}_{3\eta}$ and $\tilde{H}_{33}$ do not vanish. Later on it will turn out that their values do not affect the stresses on the boundary and can therefore be set to zero.

NOTE 2.: As regards its mathematical form multiplier $\tilde{H}_{\eta\vartheta;3}=\tilde{H}_{\vartheta\eta;3}$ is a covariant derivative. With the formulae (A.5), (A.7) and (A.8), we have

$$\tilde{H}_{\eta\vartheta;3}=\tilde{H}_{\vartheta\eta,3}+b^{\sigma}_{\eta}\tilde{H}_{\sigma\vartheta}+b^{\sigma}_{\vartheta}\tilde{H}_{\eta\sigma} \qquad\qquad \xi\in S$$

where $\tilde{H}_{\eta\vartheta,3}$ is regarded an arbitrary function. Consequently, without any loss of generality one can assume that $\tilde{H}_{\eta\vartheta;3}$ is independent of $\tilde{H}_{\eta\vartheta}(\xi)\quad \xi\in S$. (Keep in mind that we are on S and $\tilde{H}_{\eta\vartheta,3}$ is the derivative taken along the normal to S.)

NOTE 3.: With regard to the circumstance that the differential equations of compatibility (3.7a) involve three independent equations identified with the superscripts ^{RS} we can conclude that the necessary number of multipliers H_{kl} is also three and these are identified again by the same indices $_{RS}$ being considered as subscripts. In other words multipliers H_{AB} can be set to zero.

NOTE 4.: Enlarging the coefficient of multiplier $\tilde{H}_{\eta\vartheta}$ by the member

$$-b^{\beta}_{\beta}(e_{\lambda\kappa}-u_{(\lambda|\kappa)}) \qquad\qquad \xi\in S$$

we ad zero to it since (3.8a) holds.

NOTE 5.: Coefficient of $\tilde{H}_{\eta 3}$ is the identity (3.19).

NOTE 6.: Coefficient of $\tilde{H}_{33}$ is nothing but a scalar product which involves the kinematic boundary condition (3.8a).

NOTE 7.: In the light of NOTE 5 and NOTE 6 the following interpretation belongs to NOTE 1.: From the point of view of the transformations aimed to bring I_1^{VS} into a suitable form it is unimportant whether $\tilde{H}_{\eta 3}$ and $\tilde{H}_{33}$ vanish or not. In paragraph 3.11 (see NOTE 8) however, it is proved that $\tilde{H}_{\eta 3}$ and $\tilde{H}_{33}$ can always be set to zero.

3.11 Now forms with no side conditions of principle of virtual work can be obtained if we subtract I_1^{VS} from (3.4) and (3.6):

$$\int_V [t^{pl} - (g^{pq}B^l_{.;q} + g^{lq}B^p_{.;q} - g^{pl}B^k_{.;k})]\, e_{lp}\, dV \\ - \int_S n_3\, [t^{3l} - (a^{3q}B^l_{.;q} + a^{lq}B^3_{.;q} - a^{3l}B^k_{.;k})]\, u_l\, dA - I_1^{VS} \,=\, 0 \tag{3.23a}$$

$$\int_V [t^{pl} - (g^{pq}\Delta\Psi^l_{.;q} + g^{lq}\Delta\Psi^p_{.;q} - g^{pq}g^{ml}\Psi^k_{.;kmq})]\, e_{pl}\, dV \\ - \int_S n_3\, [t^{3l} - (a^{3q}\Delta\Psi^l_{.;q} + a^{lq}\Delta\Psi^3_{.;q} - a^{3q}a^{ml}\Psi^k_{.;kmq})]\, u_l\, dA - I_1^{VS} \,=\, 0 \tag{3.23b}$$

To attain a more suitable form it is expedient to transform I_1^{VS} by performing partial integrations before actually carrying out the subtraction. When transforming I_1^V we replace H_{RS} by H_{kl} and rename some dummy indices bearing in mind, however, that H_{AB} is obviously equal to zero.

By separating those terms involving strains e_{kl} and displacements u_l and observing that

$$n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{3|\kappa}b_{\vartheta\lambda}\tilde{H}_{\eta 3} = 0 \qquad\qquad \xi \in S$$

we can write:

$$I_1^{VS} = I_1^S + I_1^V = I_{1U}^S + I_{1E}^S + I_{1E}^V \tag{3.24}$$

in which

$$I_{1U}^S = -\int_S n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{(\lambda|\kappa)}\tilde{H}_{\eta\vartheta;3}dA \\ + \int_S n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[-u_{\lambda|\kappa\|\vartheta}\tilde{H}_{\eta 3} - u_{(\lambda|\kappa)}b_{\eta\vartheta}\tilde{H}_{33} + (b^\beta_\beta u_{(\lambda|\kappa)} - b^\alpha_\lambda u_{\alpha|\kappa})\tilde{H}_{\eta\vartheta}]\, dA \\ + \int_S n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[-u_{3|\kappa\|\lambda}\tilde{H}_{\eta\vartheta} - u_{3|\lambda}b_{\vartheta\kappa}\tilde{H}_{3\eta} - u_{3|\kappa}b_{\vartheta\lambda}\tilde{H}_{\eta 3}]\, dA \tag{3.25}$$

$$I_{1E}^S = \int_S n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{\, e_{\lambda\kappa}\tilde{H}_{\eta\vartheta;3} + (e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta})\tilde{H}_{\eta 3} \\ +(e_{3\kappa\|\lambda} + b^\alpha_\lambda e_{\alpha\kappa} - e_{\kappa\lambda;3} + e_{\lambda 3|\kappa} - b^\beta_\beta e_{\lambda\kappa})\tilde{H}_{\eta\vartheta} + b_{\eta\vartheta}e_{\lambda\kappa}\tilde{H}_{33}\}\, dA \tag{3.26a}$$

and

$$I_{1E}^V = \int_V \epsilon^{krm}\epsilon^{lsp}e_{rs;mp}H_{lk}\, dV \tag{3.26b}$$

Transformation of integral (3.25) requires the repeated application of the Green theorem which should be associated with suitable rearrangements. As regards the details we refer to paragraph 5.17. Finally we obtain from (A.51) that

$$I_{1U}^S = -\int_S n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa}u_l\, dA \tag{3.27}$$

NOTE 8.: Let $\tilde{w}_l(\xi)$ and $\tilde{w}_{l;3}(\xi)$ be two sufficiently smooth vector and tensor fields on S. Recalling the definition of covariant derivatives one can consider $\tilde{w}_{l;3}(\xi)$ as the covariant derivative taken along the normal to the surface of a vector field $\tilde{w}_l(x)$ which is considered as an unknown for all points $x \notin S$. Substituting

$$\tilde{H}_{kl}(\xi) - \tilde{w}_{(k;l)}(\xi) \qquad \text{for} \qquad \tilde{H}_{kl}(\xi) \qquad\qquad \xi \in S \tag{3.28}$$

we do not change the value of (3.27) since

$$\epsilon^{3\kappa\eta}\epsilon^{ldp}\tilde{w}_{(\eta;d);p\kappa} = \frac{1}{2}(\epsilon^{3\kappa\eta}\epsilon^{ldp}\tilde{w}_{\eta;d})_{;p\kappa} + \frac{1}{2}(\epsilon^{3\kappa\eta}\epsilon^{ldp}\tilde{w}_{d;\eta\kappa})_{;p}$$
$$= \frac{1}{2}(\epsilon^{3\kappa\eta}\epsilon^{3\delta\pi}\tilde{w}_{\eta;\delta\pi})_{;\kappa} + \frac{1}{2}\epsilon^{3\kappa\eta}(\epsilon^{\lambda 3\pi}\tilde{w}_{\eta;3\pi} + \epsilon^{\lambda\pi 3}\tilde{w}_{\eta;\pi 3})_{;\kappa} \equiv 0 \qquad \xi \in S$$

Observe that the expressions in parentheses do not require determination of derivatives higher than $w_{k;l\kappa}$ if one takes into consideration the interchangeability of covariant derivatives. At the same time $\tilde{w}_{k;l}(\xi)$ can always be chosen in such a way that the relation

$$\tilde{H}_{k3} - \tilde{w}_{(3;k)} = 0 \qquad \xi \in S$$

is fulfilled. This proves the correctness of the assumption $\tilde{H}_{\eta 3} = \tilde{H}_{3\eta} = \tilde{H}_{33} = 0$.

3.12 With regard to all that has been said in NOTE 8 one can really assume that the structure of $\tilde{H}_{kl}$ meets the preconditions . This choice does not affect the integral I^S_{1E} since the left expression in (3.28) can itself be renamed to $\tilde{H}_{kl}$.

Transformations of integrals (3.26a,b) are similar to that of integral (3.25) and are presented in paragraphs 5.18 and 5.19. As regards the result of the transformations mentioned it follows from (A.57),(A.59a) and (A.62) that

$$I^S_{1E} = \int_S n_3\epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}(-\tilde{H}_{\lambda\kappa}e_{\rho\vartheta;3} + \tilde{H}_{\lambda\kappa;3}e_{\rho\vartheta})\,dA \tag{3.29}$$

and

$$I^V_{1E} = \int_V \epsilon^{pyk}\epsilon^{ldr}H_{yd;kr}e_{rs}\,dV + \int_S n_3\epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}(H_{\lambda\kappa}e_{\rho\vartheta;3} - H_{\lambda\kappa;3}e_{\rho\vartheta})\,dA \tag{3.30}$$

By making use of (3.27),(3.29),(3.30) and (3.24) we can do the subtraction in (3.23a):

$$\int_V [t^{pl} - (\epsilon^{pyk}\epsilon^{ldr}H_{yd;kr} + g^{pq}B^l_{.;q} + g^{lq}B^p_{.;q} - g^{pl}B^k_{.;k})]\,e_{lp}\,dV$$
$$- \int_S n_3\,[t^{3l} - (\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa} + a^{3q}B^l_{.;q} + a^{lq}B^3_{.;q} - a^{3l}B^k_{.;k})]\,u_l\,dA$$
$$+ \int_S n_3\epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}[-(H_{\lambda\kappa} - \tilde{H}_{\lambda\kappa})e_{\rho\vartheta;3} + (H_{\lambda\kappa;3} - \tilde{H}_{\lambda\kappa;3})e_{\rho\vartheta}]\,dA = 0 \tag{3.31}$$

Since in (3.31) no conditions for

$$e_{kl}(x) \qquad x \in V$$
$$e_{\rho\vartheta}(\xi),\ e_{\rho\vartheta;3}(\xi) \qquad \xi \in S$$

and

$$u_l(\xi) \qquad \xi \in S$$

are set down they are arbitrary. Consequently, from the vanishing of (3.31) it follows the fulfilment of the field equations

$$t^{pl} = \epsilon^{pyk}\epsilon^{ldr}H_{yd;kr} + g^{pq}B^l_{.;q} + g^{lq}B^p_{.;q} - g^{pl}B^k_{.;k} \qquad x \in V \tag{3.32}$$

and the boundary conditions

$$\tilde{H}_{\lambda\kappa} - H_{\lambda\kappa} = 0 \qquad \tilde{H}_{\lambda\kappa;3} - H_{\lambda\kappa;3} = 0 \qquad \xi \in S \tag{3.33}$$

and

$$t^{3\rho} = \epsilon^{3\lambda\vartheta}\epsilon^{\rho dp}\tilde{H}_{\lambda d;p\vartheta} + a^{3q}B^\rho_{.;q} + a^{\rho q}B^3_{.;q} - a^{3\rho}B^k_{.;k}$$
$$= \epsilon^{3\lambda\vartheta}(\epsilon^{\rho 3\kappa}\tilde{H}_{\lambda 3;\kappa} - \epsilon^{\rho 3\kappa}\tilde{H}_{\lambda\kappa;3})_{;\vartheta} + a^{3q}B^\rho_{.;q} + a^{\rho q}B^3_{.;q} - a^{3\rho}B^k_{.;k} \qquad \xi \in S \tag{3.34a}$$

$$t^{33} = \epsilon^{3\lambda\vartheta}\epsilon^{3\kappa\rho}\tilde{H}_{\lambda\kappa;\rho\vartheta} + a^{3q}B^3_{.;q} + a^{3q}B^3_{.;q} - a^{33}B^k_{.;k} \qquad \xi \in S \tag{3.34b}$$

in which with regard to $(3.20)_{3,4}$, (A.5), (A.7) and (A.8):

$$\tilde{H}_{\lambda 3;\kappa} = \tilde{H}_{\lambda 3|\kappa} = \tilde{H}_{\lambda 3,\kappa} - \Gamma^{r}_{\lambda\kappa}\tilde{H}_{r3} - \Gamma^{r}_{3\kappa}\tilde{H}_{\lambda r} = b^{\rho}_{\kappa}\tilde{H}_{\lambda\rho} \qquad \xi \in S \tag{3.35a}$$

$$\tilde{H}_{\lambda\kappa;\rho} = \tilde{H}_{\lambda\kappa|\rho} = \tilde{H}_{\lambda\kappa\|\rho} - b_{\lambda\rho}\tilde{H}_{3\kappa} - b_{\kappa\rho}\tilde{H}_{\lambda 3} = \tilde{H}_{\lambda\kappa\|\rho} \qquad \xi \in S \tag{3.35b}$$

NOTE 9.: In view to the relation involved in NOTE 2 and equations (3.35a,b) it can readily be shown by using (A.9), (A.10) and (A.11) that covariant derivatives with respect to ϑ – see Eqns. (3.34a) and (3.34b) – can also be given in terms of $\tilde{H}_{\lambda\kappa}$ and $\tilde{H}_{\lambda\kappa,3}$.

If we now substitute (3.33a,b) into (3.34a,b) and compare the result to (3.32) we shall find that the stresses t^{pl} can be calculated in the same way both in V and on S, i.e., by the formula (3.32). However, it should be emphasized that following from (3.33a,b) and (3.34a,b) determination of stresses on S does not require the knowledge of $H_{k3}(\xi)$ and $H_{k3;3}(\xi)$. It can also be checked by a simple substitution into the equilibrium equations and by using (2.7) that the representation of stresses in terms of the Lagrange multipliers H_{kl} and B^l is equilibrated. In addition to that it coincides with the complete solution found by Schaefer (1953). For this reason multipliers H_{kl} will be referred to as stress functions.

Using the form (3.23b) of principle of virtual work and repeating the line of thought presented in paragraph 3.12 we find that

$$\begin{aligned}
&\int_V [t^{pl} - (\epsilon^{pyk}\epsilon^{ldr}H_{yd;kr} + g^{pq}\Delta\Psi^{l}_{.;q} + g^{lq}\Delta\Psi^{p}_{.;q} - g^{pq}g^{ml}\Psi^{k}_{.;kmq})]\, e_{lp}\, dV \\
&- \int_S n_3\, [t^{3l} - (\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa} + a^{3q}\Delta\Psi^{l}_{.;q} + a^{lq}\Delta\Psi^{3}_{.;q} - a^{3q}a^{ml}\Psi^{k}_{.;kmq})]\, u_l\, dA \\
&+ \int_S n_3\epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}[-(H_{\lambda\kappa} - \tilde{H}_{\lambda\kappa})e_{\rho\vartheta;3} + (H_{\lambda\kappa;3} - \tilde{H}_{\lambda\kappa;3})e_{\rho\vartheta}]\, dA
\end{aligned} \tag{3.36}$$

from which it follows that equilibrated stresses can be calculated both in V and on S by means of the formula

$$t^{pl} = \epsilon^{pyk}\epsilon^{ldr}H_{yd;kr} + g^{pq}\Delta\Psi^{l}_{.;q} + g^{lq}\Delta\Psi^{p}_{.;q} - g^{pq}g^{ml}\Psi^{k}_{.;kmq} \qquad x \in V \tag{3.37}$$

The above stress function solution, which was established by Gurtin (1972), is also complete, i.e., not self-equilibrated on closed boundary surfaces.

NOTE 10.: It is worthy of special mention – with reference to NOTE 3 – that because of the structure of the tensor of incompatibility or what is the same thing because of the structure of the differential equations of compatibility H_{kl} involve three scalar functions since $H_{AB} \equiv 0$. Inasmuch as H_{kl} is of six components fulfillment of the mentioned conditions can always be ensured by an appropriate choice of a vector field $v_l(x)$ $\quad x \in V$, essentially by the solution of differential equations

$$\frac{1}{2}(v_{A;B} + v_{B;A}) = H_{AB} \qquad x \in V$$

because the stress functions

$$\breve{H}_{kl} = H_{kl} - \frac{1}{2}(v_{k;l} + v_{l;k}) \qquad x \in V$$

involve only three scalar functions since $\breve{H}_{AB} \equiv 0$. Observe that the stress functions $v_{(k;l)}$ cause no stresses. This result is that of Finzi (1934). Observe further that the index pairs $_{AB}$ are to be chosen in the same way as before — see paragraph 3.5.

4 Concluding Remarks

4.1 The main result of the present work has been the proof of the possibility, that for solid bodies the general and complete solution of equilibrium equations in terms of stress functions – valid therefore not only for a self-equilibrated case, i.e. on multiply bordered bodies as well – can be derived from the general primal form of principle of virtual work provided that the necessary and sufficient conditions for the strains to be kinematically admissible are known. Since these conditions (as side conditions) can not be substituted directly into the principle of virtual work Lagrange method of undetermined multipliers should be applied.

4.2 Since the side conditions involve three field equations any state of stress can be given in terms of three stress functions. Consequently three components of the corresponding stress function tensor H_{kl} can be set to zero. In this way a solution is given to the dual counterpart of the Southwell paradox.

4.3 The long and hard transformations leading to an appropriate form of the surface integrals taken on S are also presented. It is proved that the stresses in terms of stress functions should be calculated in the same way both in V and on S.

4.4 We note that the line of thought presented herein is of methodological significance and can be applied to other cases including the micropolar one (Szeidl, 1991) provided that the necessary and sufficient conditions of compatibility are known.

5 Appendix

5.1 It is well known that

$$\epsilon_{lpr}\epsilon^{rst} = \delta^{lp}_{mr} = \delta^s_l\delta^t_p - \delta^t_l\delta^s_p. \tag{A.1}$$

Every tensor d_{km} admits the unique decomposition

$$d_{lp} = d_{(lp)} + d_{[lp]} \tag{A.2}$$

in which $d_{(lp)}$ and $d_{[lp]}$ are the symmetric and skew parts of the tensor d_{lp}:

$$d_{(lp)} = (d_{lp} + d_{pl})/2 \qquad \text{and} \qquad d_{[lp]} = (d_{lp} - d_{pl})/2\,. \tag{A.3}$$

It is obvious that

$$d_{[lp]} = \epsilon_{lpr}\epsilon^{rst} d_{st}/2\,. \tag{A.4}$$

5.2 The covariant derivative with respect to the subscript s of a tensor $d^k_{\cdot lm}$ is defined by

$$d^k_{\cdot lm;s} = d^k_{\cdot lm,s} + \Gamma^{\,k}_{ps} d^p_{\cdot lm} - \Gamma^{\,p}_{ls} d^k_{\cdot pm} - \Gamma^{\,p}_{ms} d^k_{\cdot lp} \tag{A.5}$$

where

$$\Gamma^{\,k}_{ps} = \mathbf{g}_{k,m} \cdot \mathbf{g}^s \tag{A.6}$$

is the *Christoffel* symbol of the second order. Equations (A.2-A.6) are valid in any curvilinear coordinate system.

5.3 On the surface S

$$b_{\alpha\beta}(\xi) = \Gamma^{\,3}_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^3 \qquad \text{and} \qquad b^{\mu}_{\beta}(\xi) = -\Gamma^{\,\mu}_{3\beta} = \mathbf{a}_{3,\beta} \cdot \mathbf{a}^{\mu} \qquad \xi \in S \tag{A.7}$$

are the non identically vanishing Christoffel symbols while

$$\Gamma^{\,\mu}_{33} = \Gamma^{\,3}_{3\beta} = \Gamma^{\,3}_{33} = 0\,. \qquad \xi \in S \tag{A.8}$$

Here $b_{\alpha\beta}$ and b^{μ}_{β} denote the covariant and mixed forms of the tensor of curvature.

5.4 The covariant derivative of the tensor $d^k_{\cdot lm}$ on the surface S is defined by

$$d^k_{\cdot lm;\sigma}(\xi) = d^k_{\cdot lm|\sigma}(\xi) = d^k_{\cdot lm,\sigma} + \Gamma^{\,k}_{p\sigma} d^p_{\cdot lm} - \Gamma^{\,p}_{l\sigma} d^k_{\cdot pm} - \Gamma^{\,p}_{m\sigma} d^k_{\cdot lp}\,. \qquad \xi \in S \tag{A.9}$$

Being parts of the tensor $d^k_{\cdot lm}$ the tensors $d^{\kappa}_{\cdot\lambda\mu}$; $d^3_{\cdot\lambda\mu}$; $\ldots$; $d^3_{\cdot 33}$ considered on the surface S are referred to as subtensors of order three, two,....,zero. Let the surface covariant derivative of the subtensor $d^{\kappa}_{\cdot\lambda\mu}$ be

$$d^{\kappa}_{\cdot\lambda\mu\|\sigma}(\xi) = d^{\kappa}_{\cdot\lambda\mu,\sigma} + \Gamma^{\,\kappa}_{\pi\sigma} d^{\pi}_{\cdot\lambda\mu} - \Gamma^{\,\pi}_{\lambda\sigma} d^{\kappa}_{\cdot\pi\mu} - \Gamma^{\,\pi}_{\mu\sigma} d^{\kappa}_{\cdot\lambda\pi}\,. \qquad \xi \in S \tag{A.10}$$

Upon substitution of (A.10) and (A.7) into (A.9) we obtain

$$d^{\kappa}_{.\lambda\mu;\sigma}(\xi) = d^{\kappa}_{.\lambda\mu|\sigma}(\xi) = d^{\kappa}_{.\lambda\mu\|\sigma} - b^{\kappa}_{\sigma} d^{3}_{.\lambda\mu} + b_{\lambda\sigma} d^{\kappa}_{.3\mu} + b_{\mu\sigma} d^{\kappa}_{.\lambda 3}\,. \qquad \xi \in S \tag{A.11}$$

In the sequel the above equation is considered as a rule which relates surface covariant derivative to covariant derivative on the surface. Observe that changes along ξ^3 do not affect $d^{\kappa}_{.\lambda\mu\|\sigma}$.

5.5 Let $s^{\alpha}_{.\beta}$ be a sufficiently smooth tensor field on S. By making use of (A.10) it can be shown that

$$s^{\alpha}_{.\beta\|\vartheta\lambda} - s^{\alpha}_{.\beta\|\lambda\vartheta} = -s^{\pi}_{.\beta} R^{\alpha}_{.\pi\vartheta\lambda} - s^{\alpha}_{.\pi} R^{\pi}_{.\beta\vartheta\lambda} \qquad \xi \in S \tag{A.12}$$

in which

$$R^{\pi}_{.\beta\vartheta\lambda} = \frac{\partial \Gamma^{\pi}_{\beta\lambda}}{\partial x^{\vartheta}} - \frac{\partial \Gamma^{\pi}_{\beta\vartheta}}{\partial x^{\lambda}} + \Gamma^{\pi}_{\vartheta\nu}\Gamma^{\nu}_{\beta\lambda} - \Gamma^{\lambda}_{\nu\pi}\Gamma^{\nu}_{\beta\vartheta} \qquad \xi \in S \tag{A.13}$$

is the Rieman-Christoffel tensor on the surface. It can easily be proved (Connel, 1957) that

$$R^{\pi}_{.\beta\vartheta\lambda} = b^{\pi}_{\vartheta} b_{\beta\lambda} - b^{\pi}_{\lambda} b_{\beta\vartheta} \neq 0\,. \qquad \xi \in S \tag{A.14}$$

Regarding (A.12) as a rule and applying it to the covariant derivatives of displacement u_l we can write that

$$u_{\kappa\|\mu\lambda} - u_{\kappa\|\lambda\mu} = u_{\nu}(b^{\nu}_{\mu} b_{\kappa\lambda} - b^{\nu}_{\lambda} b_{\kappa\mu}) \qquad \xi \in S \tag{A.15}$$

and

$$(u_{\lambda\|\kappa})_{\|\pi\mu} - (u_{\lambda\|\kappa})_{\|\mu\pi} = b^{\nu}_{\kappa} b_{\mu\pi}(u_{\nu\|\lambda} - u_{\lambda\|\nu}) \qquad \xi \in S \tag{A.16a}$$

$$(e_{3\lambda} - u_{3|\lambda})_{\|\kappa\mu} - (e_{3\lambda} - u_{3|\lambda})_{\|\mu\kappa} = (e_{3\nu} - u_{3|\nu})(b^{\nu}_{\kappa} b_{\lambda\mu} - b^{\nu}_{\mu} b_{\lambda\kappa}) \qquad \xi \in S \tag{A.16b}$$

where (A.14) has also taken into consideration. Relations (A.15) and (A.16a,b) show that the order of surface covariant derivatives is not interchangeable. It can also be proved that

$$b_{\alpha\beta\|\lambda} - b_{\alpha\lambda\|\beta} = 0 \qquad \text{or in other form} \qquad \epsilon^{3\beta\lambda}\, b_{\alpha\beta\|\lambda} = 0 \qquad \xi \in S \tag{A.17}$$

The above equations are known as Mainardi-Codazzi formulae (Connel, 1957).

5.6 Covariant derivatives of metric and permutation tensors are identically zero:

$$g_{kl;s} = 0; \quad g^{mn}_{..;s} = 0; \quad \delta^{k}_{l;m} = 0; \qquad \epsilon_{klm;s} = 0; \quad \epsilon^{prm}_{...;s} = 0 \qquad x \in V \tag{A.18}$$

and

$$a_{\kappa\lambda|\sigma} = a_{\kappa\lambda\|\sigma} = 0; \quad a^{\mu\nu}_{..|\sigma} = a^{\mu\nu}_{..\|\sigma} = 0; \qquad \delta^{\lambda}_{\kappa|\mu} = \delta^{\lambda}_{\kappa\|\mu} = 0 \qquad \xi \in S \tag{A.19a}$$

$$\epsilon_{\kappa\lambda 3|\sigma} = \epsilon_{\kappa\lambda 3\|\sigma} = 0; \quad \epsilon^{\pi\rho 3}_{...|\sigma} = \epsilon^{\pi\rho 3}_{...\|\sigma} = 0 \qquad \xi \in S \tag{A.19b}$$

5.7 Consider the product $d^{k}_{.lm;k}(x)\, c^{lm}(x) \quad x \in V$. Applying the Green-Gauss theorem (cf. Kellog, 1957) one can readily prove that

$$\int_V d^{k}_{.lm;k}\, c^{lm}\, dV = \int_S n_3 d^{3}_{.lm}\, c^{lm}\, dA - \int_V d^{k}_{.lm}\, c^{lm}_{..;k}\, dV\,. \tag{A.20}$$

5.8 Let S_o be an arbitrary open surface closed by the directed boundary curve g_o. Further let the positive direction on g_o be taken so that τ_{α}, n_3 and ν_{α} — ν_{α} is the normal to the boundary curve g_o that lies in the tangent plane — form a right hand triad (Figure 4). Let $b_{\alpha.}^{\ l}(\xi)$ and $c_l(\xi)$ be surface tensors. Applying the Stokes theorem it can be shown that

$$\int_{S_o} n_3\, \epsilon^{3\eta\alpha}\, b_{\alpha\,.|\eta}^{\ l}\, c_l\, dA = \oint_{g_o} b_{\alpha}^{\ l}\, c_l\, \tau^{\alpha}\, ds - \int_{S_o} n_3\, \epsilon^{3\eta\alpha}\, b_{\alpha.}^{\ l}\, c_{l|\eta}\, dA\,. \tag{A.21}$$

5.9 By making use of the Green theorem relating surface to boundary integrals (cf. Mason, 1980) one can readily check the validity of transformation

$$\int_{S_o} b_{l\,.\|\eta}^{\ \eta}\, c_l\, dA = \oint_{g_o} \nu_{\eta}\, b_{l.}^{\ \eta}\, c_l\, ds - \int_{S_o} b_{l.}^{\ \eta}\, c_{l\|\eta}\, dA\,. \tag{A.22}$$

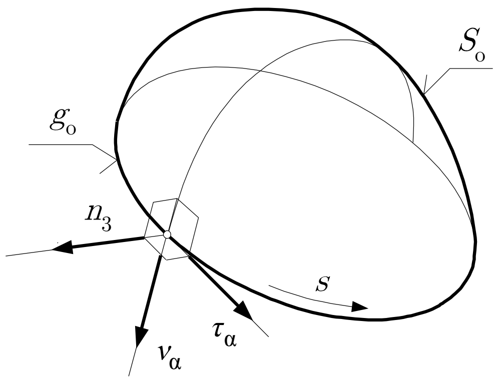

Figure 1. An Open Surface

If S_o is closed the line integrals in (A.21) and (A.22) vanish. Equations (A.20-A.22) are the rules of partial integrations.

5.10 The rigid body rotation ω^r is defined by

$$\omega^r = \frac{1}{2}\,\epsilon^{rst}\,u_{t;s} \qquad x \in V \tag{A.23}$$

from which multiplying throughout by $-\epsilon_{lpr}$ and using (A.1) we obtain

$$-\epsilon_{lpr}\omega^r = \frac{1}{2}\,(u_{l;p} - u_{p;l}) = u_{[l;p]} = \Omega_{lp}\,. \qquad x \in V \tag{A.24}$$

where Ω_{kl} is the rotation tensor. With (A.24) and (2.5), substitution of $u_{l;p}$ for d_{lp} in (A.3) gives

$$u_{l;p} = e_{lp} - \epsilon_{plr}\omega^r = e_{lp} + \Omega_{lp}\,. \qquad x \in V \tag{A.25}$$

Since $\epsilon^{rst}u_{q;ts} = 0$ with regard to (2.5) it follows from (A.23) that

$$\omega^r_{\ .;q} = \frac{1}{2}\,\epsilon^{rst}(u_{t;qs} + u_{q;ts}) = \epsilon^{str}\,e_{tq;s}\,. \qquad x \in V \tag{A.26}$$

Upon multiplying throughout by ϵ_{plr} and using (A.1) one obtains from (A.26):

$$\epsilon_{plr}\omega^r_{\ .;p} = e_{lq;p} - e_{pq;l} = \Omega_{lp;q}\,. \qquad x \in V \tag{A.27}$$

5.11 Now we shall transform the integral I^B_{V1} of equation (3.2-a) into a more suitable form. These manipulations require, however, more steps detailed as follows:

(a) Substitution of (2.8) for ΔB^l and partial integration by the use of rule (A.19);
(b) Substitution of resolution (A.24) for the gradient $u_{l:p}$ and partial integration of the term that involves ω^r;
(c) Substitution of (A.25) for $\omega^r_{\ .;p}$ and partial integrations in respect to those terms involving the gradients $e_{lq;p}$ and $e_{pq;l}$.

After carrying out the steps (a), (b) and (c) and renaming some dummy indices we have

$$I^B_{V1} = -\int_V (g^{pq}B^l_{\ .;q} + g^{lq}B^p_{\ .;q} - g^{pl}B^k_{\ .;k})\,e_{lp}\,dV + \int_S n_3 a^{3q}B^l_{\ .;q}\,u_l\,dA + I^B_{A1} \tag{A.28}$$

where

$$I^B_{A1} = -\int_S n_3\,a^{3p}B^l\epsilon_{pls}\omega^s\,dA + \int_S (n_3\,a^{3q}B^l e_{lq} - n_3\,a^{pq}B^3 e_{pq})\,dA\,. \tag{A.29}$$

To obtain the final form of I^B_{V1} we shall transform the surface integral I^B_{A1}. During the transformations use has been made of the equations

$$n_3\,a^{3p}B^l\,\epsilon_{pls}\,\omega^s = n_q\,a^{qp}B^l\,\epsilon_{pls}\,\epsilon^{skv}\,\frac{1}{2}\,u_{v;k} \tag{a}$$

$$n_3\, a^{3q} B^l e_{lq} - n_3\, a^{pq} B^3 e_{pq} = n_k\, a^{pq} B^l\, \epsilon_{pls}\, \epsilon^{skv} \frac{1}{2}\,(u_{v;q} + u_{q;v}) \tag{b}$$

and

$$n_k u_{v;q} - n_q u_{v;k} = \epsilon_{kqr}\, \epsilon^{rab}\, n_a\, u_{v;b} \tag{c}$$

whose validity can easily be shown if one takes (a*) on page 149, (A.1) and (A.19a, A.19b) into consideration.

Substitution of (a) and (b) into (A.29) and a subsequent rearrangement enables us to utilize (c):

$$I^B_{A1} = \frac{1}{2}\int_S a^{pq} B^l \epsilon_{pls} \epsilon^{skv} (n_k u_{v;q} - n_q u_{v;k})\, dA + \frac{1}{2}\int_S n_k\, a^{pq} B^l \epsilon_{pls} \epsilon^{skv} u_{q;v}\, dA\,.$$

Observe that each of the integrands involve the gradient of displacements. To complete the transformation we apply the rule for partial integrations (A.21) to the result with the aim of obtaining terms linear in the displacements. When doing the integration we keep in mind (A.18) and (A.19a, A.19b) and remember that the surface S is closed. All these manipulations yield

$$\begin{aligned} I^B_{A1} &= -\frac{1}{2}\int_S n_3\, \delta^3_a a^{pq}\, \epsilon_{pls} \epsilon^{suv}\, \epsilon_{kqr} \epsilon^{rab}\, B^l_{\ ;b}\, u_k\, dA - \frac{1}{2}\int_S n_3\, \delta^3_a\, a^{pq} \epsilon_{pls} \epsilon^{skv} B^l_{\ ;v} u_q\, dA \\ &= \int_S n_3\, (a^{lq} B^3_{\ ;q} - a^{3l} B^k_{\ ;k}) u_l\, dA \end{aligned} \tag{A.30}$$

where the identity (A.1) has also been used.

5.12 Transformation of integral I^B_{V2} of equation (3.2b)) is very similar to that of integral I^B_{V1}. During the transformation, which requires more steps,

(a) we substitute (2.9) for $\Delta\Delta\Psi$ and carry out the first partial integration;
(b) then we utilize the resolution (A.25) and integrate partially the terms involving ω^r;
(c) and finally we substitute (A.26) for $\omega^r_{\ .;n}$ and integrate those terms involving $e_{ln;q}$ and $e_{qn;l}$.

After renaming some dummy indices and a subsequent rearrangement we have

$$I^B_{V2} = -\int_V (g^{pq} \Delta\Psi^l_{\ ;q} + g^{lq} \Delta\Psi^p_{\ ;q} - g^{pq} g^{ml} \Psi^k_{\ ;kmq})\, e_{lp}\, dV + \int_S n_3\, a^{3q} a^{mn} \Delta\Psi^l_{\ ;q}\, u_l\, dA + I^B_{A2} \tag{A.31}$$

where

$$I^B_{A2} = -\int_S n_3 a^{pq} a^{3m} \Psi^l_{\ .;mq} \epsilon_{pls} \omega^s dA + \int_S (n_3 a^{3q} a^{mn} \Psi^l_{\ ;mq}\, e_{ln} - n_3 a^{pq} a^{mn} \Psi^3_{\ ;mq}\, e_{pn}) dA\,. \tag{A.32}$$

To get the final form of I^B_{V2} one should transform the integral I^B_{A2}. It can be shown in the same way as above that

$$n_3\, a^{pq} a^{3m} \Psi^l_{\ .;mq}\, \epsilon_{pls}\, \omega^s = n_n\, a^{pq} a^{mn} \Psi^l_{\ .;mq}\, \epsilon_{pls}\, \epsilon^{suv}\, \frac{1}{2}\, u_{v;u} \tag{d}$$

and

$$n_3\, a^{3q} a^{mn} \Psi^l_{\ .;mq} e_{ln} - n_3\, a^{pq} a^{mn} \Psi^l_{\ .;mq} e_{pn} = n_u\, a^{pq} a^{mn} \Psi^l_{\ .;mq}\, \epsilon_{pls}\, \epsilon^{suv} \frac{1}{2}\,(u_{v;n} + u_{n;v})\,. \tag{e}$$

With (d) and (e), we get

$$I^B_{A2} = \frac{1}{2}\int_S a^{pq} a^{mn} \Psi^l_{\ .;mq} \epsilon_{pls} \epsilon^{suv} (n_u u_{v;n} - n_n u_{v;n})\, dA + \frac{1}{2}\int_S n_u a^{pq} a^{mn} \Psi^l_{\ .;mq} \epsilon_{pls} \epsilon^{suv} u_{n;v}\, dA\,.$$

Substituting $\epsilon_{run} \epsilon^{rab} n_a u_{v;b}$ for the term in the parentheses and repeating the line of thought leading to (A.30) we arrive at

$$\begin{aligned} I^B_{A2} &= -\frac{1}{2}\int_S n_3 \delta^3_a a^{pq} a^{mn} \Psi^l_{\ .;mqb} \epsilon_{pls} \epsilon^{suv} \epsilon_{run} \epsilon^{rab} u_v\, dA - \frac{1}{2}\int_S n_3 \delta^3_u a^{pq} a^{mn} \Psi^l_{\ .;mqv} \epsilon_{pls} \epsilon^{suv} u_n\, dA \\ &= \int_S n_3 (a^{lq} \Delta\Psi^3_{\ .;q} - a^{3q} a^{ml} \Psi^k_{\ .;kmq}) u_l\, dA\,. \end{aligned} \tag{A.33}$$

5.13 Proof of equation (3.9).
By making use of (A.11) one can write that

$$e_{\kappa\mu|\lambda} - e_{\lambda\mu|\kappa} \;=\; e_{\kappa\mu\|\lambda} - e_{\lambda\mu\|\kappa} - b_{\mu\lambda} e_{\kappa 3} + b_{\mu\kappa} e_{3\lambda} \qquad \xi \in S \tag{A.34}$$

The first two terms in the right hand side can be transformed further if we substitute (3.8a) and utilize the rule (A.11) again:

$$e_{\kappa\mu\|\lambda} - e_{\lambda\mu\|\kappa} = \frac{1}{2}(u_{\kappa\|\mu\lambda} - u_{\lambda\|\mu\kappa}) + \frac{1}{2}(u_{\mu\|\kappa\lambda} - u_{\mu\|\kappa\lambda}) - (b_{\kappa\mu}u_3)_{\|\lambda} + (b_{\lambda\mu}u_3)_{\|\kappa} \qquad \xi \in S \qquad \text{(A.35)}$$

By interchanging the order of surface covariant derivatives in (A.15) we obtain

$$u_{\lambda\|\mu\kappa} = u_{\lambda\|\kappa\mu} + u_{\nu}(b^{\nu}_{\mu}b_{\kappa\lambda} - b^{\nu}_{\kappa}b_{\lambda\mu}) \qquad \xi \in S \qquad \text{(A.36a)}$$

and

$$u_{\mu\|\kappa\lambda} - u_{\mu\|\lambda\kappa} = u_{\nu}(b^{\nu}_{\kappa}b_{\mu\lambda} - b^{\nu}_{\lambda}b_{\mu\kappa})\,. \qquad \xi \in S \qquad \text{(A.36b)}$$

In addition to this it follows from the Mainardi-Codazzi formula (A.17b) that

$$-(b_{\kappa\mu}u_3)_{\|\lambda} + (b_{\lambda\mu}u_3)_{\|\kappa} = b_{\kappa\mu}u_{3\|\lambda} + b_{\lambda\mu}u_{3\|\kappa}\,. \qquad \xi \in S \qquad \text{(A.37)}$$

Substituting (A.15), (A.36a), (A.36b) and (A.37) into (A.35) and the result obtained into (A.34) we arrive at (3.18) if we also take into consideration the rule (A.11).

5.14 Proof of fulfillment of equation (3.10).
Keeping (3.14) in mind let us substitute $e_{\kappa\lambda;3}$ from (3.8b) into (3.12) and apply the rule (A.11) to the first term within the braces:

$$n_3\eta^{33} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}\{(e_{\kappa\lambda\|\pi} - b_{\kappa\pi}e_{3\lambda})_{\|\mu} - b_{\mu\pi}[(e_{3\lambda} - u_{3|\lambda})_{\|\kappa} + b^{\nu}_{\kappa}(e_{\nu\lambda} - u_{\nu|\lambda})]\} \qquad \xi \in S \qquad \text{(A.38)}$$

Now we can substitute (3.8a). If in addition to this we apply (A.11) again and take into consideration both the identity

$$-\epsilon^{3\kappa\mu}(b_{\kappa\pi}e_{3\lambda})_{\|\mu} = \epsilon^{3\kappa\mu}(b_{\mu\pi}e_{3\lambda})_{\|\kappa} \qquad \xi \in S$$

obtained by interchanging κ, μ and the Mainardi-Codazzi formula (A.17-b) we get

$$\begin{aligned} n_3\eta^{33} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}\{\frac{1}{2}(u_{\kappa\|\lambda} \;+\; u_{\lambda\|\kappa})_{\|\pi\mu} - b_{\kappa\lambda}u_{3\|\pi\mu} + b_{\mu\pi}[u_{3\|\lambda\kappa} + (b^{\nu}_{\lambda}u_{\nu})_{\|\kappa}] \\ +\; b_{\mu\pi}b^{\nu}_{\kappa}\frac{1}{2}(u_{\nu\|\lambda} - u_{\lambda\|\nu})\} \qquad \xi \in S \end{aligned} \qquad \text{(A.39)}$$

In what follows we utilize

— the identity

$$\epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}b_{\kappa\lambda}u_{3\|\pi\mu} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}b_{\mu\pi}u_{3\|\lambda\kappa} \qquad \xi \in S \qquad \text{(A.40)}$$

obtained by interchanging the index pairs κ, λ and μ, π

and

— the transformation

$$\epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}(u_{\lambda\|\kappa})_{\|\pi\mu} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}[(u_{\lambda\|\kappa})_{\|\mu\pi} - b^{\nu}_{\kappa}b_{\mu_{\pi}}(u_{\nu\|\lambda} - u_{\lambda\|\nu})] \qquad \xi \in S \qquad \text{(A.41)}$$

which follows from (A.16a).

Upon substitution (A.40) and (A.41) into (A.39) we have

$$n_3\eta^{33} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}\{\frac{1}{2}(u_{\kappa\|\lambda\mu\pi} + u_{\lambda\|\kappa\mu\pi}) + b_{\mu\pi}(b^{\nu}_{\lambda}u_{\nu})_{\|\kappa}\} \qquad \xi \in S \qquad \text{(A.42)}$$

Making use of the identity

$$\epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}u_{\lambda\|\kappa\mu\pi} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}u_{\kappa\|\lambda\pi\mu} = \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}\frac{1}{2}(u_{\kappa\|\lambda\pi} - u_{\kappa\|\pi\lambda})_{\|\mu} \qquad \xi \in S$$

obtained by renaming some indices and taking (A.15) into consideration we find that

$$\begin{aligned} n_3\eta^{33} &= \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}\{\frac{1}{2}[u_{\nu}(b^{\nu}_{\lambda}b_{\kappa\pi} - b^{\nu}_{\pi}b_{\kappa\lambda})]_{\|\mu} + b_{\mu\pi}(b^{\nu}_{\lambda}u_{\nu})_{\|\kappa}\} \\ &= \epsilon^{3\kappa\mu}\epsilon^{3\lambda\pi}\{\frac{1}{2}[u_{\nu}(b^{\nu}_{\lambda}b_{\kappa\pi} + b^{\nu}_{\lambda}b_{\kappa\pi})]_{\|\mu} - b_{\kappa\pi}(b^{\nu}_{\lambda}u_{\nu})_{\|\mu}\} \equiv 0 \qquad \xi \in S \end{aligned} \qquad \text{(A.43)}$$

if we also substitute (A.17b).

5.15 Proof of fulfillment of equation (3.11).
Upon substitution of (3.8b) and (3.9) into (3.11) and a subsequent rearrangement we have

$$n_3\eta^{3\beta} = \epsilon^{3\kappa\mu}\epsilon^{3\beta\kappa}\{(e_{3\lambda} - u_{3|\lambda})_{\|\kappa\mu} + [b^{\nu}_{\kappa}(e_{\nu\lambda} - u_{\nu|\lambda})]_{\|\mu}$$
$$b^{\nu}_{\mu}[\frac{1}{2}(u_{\lambda|\nu} - u_{\nu|\lambda})_{\|\kappa} + b_{\lambda\kappa}(e_{3\nu} - u_{3|\nu}) - b_{\nu\kappa}(e_{3\lambda} - u_{3|\lambda})]\}\,. \qquad \xi \in S$$

In order to reach the desirable result one should substitute into (A.43)

— the expression

$$\epsilon^{3\kappa\mu}(e_{3\lambda} - u_{3|\lambda})_{\|\kappa\mu} = \epsilon^{3\kappa\mu}\frac{1}{2}[(e_{3\lambda} - u_{3|\lambda})_{\|\kappa\mu} - (e_{3\lambda} - u_{3|\lambda})_{\|\mu\kappa}] =$$
$$= \epsilon^{3\kappa\mu}\frac{1}{2}(e_{3\nu} - u_{3|\nu})(b^{\nu}_{\kappa}b_{\lambda\mu} - b^{\nu}_{\mu}b_{\lambda\kappa}) = -\epsilon^{3\kappa\mu}(e_{3\nu} - u_{3|\nu}) \qquad \xi \in S$$

obtained by making use of (A.16b),

— the Mainardi-Codazzi formula $\epsilon^{3\kappa\mu}b^{\nu}_{\kappa\|\mu} \equiv 0$ [See (A.16b)],

— the transformation

$$\epsilon^{3\kappa\mu}b^{\nu}_{\kappa}(e_{\nu\lambda} - u_{\nu|\lambda})_{\|\mu} = -\epsilon^{3\kappa\mu}\frac{1}{2}b^{\nu}_{\mu}(u_{\lambda|\nu} - u_{\nu|\lambda})_{\|\kappa} \qquad \xi \in S$$

whose proof requires the use of (3.8a) and the interchange of indices κ and μ

and finally

— the identity $\epsilon^{3\kappa\mu}b^{\nu}_{\mu}b_{\nu\kappa} \equiv 0$ which expresses that the tensor of curvature is symmetric.

After the substitutions we have that the right hand side of (A.43) vanishes:

$$n_3\eta^{3\beta} \equiv 0\,. \qquad \xi \in S$$

5.16 Proof of identity (3.19).
Substituting (3.8a) into (3.19) and applying the rule (A.11) we obtain

$$\epsilon^{\lambda\vartheta 3}[(u_{\kappa|\lambda} + u_{\lambda|\kappa})_{\|\vartheta} - (u_{\lambda|\kappa})_{\|\vartheta} - u_{3|\lambda}b_{\vartheta\kappa}] =$$
$$= \epsilon^{\lambda\vartheta 3}[u_{\kappa\|\lambda\vartheta} - (b_{\kappa\lambda}u_3)_{\|\vartheta} - (u_{3\|\lambda} + b^{\nu}_{\lambda}u_{\nu})b_{\vartheta\kappa}] \qquad \xi \in S \qquad \text{(A.44)}$$

Using

— the identity

$$\epsilon^{\lambda\vartheta 3}u_{\kappa\|\lambda\vartheta} = \epsilon^{\lambda\vartheta 3}\frac{1}{2}(u_{\kappa\|\lambda\vartheta} - u_{\kappa\|\vartheta\lambda}) \qquad \xi \in S$$

obtained by renaming and interchanging indices,

– the transformation (A.15) with a suitable renaming indices

and

— the Mainardi-Codazzi formula (A.15) with a suitable renaming indices

we get from (A.44) that

$$\epsilon^{\lambda\vartheta 3}[\frac{1}{2}u_{\nu}(b^{\nu}_{\lambda}b_{\kappa\vartheta} - b^{\nu}_{\vartheta}b_{\kappa\lambda}) - b_{\kappa\lambda}u_{3\|\vartheta} - u_{3\|\lambda}b_{\vartheta\kappa} - u_{\nu}b^{\nu}_{\lambda}b_{\vartheta\kappa}] \equiv 0 \qquad \xi \in S$$

since the expression in the brackets is symmetric in λ, ϑ.

5.17 Transformation of integral I^S_{1U} of equation (3.25).
For the sake of some further transformations concerning variational principles here and in paragraphs 5.18 and 5.19 we shall assume that the surface is open – see Figure 4. for details. During the transformations use has been made of

— the identities

$$n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{[\lambda|\kappa]}\tilde{H}_{\eta\vartheta;3} \equiv 0\,, \qquad \xi \in S \tag{A.45a}$$

$$n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{[\lambda|\kappa]}b_{\eta\vartheta}\tilde{H}_{33} \equiv 0 \qquad \xi \in S \tag{A.45b}$$

and

$$n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{[\lambda|\kappa]}b^\beta_\beta\tilde{H}_{\eta\vartheta} \equiv 0 \qquad \xi \in S \tag{A.45c}$$

where the latter two follow from the equation

$$\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{[\lambda|\kappa]}\tilde{H}_{\eta\vartheta;3} = \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\epsilon_{\lambda\kappa 3}\omega^3\tilde{H}_{\eta\vartheta;3} = -\epsilon^{\vartheta\eta 3}\omega^3\tilde{H}_{\eta\vartheta;3} \equiv 0 \qquad \xi \in S$$

if one substitute $b_{\eta\vartheta}$ or $\tilde{H}_{\eta\vartheta}$ for $\tilde{H}_{\eta\vartheta;3}$,

— the manipulation

$$\begin{aligned} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}(b^\beta_\beta u_{\lambda|\kappa} - b^\alpha_\lambda u_{\alpha|\kappa})\tilde{H}_{\eta\vartheta} &= n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}(-\delta^{\alpha\beta}_{\sigma\lambda})u_{\alpha|\kappa}b^\sigma_\beta\tilde{H}_{\eta\vartheta} = \\ = n_3\epsilon^{\kappa\eta 3}\epsilon^{\alpha\beta 3}\epsilon_{\sigma\lambda 3}\epsilon^{\vartheta\lambda 3}u_{\alpha|\kappa}b^\sigma_\beta\tilde{H}_{\eta\vartheta} &= n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{\lambda|\kappa}b^\nu_\vartheta\tilde{H}_{\eta\nu} \qquad \xi \in S \end{aligned} \tag{A.46}$$

obtained by utilizing (A.2) and interchanging index pairs λ, ϑ $\rightarrow$ α, β and ϑ, λ $\rightarrow$ ν, ϑ,

and

— the equations

$$\begin{aligned} -\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{\lambda|\kappa\|\vartheta}\tilde{H}_{\eta 3}\,dA &= \int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{\lambda|\kappa}\tilde{H}_{\eta 3\|\vartheta}\,dA \\ &+ \oint_{g_o} n_3\epsilon^{\kappa\eta 3}\tau^\lambda u_{\lambda|\kappa}\tilde{H}_{\eta 3}\,ds \end{aligned} \tag{A.47}$$

and

$$\begin{aligned} -\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}(u_{3|\kappa})_{\|\lambda}\tilde{H}_{\eta\vartheta}\,dA &= \int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{3|\kappa}\tilde{H}_{\eta\vartheta\|\lambda}\,dA \\ &- \oint_{g_o} n_3\epsilon^{\kappa\eta 3}\tau^\eta u_{3|\kappa}\tilde{H}_{\eta\vartheta}\,ds \end{aligned} \tag{A.48}$$

derived by means of the Green theorem (A.22) and the relation

$$\tau^\vartheta = -\epsilon^{\lambda\vartheta 3}\nu_\vartheta \qquad \xi \in g_o \tag{A.49}$$

bearing in mind that $n_3 = 1$.

After enlarging the first surface integral in (3.25) by (A.45a) and the second one by (A.45b) and (A.45c) one can substitute (A.46), (A.47) and (A.48). Upon a subsequent rearrangement we find that

$$\begin{aligned} I^S_{1U} = -\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{\lambda|\kappa}\tilde{H}_{\eta\vartheta;3} &+ \int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{\lambda|\kappa}(\tilde{H}_{\eta 3\|\vartheta} - b_{\eta\vartheta}\tilde{H}_{33} + b^\nu_\vartheta\tilde{H}_{\eta\nu})dA \\ &+ \int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{3|\kappa}(\tilde{H}_{\eta\vartheta\|\lambda} - b_{\eta\lambda}\tilde{H}_{33} - b_{\vartheta\lambda}\tilde{H}_{\eta 3})dA \\ &+ \oint_{g_o} n_3\epsilon^{\kappa\eta 3}\tau^\vartheta(u_{\vartheta|\kappa}\tilde{H}_{\eta 3} - u_{3|\kappa}\tilde{H}_{\eta\vartheta})\,ds \end{aligned} \tag{A.50}$$

if the identity

$$n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{3|\lambda}b_{\vartheta\kappa}\tilde{H}_{\eta 3} = n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}u_{3|\kappa}b_{\vartheta\lambda}\tilde{H}_{\eta 3} \qquad \xi \in S$$

has also been taken into consideration. Upon substitution of the relations

$$\begin{aligned} \tilde{H}_{\eta 3;\vartheta} &= \tilde{H}_{\eta 3\|\vartheta} - b_{\eta\vartheta}\tilde{H}_{33} + b^\nu_\vartheta\tilde{H}_{\eta\nu} \qquad && \xi \in S \\ \tilde{H}_{\eta\vartheta;\lambda} &= \tilde{H}_{\eta\vartheta\|\lambda} - b_{\eta\lambda}\tilde{H}_{33} - b_{\vartheta\lambda}\tilde{H}_{\eta 3} \qquad && \xi \in S \end{aligned}$$

obtained from (A.11) into the second and third surface integral and a subsequent rearrangement we can apply the Stokes theorem to the sum of the three surface integrals:

$$-\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[(\tilde{H}_{\eta\vartheta;3}+\tilde{H}_{\eta 3;\vartheta})u_{\lambda;\kappa}+\tilde{H}_{\eta\lambda;\vartheta}u_{3|\lambda}]\,dA = -\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{ldp}u_{l;\kappa}\tilde{H}_{\eta d;p}\,dA =$$
$$= -\int_{S_o} n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa}u_l\,dA - \oint_{g_o}\tau^{\eta}\epsilon^{ldp}\tilde{H}_{\eta d;p}u_l\,ds$$

Rewriting the result into (A.50) we have:

$$I^S_{1U} = -\int_{S_o} n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}\tilde{H}_{\eta d;p\kappa}u_l\,dA + \oint_{g_o} n_3\epsilon^{\kappa\eta 3}\tau^{\vartheta}(u_{\vartheta|\kappa}\tilde{H}_{\eta 3} - u_{3|\kappa}\tilde{H}_{\eta\vartheta})\,ds$$
$$-\oint_{g_o}\tau^{\eta}\epsilon^{ldp}\tilde{H}_{\eta d;p}u_l\,ds\,. \qquad (A.51)$$

If the surface is closed i.e. $S_o \equiv S$ then the line integrals vanish and the above equation reduces to (3.27).

5.18 Transformation of integral I^S_{1E} of equation (3.26a).
Utilizing

— the manipulation

$$\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}(-b^{\beta}_{\beta}e_{\lambda\kappa}+b^{\alpha}_{\lambda}e_{\alpha\kappa})\tilde{H}_{\eta\vartheta} = \epsilon^{\kappa\eta 3}\epsilon^{\nu\tau 3}(e_{\sigma\kappa}b^{\sigma}_{\tau}-b^{\sigma}_{\sigma}e_{\tau\kappa})\tilde{H}_{\eta\nu} =$$
$$= \epsilon^{\kappa\eta 3}\epsilon^{\nu\tau 3}\delta^{\lambda\vartheta}_{\sigma\tau}e_{\lambda\kappa}b^{\sigma}_{\vartheta}\tilde{H}_{\eta\nu} = \qquad \xi\in S$$
$$= \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\epsilon_{\sigma\tau 3}\epsilon^{\nu\tau 3}e_{\lambda\kappa}b^{\sigma}_{\tau}\tilde{H}_{\eta\nu} = \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}e_{\lambda\kappa}b^{\nu}_{\vartheta}\tilde{H}_{\eta\nu}$$

obtained by interchanging indices $\lambda\to\tau$, $\beta\to\sigma$, $\vartheta\to\nu$ and using (A.2),

— the equation

$$e_{\kappa\lambda\|\vartheta} = e_{\kappa\lambda|\vartheta} + b_{\kappa\vartheta}e_{3\lambda} + b_{\lambda\vartheta}e_{\kappa 3} \qquad \xi\in S$$

that follows from (A.11)

and finally

— the integral transformations

$$\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}e_{3\kappa\|\lambda}\tilde{H}_{\eta\vartheta}dA = -\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}e_{3\kappa}\tilde{H}_{\eta\vartheta\|\lambda}dA$$
$$+\oint_{g_o} n_3\epsilon^{\kappa\eta 3}\tau^{\vartheta}e_{3\kappa}\tilde{H}_{\eta\vartheta}\,ds \qquad (A.52)$$

and

$$\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}e_{\lambda\kappa\|\vartheta}\tilde{H}_{\eta 3}dA = -\int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}e_{\lambda\kappa}\tilde{H}_{\eta 3\|\vartheta}dA$$
$$+\oint_{g_o} n_3\epsilon^{\kappa\eta 3}\tau^{\lambda}e_{\lambda\kappa}\tilde{H}_{\eta 3}\,ds \qquad (A.53)$$

whose derivation requires the use of the Green theorem and the equation (A.49)

we get from (3.26a) that

$$I^S_{1E} = \int_{S_o} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}[e_{\lambda\kappa}(\tilde{H}_{\eta\vartheta;3}-\tilde{H}_{\eta 3\|\vartheta}+b_{\eta\vartheta}\tilde{H}_{33}+b_{\vartheta\nu}\tilde{H}_{\eta\nu})+e_{3\lambda}b_{\kappa\vartheta}\tilde{H}_{\eta 3}+$$
$$+e_{3\kappa}(-\tilde{H}_{\eta\vartheta\|\lambda}+b_{\eta\lambda}\tilde{H}_{\vartheta 3})+e_{\kappa\lambda;3}\tilde{H}_{\eta\vartheta}+e_{\kappa 3|\lambda}\tilde{H}_{\eta\vartheta}+e_{\kappa\lambda|\vartheta}\tilde{H}_{\eta 3}]\,dA$$
$$+\oint_{g_o} n_3\epsilon^{\kappa\eta 3}(\tau^{\vartheta}e_{3\kappa}\tilde{H}_{\eta\vartheta}-\tau^{\lambda}e_{\lambda\kappa}\tilde{H}_{\eta 3})\,ds\,. \qquad (A.54)$$

It follows from (A.11) that

$$\tilde{H}_{\eta\vartheta\|\lambda} = \tilde{H}_{\eta\vartheta|\lambda}+b_{\eta\lambda}\tilde{H}_{3\vartheta}+b_{\vartheta\lambda}\tilde{H}_{\eta 3}\,, \qquad \xi\in S$$
$$\tilde{H}_{\eta 3\|\vartheta} = \tilde{H}_{\eta 3|\vartheta}+b_{\eta\vartheta}\tilde{H}_{33}+b^{\nu}_{\vartheta}\tilde{H}_{\eta\nu}\,. \qquad \xi\in S$$

Substitution of the above equations into (A.54), a subsequent rearrangement and renaming indices lead to the result

$$I^S_{1E} = \int_{S_o} n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}[-\tilde{H}_{\lambda\kappa} e_{\rho\vartheta;3} + \tilde{H}_{\lambda\kappa} e_{\rho 3;\vartheta} + \tilde{H}_{3\kappa} e_{\rho\vartheta;\lambda}$$
$$+\tilde{H}_{\lambda\kappa;3} e_{\rho\vartheta} - \tilde{H}_{\lambda\kappa|\vartheta} e_{\rho 3} - \tilde{H}_{3\kappa|\lambda} e_{\rho\vartheta}]\, dA$$
$$+ \oint_{g_o} n_3 \epsilon^{\kappa\eta 3}(\tau^\vartheta e_{3\kappa}\tilde{H}_{\eta\vartheta} - \tau^\lambda e_{\lambda\kappa}\tilde{H}_{\eta 3})\, ds\,. \tag{A.55}$$

Making use of the Stokes theorem one can write that

$$\int_{S_o} n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}\tilde{H}_{\lambda\kappa} e_{\rho 3|\vartheta}\, dA = \int_{S_o} n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}\tilde{H}_{\lambda\kappa|\vartheta} e_{\rho 3}\, dA$$
$$+ \oint_{g_o} \tau^\lambda \epsilon^{\kappa\rho 3}\tilde{H}_{\lambda\kappa} e_{\rho 3}\, ds \tag{A.56a}$$

and

$$\int_{S_o} n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}\tilde{H}_{3\kappa} e_{\rho\vartheta|\lambda}\, dA = \int_{S_o} n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}\tilde{H}_{3\kappa|\lambda} e_{\rho\vartheta}\, dA$$
$$- \oint_{g_o} \tau^\vartheta \epsilon^{\kappa\rho 3}\tilde{H}_{3\kappa} e_{\rho\vartheta}\, ds\,. \tag{A.56b}$$

Upon substitution of (A.56a,b) into (A.55) integral I^S_{1E} reduces to the form

$$I^S_{1E} = \int_{S_o} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}(-\tilde{H}_{\lambda\kappa} e_{\rho\vartheta;3} + \tilde{H}_{\lambda\kappa;3} e_{\rho\vartheta})\, dA\,. \tag{A.57}$$

Observe that the line integrals cancel each other.

If the surface is closed $S_o \equiv S$ and we obtain equation (3.29).

5.19 Transformation of integral I^V_{1E} of equation (3.26b).
By applying the *Gauss* theorem twice and renaming dummy indices we obtain from (3.26b) that

$$I^V_{1E} = I^V_{2E} + I^S_{2E} \tag{A.58}$$

where

$$I^V_{2E} = \int_V \epsilon^{pyk}\epsilon^{ldr} H_{yd;kr} e_{rs}\, dV \tag{A.59a}$$

and

$$I^S_{2E} = \int_S n_3 \epsilon^{\kappa\rho 3}\epsilon^{lsp}(e_{\rho s;p} H_{l\kappa} - e_{\rho s} H_{l\kappa;p})\, dA\,. \tag{A.59b}$$

As regards the surface integral it is worth decomposing those sums involving ϵ^{lsp}. After some manipulations we have

$$I^S_{2E} = \int_S n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}[H_{\lambda\kappa} e_{\rho\vartheta;3} - H_{\lambda\kappa} e_{\rho 3;\vartheta} - H_{3\kappa} e_{\rho\vartheta;\lambda}$$
$$-H_{\lambda\kappa;3} e_{\rho\vartheta} + H_{\lambda\kappa|\vartheta} e_{\rho 3} + H_{3\kappa|\lambda} e_{\rho\vartheta}]\, dA\,. \tag{A.60}$$

Comparison of the above integral to (A.55) enables us to repeat the line of thought leading from (A.55) to (A.57). Finally we obtain

$$I^S_{2E} = \int_{S_o} n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}(H_{\lambda\kappa} e_{\rho\vartheta;3} - H_{\lambda\kappa;3} e_{\rho\vartheta})\, dA + \oint_{g_o} n_3 \epsilon^{\kappa\rho 3}(\tau^\vartheta H_{3\kappa} e_{\rho\vartheta} - \tau^\lambda H_{\kappa\lambda} e_{\rho 3})\, ds\,. \tag{A.61}$$

Observe that we have assumed an open surface.

If the surface is closed $S_o \equiv S$, the line integrals vanish and equation (A.61) reduces to

$$I^S_{2E} = \int_S n_3 \epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}(H_{\lambda\kappa} e_{\rho\vartheta;3} - H_{\lambda\kappa;3} e_{\rho\vartheta})\, dA \tag{A.62}$$

Upon substitution of (A.59a) and (A.62) into (A.58) we arrive at (3.30).

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Adress: Professor Dr. Imre Kozák, Associate Professor Dr. György Szeidl, Department of Mathematics, University of Miskolc, H-3515 Miskolc-Egyetemváros