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On Compatibility Conditions for Mixed Boundary Value Problems

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The present paper focuses on the conditions of single-valuedness for the displacement field on multiply-connected bodies. It has been shown for a class of mixed boundary conditions that all the macro conditions of compatibility, i.e., the compatibility conditions in the large and the supplementary conditions of single valudness are also natural boundary conditions of the principle of minimum complementary energy as a variational principle.

1 Introduction

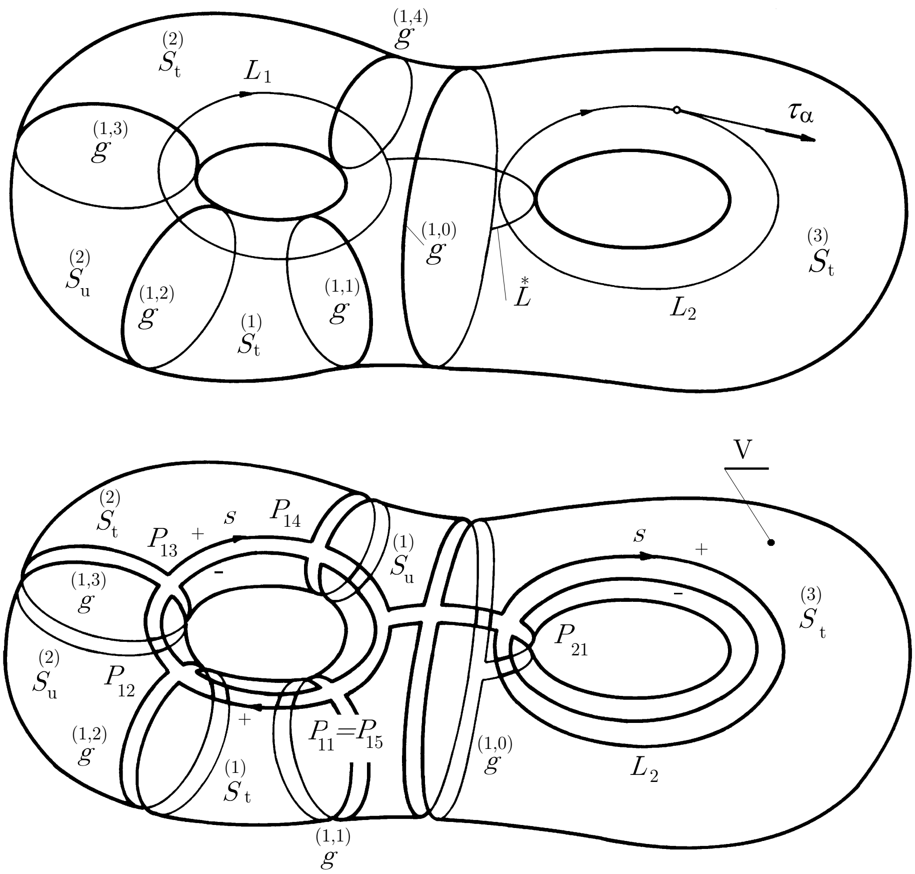
1.1 As regards the classical case it was Southwell (1936,1938) who first derived the compatibility conditions from the principle of minimum complementary energy as a variational principle. At the same time he pointed out – he utilized Maxwell's (1870) and Morera's (1892) solutions – that only three of the six Saint-Venant compatibility conditions follow from the principle of minimum complementary energy. Since any stress condition can be given in terms of three stress functions chosen appropriately he arrived at a contradiction because for the displacements to be single-valued all the six Saint-Venant compatibility conditions should be satisfied. This contradiction was named Southwell's paradox. After Southwell's papers the following problems remained unresolved: Is it sufficient for the strains to satisfy three Saint-Venant compatibility equations? If yes, which three? If yes, are there further conditions to satisfy?

1.2 A detailed description of the paradox grounded on all the possible stress function combinations is provided by Stickforth (1965) who noticed by generalizing a partial result of Washizu (1957) that the compatible strains satisfy some boundary conditions referred to as boundary conditions of compatibility. The latter term was proposed by Kozák in 1980, who pointed out in three distinct ways, i.e., in a purely mathematical one (1980b) and by utilizing the principle of minimum complementary energy as well as the dual forms of principle of virtual work (1980a,c) that for the strains to be compatible it is necessary and sufficient that three differential equation of compatibility and the boundary conditions of compatibility are fulfilled. The latter two papers are the only ones in which mixed boundary value problems are considered. It is assumed in all the papers mentioned above that the body is single-connected.

1.3 For doubly-connected domain and plane problems Prager (1946) pointed out assuming traction problems and homogeneous, isotropic material that Mitchell's conditions (1900) (cf. e.g. Gurtin, 1972), or what is the same thing, the compatibility conditions in the large regarded under the conditions mentioned are natural boundary conditions of Castigliano's principle. This result was generalized for mixed boundary value problems by Haichang (1986) and independently by Szeidl and Gemert (1991). Returning to the three dimensional case the papers by Moriguti (1948) and independently by Stickforth (1964) are concerned with three dimensional tractions problems on multiply connected regions but they provide no solution to Southwell's paradox.

By *macro conditions of compatibility* is meant the totality of those additional conditions the strains should meet to be compatible on a multiply connected body. Depending on what the boundary conditions are in the points of a simply–connected and closed curve on the surface of the body the macro conditions of compatibility are separated into two groups. If tractions are imposed in each point of the curve the condition is referred to as a *compatibility condition in the large*. If there exists at least one arc on the curve along which displacements are imposed then the corresponding condition is called *supplementary condition of single-valuedness*.

1.4 In view of the foregoing it seems to be an open question what supplementary conditions of single-valuedness are needed for mixed boundary value problems on multiply connected regions. On the bases of all that has been said the present paper is aimed at investigating the problem of what natural boundary



Figures 1(a) and (b)

conditions follow from the principle of minimum complementary energy assuming a three dimensional and multiply-connected body and a certain class of mixed boundary value problems. We shall also assume that the linear theory of deformations is valid. When applying Castigliano's principle in addition it will be assumed that the material of the body is linearly elastic.

1.5 In section 2 we collect some preliminary results and derive the supplementary conditions of single-valuedness from geometrical considerations concentrating attention on the classical case. Section 3 is devoted to the problem of how the supplementary conditions of single valuedness can be obtained from the principle of minimum complementary energy. Section 4 is a summary of the results. Finally there is section 5 where some fundamental mathematical relations being used in this paper and some longer transformations are presented.

2 Derivation of the Supplementary Conditions from Geometrical Considerations

2.1 The bounded region of the three dimensional space occupied by the multiply connected body and the surface of the body are denoted respectively by V and S. In principle the surface S of the body may consist of not only one but more closed surfaces, in which case the region is multiply-bordered, though the latter circumstance will play no role in the investigations. The surface S is divided into parts S_u and S_t whose common bounding curve is denoted by g.

The present paper restricts its attention to the triple-connected but single-bordered body represented in Figure 1(a) which contains some further notational conventions. It is clear from Figure 1(a) that both the subsurfaces S_u, S_t and the curve g consists of more parts, i.e.,

$$S_u = \overset{(1)}{S}_u \cup \overset{(2)}{S}_u; \qquad S_t = \overset{(1)}{S}_t \cup \overset{(2)}{S}_t \cup \overset{(3)}{S}_t \qquad \text{and} \qquad \overset{(1,0)}{g} \cup \overset{(1,1)}{g} \cup \overset{(1,2)}{g} \cup \overset{(1,3)}{g} \cup \overset{(1,4)}{g}$$

In a limit case any of the subsurfaces

$$\overset{(1)}{S}_u,\ \overset{(2)}{S}_u, \qquad \text{and} \qquad S_u \qquad \text{or} \qquad \overset{(1)}{S}_t,\ \overset{(1)}{S}_t,\ \overset{(1)}{S}_t \qquad \text{and} \qquad S_t$$

may be an empty set.

The non-intersecting, simple and closed curves $\mathcal{L}_1$ and $\mathcal{L}_2$ encircle the first and second holes.

It is essential for the further investigations that $\overset{(1)}{S}_u$ and $\overset{(2)}{S}_u$ are respectively triple and double connected surfaces in the way they are represented in Figure 1(b). The curves $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect the curve g and $\overset{*}{\mathcal{L}}$ in the points

$$P_{11},\ P_{12},\ P_{13},\ P_{14} \qquad \text{and} \qquad P_{21}$$

Let the parts of $\mathcal{L}_1$ be defined by

$$\mathcal{L}_{1j} = P_{1j}\, P_{1,j+1} \qquad j = 1, \ldots, 4\,.$$

When performing integral transformations by making use of Stokes' theorem one must keep in mind that the theorem is applicable under the condition that the surface considered is simple-connected. Figure 1(b) represents a possibility for cutting up the surface S into simple-connected parts by utilizing the curves g, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\overset{*}{\mathcal{L}}$.

It is worthy of mention that the restrictions we have made in connection with the body considered are not essential for the body is at least triple-connected and the partition of the boundary surface S is sufficiently general.

2.2 Indicial notations and two coordinate systems: the $(x^1x^2x^3)$ curvilinear and the $(\xi^1\xi^2\xi^3)$ curvilinear (defined on the surface S) are employed throughout this paper. Scalars and tensors, unless the opposite is stated are denoted independently of the coordinate system by the same letter. Distinction is aided by the indication of the arguments x and ξ used to denote the totality of the corresponding coordinates.

When referring to some formulas of the paper by Kozák and Szeidl (1996a) the equation number will be followed by the roman number i.

Volume and surface integrals are considered, respectively, in the coordinate systems (x^1, x^2, x^3) and (ξ^1, ξ^2, ξ^3). Consequently, in the case of integrals, arguments are omitted.

In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a semicolon denote covariant differentiation with respect to the corresponding subscripts. Latin and Greek indices range over the integers 1, 2, 3 and 1, 2, respectively. ϵ^{klm} and ϵ_{pqr} stand for the permutation tensors; δ_k^l is the Kronecker delta. In the system of coordinates (x^1, x^2, x^3) $\mathbf{g}_k$ and $\mathbf{g}^l$ are the covariant and contravariant base vectors. The corresponding metric tensors are denoted by g_{kl} and g^{pq}.

A pair of subscripts is enclosed by parentheses ( ) to indicate the symmetric part of a tensor of order two and by brackets [ ] to indicate the skew part. Covariant derivative is denoted by a Latin subscript preceded by a semicolon – see (A.4,i) for details.

2.3 Calculations carried out on the surface S can be better understood by introducing a suitable surface oriented coordinate system. Let $x^k = x^k(\xi^1, \xi^2)$ be the equation of the surface S where ξ^1 and ξ^2 are the surface coordinates. Let ξ^3 be the perpendicular distance measured on the outward unit normal $\mathbf{n}$ to the surface S. On S $\xi^3 = 0$.[Base vectors] {Metric tensors} on S are denoted by [$\mathbf{a}^k$ and $\mathbf{a}_k$] {a_{kl} and a^{kl}}. In the surface oriented coordinate system (ξ^1, ξ^2, ξ^3)

$$\mathbf{n} = \mathbf{a}_3 = \mathbf{a}^3\,, \qquad n^3 = 1 \qquad \text{and} \qquad n^\eta = 0. \tag{2.1}$$

If $|\xi^3|/(\min\{|R_1|, |R_2|\}) < 1$ in which R_1 and R_2 are the principal radii of curvature on S then the relationship $x^k = x^k(\xi^1, \xi^2, \xi^3)$ is always one-to-one. The tensor of curvature is denoted by $b_{\alpha\beta}$ – see (A.6,i). [Covariant derivative on the surface S] {The surface covariant derivative} is denoted by a Greek subscript preceded by [a semicolon (or one short vertical line) – see (A.8,i)] {two short vertical lines – see (A.9,i)}.

It will be assumed that the vector and tensor fields involved in the investigations are sufficiently smooth.

2.4 Now we shall assemble equations of elastostatics – in a form suited to our objective – in primal and dual systems as well. In accordance with this, paragraphs 2.5 and 2.6 provide a brief overview of the equations of the primal system while the following part of the present section turns its attention to the equations and boundary conditions of the dual system including the missing conditions of single-valuedness.

Let u_k, e_{kl} and t^{kl} be the displacement field, strain tensor and stress tensor (or displacements, strains and stresses for short). Body forces will be denoted by b^l.

2.5 In the primal system the three-dimensional problems under consideration are governed by the kinematic equations

$$e_{kl} = \frac{1}{2}(u_{k;l} + u_{l;k}), \qquad x \in V \tag{2.2}$$

Hook's law

$$t^{kl} = C^{klrs} e_{rs} \qquad x \in V \tag{2.3}$$

and the equilibrium equations

$$t^{kl}_{\cdot\cdot;k} + b^l = 0 \qquad x \in V \tag{2.4}$$

where C^{klrs} is the tensor of elastic coefficients.

Field equations (2.2), (2.3) and (2.4) are to be supplemented by boundary conditions. It will be assumed that [tractions] {displacements} are prescribed on $[S_t]$ $\{S_u\}$. Then

$$u_l = \hat{u}_l \qquad \xi \in S_u \tag{2.5}$$

is the displacement boundary condition and

$$n_3 t^{3l} = \hat{t}^l \qquad \xi \in S_t \tag{2.6}$$

is the traction boundary condition in which $\hat{u}_l$ and $\hat{t}^l$ are the pre-assigned displacement and traction components.

The strains e_{kl} are said to be [compatible] {kinematically admissible} if the kinematic equations (2.2) have a sufficiently smooth solution to the displacements u_l and the solution [meets no other conditions] {satisfies the displacement boundary condition (2.5)}.

Stresses t^{kl} are said to be [equilibrated] {statically admissible} if they satisfy the equilibrium equation (2.4) and [meet no other conditions] {and the traction boundary condition (2.6)}.

Every solution of equilibrium equations (2.2) admits the following representation found in its final form by Schaefer (1953) (cf. e.g. Gurtin, 1972):

$$t^{pl} = \epsilon^{pyk}\epsilon^{ldr}\tilde{H}_{yd;kr} + g^{pq}B^l_{.;q} + g^{lq}B^p_{.;q} - g^{pl}B^k_{.;k} \qquad x \in V \tag{2.7}$$

where $\tilde{H}_{yd} = \tilde{H}_{yd}$ is the stress function tensor (whose components will also be referred to as stress functions) and the vector field B^l is given by (2.8,i), i.e., it is a particular solution of the Poisson equation (2.7,i). For this reason we shall assume that the vector field $B^l(x)$ is known.

Because of its role in the further investigations an important property of the above stress representations should be mentioned here. To begin with one has to introduce some notations.

The index pairs which range over a subset of the nine possible values will be capitalized.

Let α_{ab} be a sufficiently smooth otherwise arbitrary symmetric tensor field in V. Furthermore let $w_l(x)$ be an unknown vector field on V. By $_{AB}$ we denote those subsets of the possible values of the index pairs $_{ab}$ for which the differential equation

$$\frac{1}{2}(w_{A;B} + w_{B;A}) = \alpha_{AB}(x) \qquad x \in V$$

always have a solution for the vector field $w_l(x)$. It is obvious that the index pairs $_{AB}$ may have only three different values.

Let $_{RS}$ be the supplementary subset of the index pairs whose union with $_{AB}$ is the set of index pairs $_{ab}$. Obviously, the index pairs $_{RS}$ may have six distinct values. Because of the symmetry, however, the corresponding tensor components α_{RS} represent three distinct functions only.

The stress functions

$$\mathcal{H}_{ab} = \tilde{H}_{ab} + w_{(a;b)} \qquad x \in V \tag{2.8}$$

lead to the same stress state as $\tilde{H}_{ab}$ since

$$\epsilon^{pyk}\epsilon^{ldr} w_{(y;d);kr} \equiv 0 \qquad x \in V$$

It immediately follows from this that by solving the differential equations

$$w_{(A;B)} = -\tilde{H}_{ab} \qquad x \in V \tag{2.9}$$

for $w(x)$ and substituting the result into equations (2.8) we find that the three stress functions identified by the indices $_{AB}$ will vanish. In other words: any stress state can be given in terms of the stress functions $\mathcal{H}_{RS}$, or what is the same thing, in terms of three stress functions. This result is due to Finzi (1934).

NOTE 1.: In the sequel it will be assumed, that $\mathcal{H}_{AB} = 0 \quad x \in V$ independently of the circumstance what subscripts are employed, i.e., capitalized or not.

2.6 By inverting Hook's law we obtain the equation

$$e_{kl} = \overset{-1}{C}_{klrs} t^{rs} \qquad x \in V \tag{2.10}$$

in which the fourth order tensor $\overset{-1}{C}{}^{klrs}$ follows from the condition $\overset{-1}{C}_{klrs} C^{rspq} = \delta_k^p \, \delta_l^q$. The strains e_{kl} are said to be [equilibrated] {statically admissible} if they are calculated from Hook's law (2.10) by substituting [equilibrated] {statically admissible} stresses t^{kl}.

The tensor of incompatibility $\mathcal{Y}^{ab}$ is defined by the equation

$$\mathcal{Y}^{ab} = \epsilon^{akm} \epsilon^{blp} e_{kl;mp} \, . \qquad x \in V$$

The following results are those of Kozák (1980a,b).

For strains e_{kl} to be [compatible] {kinematically admissible} in a single connected region V it is necessary and sufficient that the differential equations of compatibility

$$\mathcal{Y}^{RS} = \epsilon^{Rkm} \epsilon^{Slp} e_{kl;mp} = 0 \, , \qquad x \in V \tag{2.11}$$

the boundary conditions of compatibility

$$n_a \mathcal{Y}^{ab} = n_3 \mathcal{Y}^{3b} = n_3 \epsilon^{3km} \epsilon^{dlp} e_{kl;mp} = n_3 \epsilon^{3\eta\kappa} \epsilon^{ldp} e_{\eta d;p\kappa} = 0 \qquad [\xi \in S] \text{ or } \{\xi \in S_t\} \tag{2.12}$$

and [no further conditions] {the kinematic boundary conditions}

$$e_{\lambda\kappa} - \hat{u}_{(\lambda;\kappa)} = 0 \qquad \xi \in S_u \tag{2.13a}$$

$$(e_{3\kappa} - u_{3;\kappa})_{\|\lambda} + b_\lambda^\alpha (e_{\alpha\kappa} - u_{\alpha;\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda 3;\kappa}) = 0 \qquad \xi \in S_u \tag{2.13b}$$

should be fulfilled. From equations (2.13a,b) there follows the fulfillment of equation

$$e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta} - (u_{\lambda|\kappa})_{\|\vartheta} - u_{3|\lambda} b_{\vartheta\kappa} = 0 \, . \qquad \xi \in S \tag{2.13c}$$

For proof see the paragraphs 3.9 and 5.16 in the paper by Kozák–Szeidl (1996a).

It can also be shown that the fulfillment of kinematic boundary conditions (2.13a,b) on S_u automatically assures the fulfillment of boundary conditions of compatibility – see the paragraphs 3.7, 5.14 and 5.14 in the paper cited above.

2.7 For the equilibrated stresses

$$t^{pl} = \epsilon^{pyk} \epsilon^{ldr} \mathcal{H}_{yd;kr} + g^{pq} B^l_{.;q} + g^{lq} B^p_{.;q} - g^{pl} B^k_{.;k} \qquad x \in V \tag{2.14}$$

obtained from (2.7) by substituting $\mathcal{H}_{yd}$ for $\tilde{H}_{yd}$ to be statically admissible it is sufficient if the traction boundary condition

$$\begin{aligned} \hat{t}^l &= n_p(\epsilon^{pyk} \epsilon^{ldr} H_{yd;kr} + g^{pq} B^l_{.;q} + g^{lq} B^p_{.;q} - g^{pl} B^k_{.;k}) \\ &= n_3(\epsilon^{3\eta\kappa} \epsilon^{ldp} H_{\eta d;p\kappa} + a^{3q} B^l_{.;q} + a^{lq} B^3_{.;q} - a^{3l} B^k_{.;k}) \qquad \xi \in S_t \end{aligned} \tag{2.15}$$

is fulfilled.

2.8 For single-connected bodies the three dimensional problems of elasticity in the dual system are governed by the dual kinematic equation (2.14), the constitutive equation (2.10) and the dual balance equation (2.11).

Observe that we have as many unknowns — the dual configuration variable H_{RS}, the first intermediate variable t^{kl} and the second intermediate variable e_{kl} — as there are field equations, viz. (2.14), (2.10) and (2.11); both numbers are 15.

Field equations (2.14), (2.10) and (2.11) are associated with the boundary conditions (2.12) of compatibility and the traction boundary condition (2.15) on S_t and with the strain boundary conditions (2.13a,b) on S_u.

The terminology used in the present paragraph was proposed by Tonti (1972).

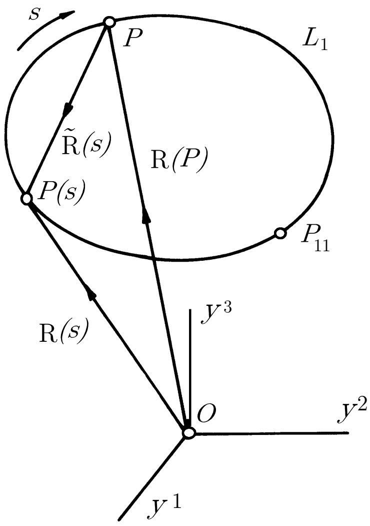


Figure 2.

2.9 For multiply-connected domains fulfilment of the differential equation (2.11) of compatibility and the compatibility boundary conditions (2.12) is, however, not sufficient for strains e_{kl} to be compatible, but some supplementary conditions, referred to as compatibility conditions in the large, are also to be satisfied. Now we turn our attention to the supplementary conditions. We shall consider the single, closed curve $\mathcal{L}_1$ – see Figure 2. Let P be an arbitrary but fixed point on $\mathcal{L}_1$. Position vector of point $P(s)$ with respect to P is denoted by $\tilde{\mathbf{R}}(s) = \tilde{R}^v \mathbf{a}_v(s)$. Observe that the basis is taken at the point s. It is apparent that

$$\tau^\alpha \mathbf{a}_\alpha = \frac{d\tilde{\mathbf{R}}}{ds} = \frac{d\,(\mathbf{R}(s) - \mathbf{R}(P))}{ds} = \frac{\partial \mathbf{R}}{\partial \xi^\alpha}\frac{d\xi^\alpha}{ds} = \frac{d\xi^\alpha}{ds}\mathbf{a}_\alpha \qquad s \in L_1 \tag{2.16}$$

With the rotation field

$$\omega^l = \frac{1}{2}\epsilon^{lpd} u_{d;p} \qquad x \in V \tag{2.17}$$

and equations (2.2) one can write

$$\omega^3 = \frac{1}{2}\epsilon^{3\lambda\vartheta} u_{\vartheta|\lambda} \qquad \xi \in S \tag{2.18a}$$

and

$$\omega^\lambda = \epsilon^{\lambda 3\vartheta}[\frac{1}{2}(u_{\vartheta;3} + u_{3|\vartheta}) - u_{3|\vartheta}] = \epsilon^{\lambda 3\vartheta}(e_{\vartheta 3} - u_{3|\vartheta}) \qquad \xi \in S \tag{2.18b}$$

NOTE 2.: Assume for clarity that u_k is a displacement field given on S. Equation (2.18a) reflects that $u_k(\xi)$ uniquely determines the rotation ω^3. On the contrary equation (2.18b) involves the derivative $u_{\lambda;3}$ taken along the normal to the surface S consequently ω^λ is only partly determined by $u_k(\xi)$.

2.10 By making use of the kinematic equation (2.2) from (2.17) it follows

$$\omega^l{}_{.;y} = \frac{1}{2}\epsilon^{lpd}(u_{d;py} + u_{y;dp}) = \epsilon^{lpd} e_{yd;p} \qquad x \in V \tag{2.19}$$

which, with regard to (2.16) and the notations of Figure 3. leads to

$$\frac{\partial \omega^l \mathbf{a}_l}{\partial s} = \tau^\eta \epsilon^{lpd} e_{\eta d;p}\,\mathbf{a}_l \qquad \xi \in S \tag{2.20}$$

and consequently

$$(\omega^\lambda \mathbf{a}_\lambda + \omega^3 \mathbf{a}_3)|_P = (\omega^\lambda \mathbf{a}_\lambda + \omega^3 \mathbf{a}_3)|_{P_{11}} + \int_{P_{11}}^{P} \tau^\eta \epsilon^{lpd} e_{\eta d;p}\, \mathbf{a}_l ds\,. \tag{2.21}$$

A comparison of equations (2.2) and (2.17) yields

$$u_{k;l} = e_{kl} - \epsilon_{klr}\omega^r \qquad\qquad x \in V$$

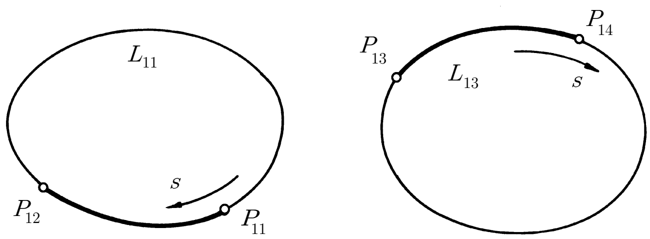
from which utilizing again the notations of Figure 3. and substituting the equations (2.16) and (2.19) one obtains

$$\frac{\partial u_l \mathbf{a}^l}{\partial s} = -\frac{\partial}{\partial s}\{(R^v(s) - R^v(P))\epsilon_{kvl}\omega^l \mathbf{a}^k(s)\} + [\tau^\eta e_{\eta k} + \tau^\eta \epsilon_{kvl}\epsilon^{lpd} e_{\eta d;p}(R^v(s) - R^v(P))]\,\mathbf{a}^k(s) \qquad s \in \mathcal{L}_1 \tag{2.22}$$

or after integration

$$u_k\, \mathbf{a}^k|_P = u_k\, \mathbf{a}^k|_{P_{11}} + \epsilon_{kvl}\omega^l (R^v(s) - R^v(P))\mathbf{a}^k|_{P_{11}} + \int_{P_{11}}^{P} \tau^\eta [e_{\eta k} + \epsilon_{kvl}\epsilon^{lpd}(R^v(s) - R^v(P)) e_{\eta d;p}]\,\mathbf{a}^k\, ds \tag{2.23}$$

Equations (2.21) and (2.23) coincide with Cesaro's formulae. Outline of their derivation is presented herein for the sake of those further transformations in which same partial results of the derivation will be utilized.



(a) (b)

Figure 3.

Returning to the three dimensional elasticity problem considered on the triple-connected region V — see Figure 1 — for the strains e_{kl} to be compatible it is also necessary, in addition to the fulfilment of field equation (2.11), that the rotation field ω^l and the displacement u_k should be single-valued on all closed simple curves which encircle the two holes. It can readily be shown that the fulfilment of the supplementary conditions of single-valuedness on one-one single curve encircling the holes — one can chose $\mathcal{L}_1$ and $\mathcal{L}_2$ for instance — is sufficient for the rotation ω^l and the displacements u_k to be single-valued along all the other curves that encircle the holes provided that the differential equations of compatibility hold.

In the light of this circumstance we confine ourselves to the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ during further investigations. Observing that $\mathcal{L}_2$ lies inside S_t for the single-valuedness of the rotation ω^l there follows from equation (2.21) the compatibility conditions in the large

$$\oint_{\mathcal{L}_2} \tau^\eta \epsilon^{lpd} e_{\eta d;p}\, \mathbf{a}_l\, ds \;=\; \oint_{\mathcal{L}_2} \tau^\eta \epsilon^{\rho\vartheta 3} e_{\eta\vartheta|\rho}\, \mathbf{a}_3\, ds \;+\; \oint_{\mathcal{L}_2} \tau^\eta \epsilon^{\lambda\vartheta 3}(e_{\eta 3|\vartheta} - e_{\eta\vartheta;3})\, \mathbf{a}_\lambda\, ds \;=\; 0 \tag{2.24}$$

Using exactly analogous reasoning from equation (2.23) one obtains the compatibility condition in the large

$$\oint_{\mathcal{L}_2} \tau^\eta [e_{\eta k} + \epsilon_{kvl}\epsilon^{lpd}(R^v(s) - R^v(P_{21})) e_{\eta d;p}]\, \mathbf{a}^k\, ds \;=\; 0 \tag{2.25}$$

for the displacements u_k to be single-valued along $\mathcal{L}_2$.

As regards the curve $\mathcal{L}_1$ its parts [$\mathcal{L}_{11}$ and $\mathcal{L}_{13}$] {$\mathcal{L}_{12}$ and $\mathcal{L}_{13}$} lie wholly [in S_t] {in S_u}. In view of this fact the problem raised in paragraph 1.4 can be reworded in the following way: What is the effect on the compatibility conditions in the large of the circumstance that any simple closed curve encircling the first hole on S passes through the boundary part S_u twice. Solution to the problem posed is sought by means of a geometrical line of thought. Assume that the curve $\mathcal{L}_1$ is divided into two parts as shown in Figure 3. On arc $\mathcal{L}_{1i}$ $(i = 1, 3)$ drawn in heavy line the tractions are given while displacements are prescribed on the parts of arc $P_{1,i+1}, P_{1i}$ lying in the neighborhood of the endpoints. For the rotation ω^l to be continuous on the closed curve $\mathcal{L}_1$ being divided into two parts in two ways it is necessary that the continuity conditions

$$\omega^l \mathbf{a}_l\Big|_{P_{11}}^{P_{12}} + \omega^l \mathbf{a}_l\Big|_{P_{12}}^{P_{11}} = 0 \qquad \text{and} \qquad \omega^l \mathbf{a}_l\Big|_{P_{13}}^{P_{14}} + \omega^l \mathbf{a}_l\Big|_{P_{14}}^{P_{13}} = 0 \tag{2.26}$$

be satisfied. The first term in the left hand sides can be calculated from equation (2.21) by an appropriate change of limits. As regards the second term one has to utilize equations (2.18a,b) by taking into consideration the circumstance that the displacements are known if

$$s \in (s(P_{1j}) - \epsilon,\, s(P_{1j})] \qquad \text{or} \qquad s \in [s(P_{1,j+1}), s(P_{1,j+1}) + \epsilon) \qquad j = 1, 3 \tag{2.27}$$

in which ϵ is a small positive otherwise arbitrary quantity. In this way one obtains

$$\left[\frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta|\lambda}\,\mathbf{a}_3 + \epsilon^{\lambda 3\vartheta}(e_{\vartheta 3} - \hat{u}_{3|\vartheta})\,\mathbf{a}_\lambda\right]\Bigg|_{P_{1,i+1}}^{P_{1i}} + \int_{\mathcal{L}_{1i}} \tau^\eta \epsilon^{\rho\vartheta 3} e_{\eta\vartheta|\rho}\,\mathbf{a}_3\, ds + \int_{\mathcal{L}_{1i}} \tau^\eta \epsilon^{\lambda\vartheta 3}(e_{\eta 3|\vartheta} - e_{\eta\vartheta;3})\,\mathbf{a}_\lambda\, ds = 0 \quad i = 1, 3 \tag{2.28}$$

One should notice, that the rotation is continuous along the whole curve $\mathcal{L}_1$ if the condition (2.28) is fulfilled.

Repeating the line of thought leading to equation (2.28) we find that the displacements should meet the continuity conditions

$$u_k\,\mathbf{a}^k\Big|_{P_{11}}^{P_{12}} + u_k\,\mathbf{a}^k\Big|_{P_{12}}^{P_{11}} = 0 \quad \text{and} \quad u_k\,\mathbf{a}^k\Big|_{P_{13}}^{P_{14}} + u_k\,\mathbf{a}^k\Big|_{P_{14}}^{P_{13}} = 0 \tag{2.29}$$

In order to obtain a more suitable form the above continuity conditions will be transformed further in three steps:

(a) First one has to take into consideration again that the displacements are prescribed if s meets the conditions (2.27); this circumstance affects the second term in the left hand sides.

(b) Secondly Cesaro's formula (2.23) should be used to determine the first difference (term) in the left hand sides.

(c) Finally one has to substitute equations (2.18a,b) so as to determine the rotations ω^3 and ω^λ involved in the Cesaro formula at the points P_{11} and P_{13}.

After all these manipulations one obtains

$$\begin{aligned} &\hat{u}_k\,\mathbf{a}^k\Big|_{P_{1,i+1}}^{P_{1i}} + \epsilon^{\lambda 3\vartheta}\,(e_{3\vartheta} - \hat{u}_{3|\upsilon})\epsilon_{\lambda kl}\,(R^k(P_{1,i+1}) - R^k(P_{1i}))\,\mathbf{a}^l\Big|_{P_{1i}} \\ &\qquad + \frac{1}{2}\,\epsilon^{3\lambda\vartheta}\hat{u}_{\upsilon;\lambda}\,\epsilon_{3\psi\rho}\,(R^\psi(P_{1,i+1}) - R^\psi(P_{1i}))\,\mathbf{a}^\rho\Bigg|_{P_{1i}} \\ &\qquad + \int_{\mathcal{L}_{1i}} \tau^\eta\{e_{\eta k} + \epsilon^{lpd}\,\epsilon_{kvl}\,[R^v(s) - R^v(P_{1,i+1})]e_{\eta d;p}\}\,\mathbf{a}^k\, ds = 0 \qquad i = 1, 3 \end{aligned} \tag{2.30}$$

NOTE 3.: Conditions of single-valuedness (2.28) and (2.30) reduce formally to the compatibility conditions e2.270) and (2.25) in the large if moving in clockwise direction the point $P_{1,i+1}$ reaches the point P_{1i} or, what is the same thing, if the arc $\mathcal{L}_{1i}$ coincides with the closed curve $\mathcal{L}_1$.

2.11 For triple-connected domains the dual equation system (2.14), (2.10) and (2.11), which is associated with the boundary conditions (2.12), (2.15) and (2.13a,b), should be supplemented by the compatibility conditions in the large (2.24), (2.25) and supplementary conditions (2.28) and (2.30) of single-valuedness.

3 Derivation of the Supplementary Conditions from Casigliano's Principle

3.1 All conditions the strains e_{kl} should satisfy in order to be [compatible] {kinematically admissible} in a simply-connected three dimensional region can be derived from Castigliano's principle – see the paper by Kozák (1980a) for details. Consequently the present section follows the line of thought of Kozák except in two things:

1. Stress functions satisfying the side condition (2.15) can be varied provided that there are no stresses due to the variations. In this respect the paper will utilize such a form which does not require the knowledge of derivatives taken along the normal to the surface S_t of the vector field in terms of which the variations causing no stresses can be given.
2. Since now the volume region is multiply-connected, special care will be taken with those terms obtained by applying Stoke's theorem on the simple-connected parts of S in order to find a proper form for the jumps of stress functions and to ensure in this way the fulfilment of the subsidiary conditions of single-valuedness.

3.2 Let $-K$ be the total complementary energy functional. Then the functional

$$K = -\frac{1}{2}\int_V t^{kl} e_{kl}\, dA + \int_{S_u} n_3 t^{3l} \hat{u}_l\, dA \tag{3.1}$$

as a function of the statically admissible stresses t_{kl} and strains e_{kl} has a strict maximum at the actual solution.

The necessary condition for extremum of functional (3.1) is the vanishing of the first variation

$$\delta K = -\int_V e_{kl}\delta t^{kl}\, dV + \int_{S_u} n_3 \delta t^{kl} \hat{u}_\lambda\, dA = 0 \tag{3.2}$$

obtained from (3.1) by making use of the fact that

$$\frac{1}{2}\delta(t^{kl}e_{kl}) = e_{kl}\delta t^{kl}\,.$$

Paragraphs 3.3 to 3.8 are devoted to the long formal transformations aimed at casting the extremum condition $-\delta K = 0$ into an appropriate form. In order to clarify the nature of the various steps a short outline of the transformations is presented below.

(a) In paragraph 3.3 the variations of stresses are given in such a form that the side conditions are satisfied both in V and on S_t.

(b) paragraph 3.4 is devoted to the integral transformations. When applying Stoke's theorem we must keep in mind that the surface S has been cut up and the integrals have to be taken on simple-connected subsurfaces. Consequently line integrals are also obtained. From these some line integrals cancel each other if the following conditions hold:
1. All the variables in the integrals are continuous on the part of the boundary curve regarded.
2. We go twice along this part of the boundary curve. (When applying Stokes' theorem on S_u for instance we go along the arc P_{12}, P_{13} twice.)
It is worthy of mention here that except δw_l all the other variables are continuous on the entire boundary S. As regards δw_l it is continuous on the simple-connected subsurfaces

$$\overset{(1)}{S}_t,\ \overset{(2)}{S}_t \qquad \text{and} \qquad \overset{(3)}{S}_t$$

but has a jump on the arc

$$\mathcal{L} = \mathcal{L}_{11} \cup \mathcal{L}_{13} \cup \mathcal{L}_2\,.$$

The first integral to contain δw_l is the one resulting from equation (3.14) after substituting the side conditions (3.6a,b). In the subsequent steps special care should be taken with those integrals performed along $\mathcal{L}_{11}$, $\mathcal{L}_{13}$ and $\mathcal{L}_2$.

(c) paragraph 3.5 contains the final form of the extremum condition in respect of the volume and surface integrals.

(d) paragraphs 3.6 to 3.8 are devoted to the analysis of the line integrals taken along g and L.

3.3 Being statically admissible the stresses t^{kl} are to be varied under the subsidiary conditions (2.4) and (2.6). Consequently the variations δt^{kl} can not be taken at will but should meet the conditions

$$\delta t^{kl}_{\ ..;k} = 0 \qquad x \in V \qquad \text{and} \qquad n_3 \delta t^{3l} = 0 \qquad \xi \in S_t \tag{3.3}$$

which follow from equations (2.4) and (2.6) by taking into consideration that the variations of body forces b_l and prescribed tractions $\hat{t}^l$ are equal to zero. It is also assumed that the variation δB^l is equal to zero.

It follows from equations (2.14) and (2.15) that the variations of stresses satisfying (3.3) can be given in terms of variations of stress functions:

$$\delta t^{kl} = \epsilon^{krm}\epsilon^{lsp}\delta\mathcal{H}_{rs;mp} \qquad x \in V \tag{3.4}$$

where δH_{rs} is arbitrary on V and S_u but should satisfy the side condition

$$\delta \hat{t}^{l} = n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}\delta\mathcal{H}_{\eta d;p\kappa} = 0 \qquad \xi \in S_t \tag{3.5}$$

Let δw_l be a vector field defined on S_t. If $\delta\mathcal{H}_{\eta d}$ satisfies the conditions

$$\delta\mathcal{H}_{\eta d} = \delta w_{(\lambda|\kappa)} \qquad \xi \in S_t \tag{3.6a}$$

$$(\delta\mathcal{H}_{\kappa\lambda} - \delta w_{3|\kappa})_{\|\lambda} + b^{\alpha}_{\lambda}(\delta\mathcal{H}_{\alpha\kappa} - \delta w_{\alpha|\kappa}) - (\delta H_{\kappa\lambda;3} - \delta H_{\lambda 3;\kappa}) = 0 \qquad \xi \in S_t \tag{3.6b}$$

then condition (3.5) holds. The proof of this statement is very simple if one notices that equations (3.5) and (3.6a,b) coincide with the compatibility boundary condition (2.12) and kinematic boundary conditions (2.13a,b) provided that $e_{\eta d}$ and u_l are substituted for $\delta\mathcal{H}_{\eta d}$ and δw_l in equations (3.5) and (3.6a,b), respectively. Then recalling the assertion that the fulfilment of the strain boundary conditions (2.13a,b) implies the fulfilment of the boundary conditions of compatibility one can come to the conclusion that the original statement is true.

NOTE 4.: The relatively long and cumbersome proof of the assertion mentioned is published in a paper by Kozák and Szeidl (1996), but originally it appeared in a thesis written by Kozák (1980d).

NOTE 5.: The static-kinematic analogy, which makes possible the use of the proof cited above, was found by Kozák and Szeidl (1996b). This analogy also involves the equation

$$\delta\mathcal{H}_{\kappa\lambda\|\vartheta} + \delta\mathcal{H}_{\lambda\kappa\|\vartheta} - \delta w_{\lambda|\kappa\|\vartheta} - \delta w_{3|\lambda} b_{\vartheta\kappa} = 0 \qquad \xi \in S_t \tag{3.6c}$$

which is the dual counterpart of (2.13c).

NOTE 6.: Kozák (1980) assumes that

$$\delta\mathcal{H}_{kl} = \delta w_{(k;l)} \qquad \xi \in S_t \tag{3.7}$$

which, contrary to conditions (3.6a,b), involves the derivatives $\delta w_{k;3}$ taken along the normal to the surface S_t. It has also been shown — see the NOTE 11 in the paper by Kozák and Szeidl (1996b) — that equations (3.7) imply equations (3.6a,b) but the opposite statement is not true. Consequently condition (3.7) is less rigorous than conditions (3.6a,b).

NOTE 7.: Conditions (3.6a,b) are given in terms of $\delta w_l(\xi)$, therefore, as already mentioned they say nothing about $\delta w_{\lambda;3}$ on S_t. With regard to the assumed continuity of δH_{kl} on curve g which separates S_u and S_t one may write by using equations (3.7) that

$$\delta\mathcal{H}_{\kappa 3} = \frac{1}{2}(\delta w_{\lambda;3} + \delta w_{3;\lambda}) \qquad \xi \in g \tag{3.8a}$$

On the bases of the latter equation it will be assumed that $\delta H_{\lambda 3}$ can freely be varied on g or, what is the same thing, it can be given in terms of $\delta w_{\lambda;3}$ which is considered to be arbitrary.

NOTE 8.: With regard to conditions (3.6a,b) and NOTE 7 one has three functions, i.e., δw_l which can be varied freely on S_t and five functions, i.e., δw_l and $\delta w_{\lambda;3}$ which can be varied freely on g.

NOTE 9.: Without any loss of generality one may assume that

$$\delta H_{3;3} = \delta w_{3;3}\,. \qquad \xi \in g \tag{3.8b}$$

3.4 Upon substitution of condition (3.5) into variation (3.2) one obtains

$$-\delta K = I_1^V + I_1^{S_u} = \int_V \epsilon^{krm}\epsilon^{lsp}\delta\mathcal{H}_{rs;mp}\, e_{kl} dV - \int_{S_u} \epsilon^{3\eta\kappa}\epsilon^{ldp}\delta\mathcal{H}_{\eta d;p\kappa}\hat{u}_l\, dA = 0\,. \tag{3.9}$$

Since the surface integral in equation (3.9) coincides with the left hand side of equation (5.1) if in the latter $\hat{u}_l$, $\delta\mathcal{H}_{kl}$, S_u and g are substituted for u_l, $\mathcal{H}_{kl}$, S_o and g_o, respectively, one can write

$$I_1^{S_u} = I_2^{S_u} + I_1^G = -\int_{S_u} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{\hat{u}_{\lambda;\kappa}\delta\mathcal{H}_{\eta\vartheta;3} + [(\hat{u}_{3|\kappa})_{\|\lambda} + b_\lambda^\alpha \hat{u}_{\alpha;\kappa} + b_\beta^\beta \hat{u}_{(\lambda;\kappa)}]\delta\mathcal{H}_{\eta\vartheta}$$
$$+[(\hat{u}_{\lambda|\kappa})_{\|\vartheta} + \hat{u}_{3|\lambda} b_{\vartheta\kappa}]\delta\mathcal{H}_{\eta 3} + b_{\eta\vartheta}\hat{u}_{(\lambda;\kappa)}\delta\mathcal{H}_{33}\}\,dA$$
$$+\int_g n_3\epsilon^{\kappa\eta 3}\tau^\vartheta(\hat{u}_{\vartheta|\kappa}\delta\mathcal{H}_{\eta 3} - \hat{u}_{3|\kappa}\delta\mathcal{H}_{\eta\vartheta})\,ds - \int_g \tau^\eta \epsilon^{ldp}\delta\mathcal{H}_{\eta d;p}\hat{u}_l\,ds \tag{3.10}$$

for the surface S_u lies on the right hand side with respect to the positive direction chosen on curve g — see Figure 1(b).

By applying Gauss' theorem twice and renaming dummy indices one obtains for I_1^V that

$$I_1^V = I_2^V + I_1^S = \int_V \epsilon^{rkm}\epsilon^{slp} e_{kl;mp}\delta\mathcal{H}_{rs}dV + \int_S n_3\epsilon^{\kappa\rho 3}\epsilon^{lsp}(e_{l\kappa}\delta\mathcal{H}_{\rho s;p} - e_{l\kappa;p}\delta\mathcal{H}_{\rho s})dA \tag{3.11}$$

As regards the surface integral it is worth decomposing those sums involving ϵ^{lsp}. After some manipulations we have

$$I_1^S = I_1^{S_t} + I_3^{S_u} = \int_S n_3\epsilon^{\kappa\rho 3}\epsilon^{\lambda\vartheta 3}[e_{\lambda\kappa}\delta\mathcal{H}_{\rho\vartheta;3} - e_{\lambda\kappa}\delta\mathcal{H}_{\rho 3;\vartheta} - e_{3\kappa}\delta\mathcal{H}_{\rho\vartheta;\lambda}$$
$$-e_{\lambda\kappa;3}\delta\mathcal{H}_{\rho\vartheta} + e_{\lambda\kappa|\vartheta}\delta\mathcal{H}_{\rho 3} + e_{3\kappa|\lambda}\delta\mathcal{H}_{\rho\vartheta}]\,dA\,. \tag{3.12}$$

Integral $I_3^{S_u}$ can be transformed into a more suitable form by making use of equation (5.2). First the dummy indices ρ, ϑ in equation (5.2) should be renamed κ, λ and vice versa, i.e., κ, λ are to be renamed ρ, ϑ. Then the desired result can readily be achieved if one follows the way leading to equation (3.10), i.e., by substituting $\hat{u}_l$, $\delta\mathcal{H}_{kl}$, S_u and g for u_l, $\mathcal{H}_{kl}$, S_o and g_o and taking into consideration that the positive description on g is the one which leaves S_u on the right; the latter convention affects the sign of the line integrals. After renaming some dummy indices in the line integrals in order to factor out the vector τ^ϑ one has

$$I_3^{S_u} = I_4^{S_u} + I_2^G = \int_{S_u} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{\, e_{\lambda\kappa}\delta\mathcal{H}_{\rho\vartheta;3} + (e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta})\delta\mathcal{H}_{\eta 3}$$
$$+(e_{3\kappa\|\lambda} + b_\lambda^\alpha e_{\alpha\kappa} - e_{\kappa\lambda;3} + e_{\lambda 3|\kappa} - b_\beta^\beta e_{\lambda\kappa})\delta\mathcal{H}_{\eta\vartheta} + b_{\eta\vartheta}e_{\lambda\kappa}\delta\mathcal{H}_{33}\}\,dA$$
$$-\int_g n_3\epsilon^{\kappa\eta 3}\tau^\vartheta(\delta\mathcal{H}_{\vartheta\kappa}e^U_{\eta 3} - \delta\mathcal{H}_{3\kappa}e^U_{\eta\vartheta})ds \tag{3.13}$$

where U is not upper index but denotes the limit $\lim_{\xi\in S_u\to g} e_{kl}$.

Now we intend to manipulate integral $I_1^{S_t}$ into a form which makes possible the direct substitution of side conditions (3.6a,b,c). To achieve this goal transformation (5.3) should be applied. The procedure is the same as above except three things: (a) There is no need to rename dummy indices in the surface integral; (b) S_t should be substituted for S_o, and (c) because of that the sign of the line integrals will remain unchanged.

Finally we have

$$I_1^{S_t} = -\int_{S_t} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{\delta\mathcal{H}_{\lambda\kappa}e_{\eta\vartheta;3} + (\delta\mathcal{H}_{\kappa\lambda||\vartheta} + \delta\mathcal{H}_{\lambda\kappa||\vartheta})e_{\eta 3}$$
$$+(\delta\mathcal{H}_{3\kappa||\lambda} + b_\lambda^\alpha\delta\mathcal{H}_{\alpha\kappa} - \delta\mathcal{H}_{\kappa\lambda;3} + \delta\mathcal{H}_{\lambda 3;\kappa} - b_\beta^\beta\delta\mathcal{H}_{\lambda\kappa})e_{\eta\vartheta} + b_{\eta\vartheta}\delta\mathcal{H}_{\lambda\kappa}e_{33}\}\,dA$$
$$+\int_g n_3\epsilon^{\kappa\eta 3}(\tau^\vartheta\delta\mathcal{H}_{3\kappa}e^T_{\eta\vartheta} - \tau^\lambda\delta\mathcal{H}_{\lambda\kappa}e^T_{\eta 3})ds \tag{3.14}$$

from which by substituting from conditions (3.6a,b,c) and renaming dummy indices in the line integral it follows

$$I_1^{S_t} = I_2^{S_t} + I_3^G = -\int_{S_t} n_3\epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{\delta w_{(\lambda|\kappa)}e_{\eta\vartheta;3} + [(\delta w_{3|\kappa})_{\|\lambda} + b_\lambda^\alpha\delta w_{\alpha|\kappa} + b_\beta^\beta\delta w_{(\lambda|\kappa)}]e_{\eta\vartheta}$$
$$+[(\delta w_{\lambda|\kappa})_{\|\vartheta} + \delta w_{3|\lambda}b_{\vartheta\kappa}]e_{\eta 3} + b_{\eta\vartheta}\delta w_{(\lambda|\kappa)}e_{33}\}\,dA$$
$$-\int_g n_3\epsilon^{\kappa\eta 3}\tau^\vartheta(\delta\mathcal{H}_{3\kappa}e^T_{\eta 3} - \delta\mathcal{H}_{\lambda\kappa}e^T_{\eta\vartheta})ds \tag{3.15}$$

Here T stands for the limit $\lim_{\xi\in S_t\to g} e_{kl}$. In the next step integral $I_2^{S_t}$ will be transformed further into a final form in respect to the surface integral in the resulting equation.

It is worthy of mention that $I_2^{S_t}$ coincides with the surface integral on the right hand side of equation (5.1) if in the latter δw_l, e_{kl}, and S_t are substituted for u_l, H_{kl}, and S_o. In addition to this, special care should be taken with the line integrals on the boundary of S_t.

Recalling that the union of curves $\mathcal{L}_{11}$, $\mathcal{L}_{13}$, and $\mathcal{L}_2$ is $\mathcal{L}$ we should also remember that we go twice along $\mathcal{L}$ when we apply Stokes' theorem. Let the jump of δw_l be denoted by $[\delta w_l]$. Its definition is as follows

$$[\delta w_l] = \delta w_l^+ - \delta w_l^- \qquad \xi \in L \tag{3.16}$$

where the [positive] {negative} sign denotes the value of δw_l taken on the [positive] {negative} side of $\mathcal{L}$ – see Figure 1(b) for details. On the bases of all that has been said one has

$$\begin{aligned} I_2^{S_t} = I_3^{S_t} + I_1^{G\delta w} + I_1^{\mathcal{L}[w]} = &\int_{S_t} \epsilon^{3\eta\kappa}\epsilon^{ldp} e_{\eta d;p\kappa}\delta w_l \, dA \\ &+ \int_g n_3 \epsilon^{\kappa\eta 3}\tau^{\vartheta}(\delta w_{\vartheta|\kappa} e^T_{\eta 3} - \delta w_{3|\kappa} e^T_{\eta\vartheta}) \, ds - \int_g \tau^{\eta}\epsilon^{ldp} e^T_{\eta d;p}\delta w_l \, ds \\ &+ \int_{\mathcal{L}} n_3 \epsilon^{\kappa\eta 3}\tau^{\vartheta}\left([\delta w_{\vartheta|\kappa}] \, e^T_{\eta 3} - [\delta w_{3|\kappa}] \, e^T_{\eta\vartheta}\right) \, ds - \int_{\mathcal{L}} \tau^{\eta}\epsilon^{ldp} e^T_{\eta d;p} [\delta w_l] \, ds \end{aligned} \tag{3.17}$$

3.5 Comparing the left hand sides of equations (3.9), (3.11), (3.12), (3.13), (3.14) and (3.15) we may write

$$-\delta K = I_2^V + I_2^{S_u} + I_4^{S_u} + I_3^{S_t} + I_1^G + I_2^G + I_3^G + I_1^{G\delta w} + I_1^{\mathcal{L}[w]} = 0 \tag{3.18}$$

or after substituting the right hand sides

$$\begin{aligned} -\delta K \;=\; &\int_V \epsilon^{Rkm}\epsilon^{Slp} e_{kl;mp}\delta\mathcal{H}_{RS} dV \\ &+ \int_{S_u} n_3 \epsilon^{\kappa\eta 3}\epsilon^{\lambda\vartheta 3}\{(e_{\lambda\kappa} - \hat{u}_{(\lambda|\kappa)})\delta\mathcal{H}_{\eta\vartheta;3} \\ &+[(e_{3\kappa} - \hat{u}_{3;\kappa})_{\|\lambda} + b^{\alpha}_{\lambda}(e_{\alpha\kappa} - \hat{u}_{\alpha|\kappa}) - (e_{\kappa\lambda;3} - e_{\lambda 3;\kappa}) - b^{\beta}_{\beta}(e_{\lambda\kappa} - \hat{u}_{(\lambda|\kappa)})]\delta\mathcal{H}_{\eta\vartheta} \\ &+[e_{\kappa\lambda\|\vartheta} + e_{\lambda\kappa\|\vartheta} - (\hat{u}_{\lambda|\kappa})_{\|\vartheta} - \hat{u}_{3|\lambda} b_{\vartheta\kappa}]\delta\mathcal{H}_{\eta 3} - b_{\eta\vartheta}(e_{\lambda\kappa} - \hat{u}_{(\lambda|\kappa)})\delta\mathcal{H}_{33}\} \, dA \\ &+ \int_{S_t} \epsilon^{3\eta\kappa}\epsilon^{ldp} e_{\eta d;p\kappa}\delta w_l \, dA + I_I^G + I_1^{\mathcal{L}[w]} = 0 \end{aligned} \tag{3.19a}$$

where

$$I_I^G = I_1^G + I_2^G + I_3^G + I^{G\delta w} \tag{3.19b}$$

Since in equations (3.19a,b) no conditions for

$$\delta\mathcal{H}_{RS}, \ \delta\mathcal{H}_{\eta\vartheta;3}, \ \delta\mathcal{H}_{\eta\vartheta}, \ \delta\mathcal{H}_{\eta 3}, \ \delta\mathcal{H}_{33}, \text{ and } \delta w_l$$

are set down, they are arbitrary. Consequently from the vanishing of volume and surface integrals in equations (3.19a,b) there follows the fulfilment of differential equations (2.11) of compatibility, the fulfilment of compatibility boundary conditions (2.12) and the fulfilment of kinematic boundary conditions (2.13a,b).

In the latter case one has to recall that on one hand the condition (2.13c) is not independent of conditions (2.13a,b) and on the other hand the coefficient of $\delta\mathcal{H}_{33}$ coincides with conditions (2.13a,b). Among others this is one of the reasons why the corresponding stress functions $H_{\lambda 3}$ and H_{33} can be set to zero. It is worthy of mention that the Euler equation and natural boundary conditions obtained from the extremum condition $-\delta K = 0$ are those the strains e_{kl} should satisfy in order to be kinematically admissible on a single-connected volume region.

3.6 Analysis of line integrals involved in equation (3.19b) requires some preparations. Let the jump $[\delta w_b]$ be

$$[\delta w_b] \;=\; \delta \overset{(1i)}{c}_b + \epsilon_{svb}\,\delta \overset{(1i)}{C}{}^s [R^v - R^v(P_{1,i+1})] \qquad i = 1,3 \qquad \xi \in L_{1i} \tag{3.20}$$

$$[\delta w_b] \;=\; \delta \overset{(21)}{c}_b + \epsilon_{svb}\,\delta \overset{(21)}{C}{}^s [R^v - R^v(P_{21})] \qquad \xi \in L_2 \tag{3.21}$$

where $\delta \overset{(1i)}{c}_b$, $\delta \overset{(1i)}{C}{}^s$, $\delta \overset{(21)}{c}_b$ and $\epsilon_{svb}\,\delta \overset{(21)}{C}{}^s$ are arbitrary constants.

NOTE 10.: Function

$$[\delta w_b] = \delta c_b + \epsilon_{svb}\delta C^s[R^v - R^v(P)]\,, \qquad \xi \in S$$

in which δc_b and δC^s are arbitrary constants and P is an arbitrary but fixed point, causes no 'stress functions' since the gradient of $[\delta w_b]$ is skew-symmetric:

$$[\delta w_{\vartheta;\kappa}] = \epsilon_{s\kappa\vartheta}\delta C^s \qquad [\delta w_{3;\kappa}] = \epsilon_{\lambda\kappa 3}\delta C^\lambda \qquad \xi \in S \tag{3.22}$$

3.7 As regards the line integral I_I^G the necessary transformations are presented in the Appendix – see Section 5. Assuming continuity of strains on S it is plain that

$$e^U_{kl} = e^T_{kl} \qquad \xi \in g \tag{3.23}$$

By using the equation

$$\delta w_{(3|\lambda)} - \delta w_{[3|\lambda]} = \delta w_{(\lambda|3)} + \delta w_{[\lambda|3]} = \delta w_{\lambda|3} \qquad \xi \in g$$

and omitting the distinguishing U and T from (5.13) and (5.14) one obtains

$$\begin{aligned} I_I^G \;=\; & \int_g \{\tau^\eta \epsilon^{\vartheta\lambda 3}(e_{\vartheta\eta;3} - e_{3\eta|\vartheta}) - \frac{d}{ds}[\epsilon^{\vartheta\lambda 3}(e_{\vartheta 3} - \hat{u}_{3;\vartheta})]\}\delta w_\lambda ds \\ & + \int_g [\tau^\eta \epsilon^{3\rho\vartheta} e_{\vartheta\eta|\rho} - \frac{d}{ds}(\frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta;\lambda})]\delta w_3\, ds \\ & + \int_g \tau^\eta \epsilon^{\vartheta 3\lambda}(e_{\eta\vartheta} - \hat{u}_{(\vartheta;\eta)})\delta w_{\lambda|3}\, ds + \Sigma^1 + \Sigma^2 + \Sigma^3 \end{aligned} \tag{3.24}$$

Vanishing of the line integrals for arbitrary δw_λ, δw_3 and $\delta w_{\lambda|3}$ yields the continuity conditions

$$\tau^\eta \epsilon^{\vartheta\lambda 3}(e_{\vartheta\eta;3} - e_{3\eta|\vartheta}) = \frac{d}{ds}[\epsilon^{\vartheta\lambda 3}(e_{\vartheta 3} - \hat{u}_{3;\vartheta})] \qquad \xi \in g \tag{3.25a}$$

$$\tau^\eta \epsilon^{3\rho\vartheta} e_{\vartheta\eta|\rho} = \frac{d}{ds}(\frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta;\lambda}) \qquad \xi \in g \tag{3.25b}$$

$$\tau^\eta(e_{\eta\vartheta} - \hat{u}_{(\vartheta;\eta)}) = 0 \qquad \xi \in g \tag{3.25c}$$

If we had not omitted the distinguishing U and T then with respect to equations (2.18a,b) and (2.20) continuity conditions (3.25a,b) would assume the forms

$$(\frac{d\,\omega^\lambda}{ds})^T = (\frac{d\,\omega^\lambda}{ds})^U \qquad \text{and} \qquad (\frac{d\,\omega^3}{ds})^T = (\frac{d\,\omega^3}{ds})^U \qquad \xi \in g \tag{3.26}$$

NOTE 11.: Fulfilment of conditions (3.25a,b,c) is sufficient for the determination of ω^3, $\hat{u}_\vartheta$ and $\hat{u}_{3|\lambda}$ in terms of $e_{\vartheta\eta;3}$, $e_{3\eta|\vartheta}$, $e_{\vartheta\eta|\rho}$ and $e_{\eta\vartheta}$. After integrating equation (3.25b) we obtain ω^3. With ω^3 and equation (3.25c) it follows

$$\tau^\eta(e_{\eta\vartheta} - \hat{u}_{\vartheta;\eta} + \hat{u}_{[\vartheta|\eta]}) = 0 \qquad \xi \in g$$

or what is the same thing

$$\frac{d\hat{u}_\vartheta}{ds} = \tau^\eta(e_{\eta\vartheta} - \epsilon_{\vartheta\eta 3}\omega^3) \qquad \xi \in g$$

from which the displacements $\hat{u}_\vartheta$ are obtained immediately by integration with respect to s.

As regards the equation (3.25a), it can be solved for $\hat{u}_{3|\vartheta}$ by integration with respect to s. If $\hat{u}_{3|\vartheta}$ is known then

$$\frac{d\hat{u}_3}{ds} = \tau^\vartheta \hat{u}_{3|\vartheta} \qquad \xi \in g$$

from which $\hat{u}_3$ can be determined by integration.

3.8 The last part of the line integrals consists of those terms whose presence expresses that the volume region is triple connected. Substituting $I_1^{\mathcal{L}[w]}$, Σ^1, Σ^2, and Σ^3 from equations (3.17), (5.9),(5.11) and (5.12) and comparing equations (3.19a) and (3.24) we have

$$
\begin{aligned}
I_1^{\mathcal{L}[w]} &+ \Sigma^1 + \Sigma^2 + \Sigma^3 = \\
= \sum_{i=1,3} \Big\{ & \int_{\mathcal{L}_{1i}} n_3 \epsilon^{\kappa\eta 3} \tau^{\vartheta} \left([\delta w_{\vartheta|\kappa}] e^T_{\eta 3} - \left[\delta w_{3|\kappa}\right] e^T_{\eta\vartheta} \right) ds - \int_{L_{1i}} \tau^{\eta} \epsilon^{ldp} e^T_{\eta d;p} \left[\delta w_l\right] ds \\
& + \epsilon^{\vartheta 3\lambda} \left([\delta w_{(\lambda|3)}] - \left[\delta w_{\lambda|3}\right] \right) \hat{u}_{\vartheta} \big|_{P_{1i}} - \epsilon^{\vartheta 3\lambda} \left([\delta w_{(\lambda|3)}] - \left[\delta w_{\lambda|3}\right] \right) \hat{u}_{\vartheta} \big|_{P_{1,i+1}} - \\
& + \epsilon^{\lambda 3\vartheta} (e_{\vartheta 3} - \hat{u}_{3;\vartheta}) \left[\delta w_{\vartheta}\right] \big|_{P_{1i}} - \epsilon^{\lambda 3\vartheta} (e_{\vartheta 3} - \hat{u}_{3;\vartheta}) \left[\delta w_{\vartheta}\right] \big|_{P_{1,i+1}} \\
& + \frac{1}{2} \epsilon^{3\lambda\vartheta} \hat{u}_{\vartheta;\lambda} \left[\delta w_3\right] \Big|_{P_{1i}} - \frac{1}{2} \epsilon^{3\lambda\vartheta} \hat{u}_{\vartheta;\lambda} \left[\delta w_3\right] \Big|_{P_{1,i+1}} \Big\} \\
+ \int_{\mathcal{L}_2} & n_3 \epsilon^{\kappa\eta 3} \tau^{\vartheta} ([\delta w_{\vartheta;\kappa}] e^T_{\eta 3} - \left[\delta w_{3;\kappa}\right] e^T_{\eta\vartheta}) \, ds - \int_{L_2} \tau^{\eta} \epsilon^{ldp} e^T_{\eta d;p} \left[\delta w_l\right] \, ds.
\end{aligned}
$$

To obtain the supplementary conditions of single-valuedness let us substitute equations (3.20) and (3.21) then gather the coefficients of

$$
\delta \overset{(21)}{c}_k \mathbf{a}^k, \; \delta \overset{(21)}{C}{}^l \mathbf{a}_l \qquad \text{and} \qquad \delta \overset{(1i)}{c}_k \mathbf{a}^k, \; \delta \overset{(1i)}{C}{}^l \mathbf{a}_l \, .
$$

Upon a subsequent rearrangement we have

$$
\begin{aligned}
& I_1^{\mathcal{L}[w]} + \Sigma^1 + \Sigma^2 + \Sigma^3 = \\
= & \left[\oint_{\mathcal{L}_2} \tau^{\eta} \epsilon^{\rho\vartheta 3} e_{\eta\vartheta|\rho} \, \mathbf{a}_3 \, ds + \oint_{L_2} \tau^{\eta} \epsilon^{\lambda\vartheta 3} (e_{\eta 3|\vartheta} - e_{\eta\vartheta;3}) \, \mathbf{a}_{\lambda} \, ds \right] \cdot \mathbf{a}^k \, \delta \overset{(21)}{c}_k \\
& + \oint_{\mathcal{L}_2} \tau^{\eta} [e_{\eta k} + \epsilon_{kvl} \epsilon^{lpd} (R^v(s) - R^v(P_{21})) e_{\eta d;p}] \, \mathbf{a}^k \, ds \; \cdot \mathbf{a}_l \, \delta \overset{(21)}{C}{}^l \\
& \sum_{i=1,3} \left\{ \left[\frac{1}{2} \epsilon^{3\lambda\vartheta} \hat{u}_{\vartheta|\lambda} \, \mathbf{a}_3 + \epsilon^{\lambda 3\vartheta} (e_{\vartheta 3} - \hat{u}_{3|\vartheta}) \, \mathbf{a}_{\lambda} \right] \right|_{P_{1,i+1}}^{P_{1i}} \\
& + \int_{\mathcal{L}_{1i}} \tau^{\eta} \epsilon^{\rho\vartheta 3} e_{\eta\vartheta|\rho} \, \mathbf{a}_3 \, ds + \int_{L_{1i}} \tau^{\eta} \epsilon^{\lambda\vartheta 3} (e_{\eta 3|\vartheta} - e_{\eta\vartheta;3}) \, \mathbf{a}_{\lambda} \, ds \Big\} \cdot \mathbf{a}^k \, \delta \overset{(1i)}{c}_k \\
& \sum_{i=1,3} \Big[\hat{u}_k \, \mathbf{a}^k \big|_{P_{1,i+1}}^{P_{1i}} + \epsilon^{\lambda 3\vartheta} \, (e_{3\vartheta} - \hat{u}_{3|\vartheta}) \epsilon_{\lambda kl} \, (R^k(P_{1,i+1}) - R^k(P_{1i})) \, \mathbf{a}^l \big|_{P_{1i}} \\
& \qquad + \frac{1}{2} \epsilon^{3\lambda\vartheta} \hat{u}_{\vartheta|\lambda} \, \epsilon_{3\psi\rho} \, (R^{\psi}(P_{1,i+1}) - R^{\psi}(P_{1i})) \, \mathbf{a}^{\rho} \Big|_{P_{1i}} \\
& + \int_{\mathcal{L}_{1i}} \tau^{\eta} \{ e_{\eta k} + \epsilon^{lpd} \, \epsilon_{kvl} \, [R^v(s) - R^v(P_{1,i+1})] e_{\eta d;p} \} \, \mathbf{a}^k \, ds \Big] \cdot \mathbf{a}_l \delta \overset{(1i)}{C}{}^l \mathbf{a}_l
\end{aligned}
$$

Vanishing of the above expression for arbitrary $\delta \overset{(21)}{c}_k, \ldots, \delta \overset{(1i)}{C}{}^l$ results in the fulfillment of compatibility conditions (2.24) and (2.25) in the large and the supplementary conditions (2.28) and (2.30) of single valuedness.

4 Concluding Remarks

4.1 In accordance with the aims detailed in Paragraph 1.5 the present paper has studied the question as to what further conditions the strain fields should meet in addition to the usual ones in order to be kinematically admissible on multiply connected regions providing three dimensional and mixed boundary value problems. The mathematical form of the supplementary conditions of single-valuedness has been derived from a geometrical line of thought. Consequently the conditions are independent of the material law.

4.2 It has also been proved that both the compatibility conditions in the large and the supplementary conditions of single valuedness are natural boundary conditions that follow from the minimum complementary energy The significance of this conclusion is inherent in the applications. In other words application of direct methods — finite element method for instance — to find approximate solutions from the extremum condition do not require that the admissible fields should satisfy these conditions in advance.

5 Mathematical Transformations

5.1 Let u_l and H_{kl} be sufficiently smooth tensors on S_o. Comparing equations (A50,i) and (3.25,i) one can write

$$\begin{aligned}\int_{S_o} n_3\epsilon^{3\eta\kappa}\epsilon^{ldp}\mathcal{H}_{\eta d;p\kappa}u_l\,dA \;=\; & -\int_{S_o} n_3\epsilon^{\kappa\eta3}\epsilon^{\lambda\vartheta3}\{u_{(\lambda|\kappa)}\mathcal{H}_{\eta\vartheta;3}+[(u_{3|\kappa})_{\|\lambda}+b^{\alpha}_{\lambda}u_{\alpha|\kappa}+b^{\beta}_{\beta}u_{(\lambda|\kappa)}]\mathcal{H}_{\eta\vartheta}\\ & \qquad +[(u_{\lambda|\kappa})_{\|\vartheta}+u_{3|\lambda}b_{\vartheta\kappa}]\mathcal{H}_{\eta3}+b_{\eta\vartheta}u_{(\lambda|\kappa)}\mathcal{H}_{33}\}\,dA\\ & -\oint_{g_o} n_3\epsilon^{\kappa\eta3}\tau^{\vartheta}(u_{\vartheta|\kappa}\mathcal{H}_{\eta3}-u_{3|\kappa}H_{\eta\vartheta})\,ds+\oint_{g_o}\tau^{\eta}\epsilon^{ldp}\mathcal{H}_{\eta d;p}u_l\,ds\,. \qquad (5.1)\end{aligned}$$

A detailed proof of the above equation is presented in the paper by Kozák–Szeidl (1996a).

5.2 Using the above notations and comparing equations (A54,i) and (3.26a,i) one obtains

$$\begin{aligned}&\int_{S_o} n_3\epsilon^{\kappa\rho3}\epsilon^{\lambda\vartheta3}[-\mathcal{H}_{\lambda\kappa}e_{\rho\vartheta;3}+\mathcal{H}_{\lambda\kappa}e_{\rho3;\vartheta}+\mathcal{H}_{3\kappa}e_{\rho\vartheta;\lambda}+\mathcal{H}_{\lambda\kappa;3}e_{\rho\vartheta}-\mathcal{H}_{\lambda\kappa|\vartheta}e_{\rho3}-\mathcal{H}_{3\kappa|\lambda}e_{\rho\vartheta}]\,dA=\\ &=\int_{S_o} n_3\epsilon^{\kappa\eta3}\epsilon^{\lambda\vartheta3}\{\,e_{\lambda\kappa}\mathcal{H}_{\eta\vartheta;3}+(e_{\kappa\lambda\|\vartheta}+e_{\lambda\kappa\|\vartheta})\mathcal{H}_{\eta3}\\ &\qquad +(e_{3\kappa\|\lambda}+b^{\alpha}_{\lambda}e_{\alpha\kappa}-e_{\kappa\lambda;3}+e_{\lambda3|\kappa}-b^{\beta}_{\beta}e_{\lambda\kappa})\mathcal{H}_{\eta\vartheta}+b_{\eta\vartheta}e_{\lambda\kappa}\mathcal{H}_{33}\}\,dA\\ &+\oint_{g_o} n_3\epsilon^{\kappa\eta3}(\tau^{\vartheta}e_{3\kappa}\mathcal{H}_{\eta\vartheta}-\tau^{\lambda}e_{\lambda\kappa}\mathcal{H}_{\eta3})\,ds\,. \qquad (5.2)\end{aligned}$$

As regards the transformations leading to (A54,i) the reader is referred to Paragraph 5.18 in the paper by Kozák–Szeidl (1996a). By interchanging the letters e and H in (5.2) the sign of surface integral turns to the opposite. Keeping this in mind we may write

$$\begin{aligned}&\int_{S_o} n_3\epsilon^{\kappa\rho3}\epsilon^{\lambda\vartheta3}[-e_{\lambda\kappa}\mathcal{H}_{\rho\vartheta;3}+e_{\lambda\kappa}\mathcal{H}_{\rho3;\vartheta}+e_{3\kappa}\mathcal{H}_{\rho\vartheta;\lambda}+e_{\lambda\kappa;3}\mathcal{H}_{\rho\vartheta}-e_{\lambda\kappa|\vartheta}\mathcal{H}_{\rho3}-e_{3\kappa|\lambda}\mathcal{H}_{\rho\vartheta}]\,dA=\\ &=\int_{S_o} n_3\epsilon^{\kappa\eta3}\epsilon^{\lambda\vartheta3}\{\,\mathcal{H}_{\lambda\kappa}e_{\eta\vartheta;3}+(\mathcal{H}_{\kappa\lambda\|\vartheta}+\mathcal{H}_{\lambda\kappa\|\vartheta})e_{\eta3}\\ &\qquad +(\mathcal{H}_{3\kappa\|\lambda}+b^{\alpha}_{\lambda}\mathcal{H}_{\alpha\kappa}-\mathcal{H}_{\kappa\lambda;3}+\mathcal{H}_{\lambda3|\kappa}-b^{\beta}_{\beta}\mathcal{H}_{\lambda\kappa})e_{\eta\vartheta}+b_{\eta\vartheta}\mathcal{H}_{\lambda\kappa}e_{33}\}\,dA\\ &+\oint_{g_o} n_3\epsilon^{\kappa\eta3}(\tau^{\vartheta}\mathcal{H}_{3\kappa}e_{\eta\vartheta}-\tau^{\lambda}\mathcal{H}_{\lambda\kappa}e_{\eta3})\,ds. \qquad (5.3)\end{aligned}$$

5.3 Transformation of intgeral I^G_I.

By making use of equations (3.10), (3.13), (3.15), (3.17) and (3.19b) we may write

$$\begin{aligned}I^G_I &= I^G_1+I^G_2+I^G_3+I^{G\delta w}_1=\\ &=\int_g n_3\epsilon^{\kappa\eta3}\tau^{\vartheta}(\delta\mathcal{H}_{\vartheta\kappa}\hat{u}_{3|\eta}-\delta\mathcal{H}_{3\kappa}\hat{u}_{\vartheta|\eta})\,ds-\int_g\tau^{\eta}\epsilon^{ldp}\delta\mathcal{H}_{\eta d;p}\hat{u}_l\,ds\\ &-\int_g n_3\epsilon^{\kappa\eta3}\tau^{\vartheta}(\delta\mathcal{H}_{\vartheta\kappa}e^U_{\eta3}-\delta\mathcal{H}_{3\kappa}e^U_{\eta\vartheta})ds-\int_g n_3\epsilon^{\kappa\eta3}\tau^{\vartheta}(\delta\mathcal{H}_{\vartheta\kappa}e^T_{\eta3}-\delta\mathcal{H}_{3\kappa}e^T_{\eta\vartheta})ds\\ &+\int_g n_3\epsilon^{\kappa\eta3}\tau^{\vartheta}(\delta w_{\vartheta|\kappa}e^T_{\eta3}-\delta w_{3|\kappa}e^T_{\eta\vartheta})\,ds-\int_g\tau^{\eta}\epsilon^{ldp}e^T_{\eta d;p}\delta w_l\,ds\,. \qquad (5.4)\end{aligned}$$

The above equation clearly shows the structure of those line integrals resulted from the transformations if the volume region under consideration is simply-connected.

Now it is our aim to manipulate integral (5.4) into a more suitable form.

During the transformations we shall utilize, among others, the following equations:

$$\delta w_{[\vartheta|\kappa]} = -\epsilon_{\vartheta\kappa 3}\delta r^3 \qquad \delta w_{(\eta|\vartheta)} = \delta w_{\vartheta|\eta} + \epsilon_{\vartheta\eta 3}\delta r^3 \qquad \xi \in S_t \tag{5.5}$$

where δr^3 is the third component of the corresponding axial vector.

It follows from equation (3.6b) that

$$\delta\mathcal{H}_{\lambda\eta;3} - \delta\mathcal{H}_{3\eta;\lambda} = \delta\mathcal{H}_{3|\lambda||\eta} + b^\alpha_\eta \delta\mathcal{H}_{\alpha\lambda} - \delta w_{3|\lambda\|\eta} - b^\alpha_\eta \delta w_{\alpha|\lambda} \qquad \xi \in S_t$$

from which by adding $0 = b_{\lambda\eta}(\delta\mathcal{H}_{33} - \delta w_{3;3})$ to the right hand side – see (3.8b) — and taking into consideration the rule (A9,i) we obtain

$$\begin{aligned}\delta\mathcal{H}_{\lambda\eta;3} - \delta\mathcal{H}_{3\eta;\lambda} &= \delta\mathcal{H}_{3|\lambda||\eta} + b^\alpha_\eta \delta\mathcal{H}_{\alpha\lambda} - b_{\lambda\eta}\delta H_{33} - (\delta w_{3|\lambda\|\eta} + b^\alpha_\eta \delta w_{\alpha|\lambda} - b_{\lambda\eta}\delta w_{3;3}) \\ &= \delta H_{3\lambda|\eta} - \delta w_{3;\lambda\eta}\end{aligned} \tag{5.6}$$

In view of (3.6a) it is plain that

$$\epsilon^{3\lambda\vartheta}\delta\mathcal{H}_{\vartheta\eta;\lambda} = \epsilon^{3\lambda\vartheta}\frac{1}{2}(\delta w_{\vartheta;\lambda\eta} + \delta w_{\eta;\vartheta\lambda}) = \epsilon^{3\lambda\vartheta}\frac{1}{2}\delta w_{\vartheta;\lambda\eta}\,. \qquad \xi \in g \tag{5.7}$$

Care should be taken to the partial integrations carried out along the curves $i = 1, ..., 4$ since δw_l has a jump at the points P_{1i}.

Now we shall consider the line integral I^G_1 — see equation (3.1) or the first line of expression (5.4). After making use of the resolution

$$-\tau^\eta\epsilon^{ldp}\delta\mathcal{H}_{\eta d;p}\hat{u}_l = \tau^\eta\epsilon^{3\lambda\vartheta}\hat{u}_3 + \tau^\eta\epsilon^{\vartheta 3\lambda}(\delta\mathcal{H}_{\lambda\eta;3} - \delta\mathcal{H}_{3\eta|\lambda})\hat{u}_\vartheta \qquad \xi \in S_u \tag{5.8}$$

one can substitute equations (5.6) and (5.7). Then it becomes possible to carry out partial integrations along the curve g. Finally substitute (3.6a) and (3.8a) for $\delta\mathcal{H}_{\lambda\kappa}$ and $\delta\mathcal{H}_{3\lambda}$. Renaming some dummy indices where necessary we have

$$\begin{aligned}I^G_1 = I^G_5 + \Sigma^1 = &\int_g \tau^\eta\epsilon^{3\lambda\vartheta}\frac{1}{2}\delta w_{\lambda|\vartheta}\hat{u}_{3|\eta}ds - \int_g \tau^\eta\epsilon^{\vartheta 3\lambda}(\delta w_{(\lambda;3)} - \delta w_{3;\lambda})\hat{u}_{\vartheta|\eta}\,ds \\ &- \int_g \tau^\eta\epsilon^{\vartheta 3\lambda}\delta w_{\lambda;3}\hat{u}_{\vartheta|\eta}ds - \int_g \tau^\eta\epsilon^{3\kappa\eta}\delta w_{(\eta|\vartheta)}\hat{u}_{3|\kappa}\,ds \\ &+ \sum_{i=1,3}\left\{\epsilon^{\vartheta 3\lambda}(\delta w_{(\lambda;3)} - \delta w_{3|\lambda})\hat{u}_\vartheta\big|_{P_{1i}} - \epsilon^{\vartheta 3\lambda}(\delta w_{(\lambda;3)} - \delta w_{3|\lambda})\hat{u}_\vartheta\big|_{P_{1,i+1}}\right\}\end{aligned} \tag{5.9}$$

Observe that the terms denoted by Σ^1 are those one obtains for multiply-connected regions and mixed boundary value problems.

Now we turn our attention to the line integral $I^G_3 + I^{G\delta w}_1$ — see equations (3.15) and (3.17) or the last line of expression (5.4). Decomposing the sum $\tau^\eta\epsilon^{ldp}e_{\eta d;p}\delta w_l$ — see equation (5.8) for details — and substituting equations (3.6a), (3.8a) and (5.5) we find after some rearrangement that

$$\begin{aligned}I^G_3 + I^{G\delta w}_1 = &\int_g \tau^\eta\epsilon^{3\lambda\vartheta}e_{\vartheta\eta;\lambda}\delta w_3\,ds + \int \tau^\eta\epsilon^{\vartheta 3\lambda}(e^T_{\lambda\eta;3} - e^T_{3\eta|\lambda})\delta w_\vartheta\,ds \\ &+ \int_g \tau^\eta e^T_{3\eta}\delta r^3 ds + \int_g \tau^\eta\epsilon^{\kappa\eta 3}e^T_{\eta\vartheta}\delta w_{[\kappa;3]}\,ds\end{aligned} \tag{5.10}$$

A more suitable form of line integral I^G_I can be derived if one substitutes equations (5.9) and (5.10) for I^G_1 and $I^G_3 + I^{G\delta w}_1$ and then equations (3.6a) and (3.8a) for $\delta\mathcal{H}_{\lambda\kappa}$ and $\delta\mathcal{H}_{\lambda 3}$ in I^G_2 – see the third line in expression (5.4). Next substitute equation (5.5b) for $\delta w_{(\eta|\vartheta)}$ in the [last] {first} line integral in [I^G_1 (5.9)] {I^G_2}. After some rearrangement it becomes possible to utilize the transformation

$$\int_g \tau^\eta\epsilon^{\kappa\vartheta 3}(e^U_{\kappa 3} - \hat{u}_{3|\kappa})\delta w_{\vartheta|\eta}\,ds = -\int_g \frac{d}{ds}[\epsilon^{\kappa\vartheta 3}(e^U_{\kappa 3} - \hat{u}_{3|\kappa})]\delta w_\vartheta\,ds + \Sigma^2$$

where

$$\Sigma^2 = \sum_{i=1,3}\left\{\epsilon^{\kappa\vartheta 3}(e^U_{\kappa 3} - \hat{u}_{3|\kappa})\,[\delta w_\vartheta]\big|_{P_{1i}} - \epsilon^{\kappa\vartheta 3}(e^U_{\kappa 3} - \hat{u}_{3|\kappa})\,[\delta w_\vartheta]\big|_{P_{1,i+1}}\right\} \tag{5.11}$$

If in addition to this the right hand side of I_I^G is 'enlarged' by

$$0 = -\int_g \frac{d}{ds}\left\{\frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta|\lambda}\right\}\delta w_3\, ds + \int_g \tau^\eta \frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta|\lambda}\delta w_{3|\eta}\, ds + \Sigma^3$$

in which

$$\Sigma^3 = \sum_{i=1,3}\left\{ \frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta|\lambda}\left[\delta w_3\right]\Big|_{P_{1,i}} - \frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta|\lambda}\left[\delta w_3\right]\Big|_{P_{1,i+1}}\right\} \tag{5.12}$$

then, upon a subsequent rearrangement, we have

$$\begin{aligned} I_I^G &= I_5^G + I_2^G + I_3^G + I_1^{G\delta w} + \Sigma^1 + \Sigma^2 + \Sigma^3 = \\ &= \int_g \{\tau^\eta \epsilon^{\vartheta\lambda 3}(e_{\vartheta\eta;3} - e_{3\eta|\vartheta}) - \frac{d}{ds}[\epsilon^{\vartheta\lambda 3}(e_{\vartheta 3} - \hat{u}_{3|\vartheta})]\}\delta w_\lambda ds \\ &\quad + \int_g [\tau^\eta \epsilon^{3\rho\vartheta} e_{\vartheta\eta|\rho} - \frac{d}{ds}(\frac{1}{2}\epsilon^{3\lambda\vartheta}\hat{u}_{\vartheta|\lambda})]\delta w_3\, ds \\ &\quad + \int_g (e^U_{\eta 3} - e^T_{\eta 3})\delta r^3 \tau^\eta ds + I_6^G + \Sigma^1 + \Sigma^2 + \Sigma^3 \end{aligned} \tag{5.13}$$

In the above equation

$$\begin{aligned} I_6^G &= -\int_g \tau^\eta \epsilon^{\vartheta 3\lambda} 2\hat{u}_{(\eta|\vartheta)}\delta w_{(\lambda|3)} ds - \int_g \tau^\eta \epsilon^{\vartheta 3\lambda} e^T_{(\eta|\vartheta)}\delta w_{[3|\lambda]} ds \\ &\quad + \int_g \tau^\eta \epsilon^{\vartheta 3\lambda}(\hat{u}_{(\vartheta|\eta)} - \hat{u}_{[\vartheta|\eta]})\delta w_{3|\lambda} ds \\ &\quad + \int_g \tau^\eta \epsilon^{\vartheta 3\lambda}\hat{u}_{[\eta|\vartheta]}\delta w_{[3|\lambda]} ds + \int_g \tau^\eta \epsilon^{\vartheta 3\lambda} e^U_{(\eta|\vartheta)}\delta w_{(3|\lambda)} ds = \\ &= \int_g \tau^\eta \epsilon^{\vartheta 3\lambda}(e^U_{(\eta|\vartheta)} - \hat{u}_{(\eta|\vartheta)})\delta w_{(3|\lambda)} ds - \int_g \tau^\eta \epsilon^{\vartheta 3\lambda}(e^T_{(\eta|\vartheta)} - \hat{u}_{(\eta|\vartheta)})\delta w_{[3|\lambda]} ds \end{aligned} \tag{5.14}$$

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