On Compatibility Conditions for Mixed Boundary Value Problems

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The present paper focuses on the conditions of single-valuedness for the displacement field on multiply-connected bodies. It has been shown for a class of mixed boundary conditions that all the macro conditions of compatibility, i.e., the compatibility conditions in the large and the supplementary conditions of single-valuedness are also natural boundary conditions of the principle of minimum complementary energy as a variational principle.

1 Introduction

1.1 As regards the classical case it was Southwell (1936,1938) who first derived the compatibility conditions from the principle of minimum complementary energy as a variational principle. At the same time he pointed out – he utilized Maxwell’s (1870) and Morera’s (1892) solutions – that only three of the six Saint-Venant compatibility conditions follow from the principle of minimum complementary energy. Since any stress condition can be given in terms of three stress functions chosen appropriately he arrived at a contradiction because for the displacements to be single-valued all the six Saint-Venant compatibility conditions should be satisfied. This contradiction was named Southwell’s paradox. After Southwell’s papers the following problems remained unresolved: Is it sufficient for the strains to satisfy three Saint-Venant compatibility equations? If yes, which three? If yes, are there further conditions to satisfy?

1.2 A detailed description of the paradox grounded on all the possible stress function combinations is provided by Stickforth (1965) who noticed by generalizing a partial result of Washizu (1957) that the compatible strains satisfy some boundary conditions referred to as boundary conditions of compatibility. The latter term was proposed by Kozák in 1980, who pointed out in three distinct ways, i.e., in a purely mathematical one (1980b) and by utilizing the principle of minimum complementary energy as well as the dual forms of principle of virtual work (1980a,c) that for the strains to be compatible it is necessary and sufficient that three differential equation of compatibility and the boundary conditions of compatibility are fulfilled. The latter two papers are the only ones in which mixed boundary value problems are considered. It is assumed in all the papers mentioned above that the body is single-connected.

1.3 For doubly-connected domain and plane problems Prager (1946) pointed out assuming traction problems and homogeneous, isotropic material that Mitchell’s conditions (1900) (cf. e.g. Gurtin, 1972), or what is the same thing, the compatibility conditions in the large regarded under the conditions mentioned are natural boundary conditions of Castigliano’s principle. This result was generalized for mixed boundary value problems by Haichang (1986) and independently by Szeidl and Gemert (1991). Returning to the three dimensional case the papers by Moriguti (1948) and independently by Stickforth (1964) are concerned with three dimensional tractions problems on multiply connected regions but they provide no solution to Southwell’s paradox.

By macro conditions of compatibility is meant the totality of those additional conditions the strains should meet to be compatible on a multiply connected body. Depending on what the boundary conditions are in the points of a simply-connected and closed curve on the surface of the body the macro conditions of compatibility are separated into two groups. If tractions are imposed in each point of the curve the condition is referred to as a compatibility condition in the large. If there exists at least one arc on the curve along which displacements are imposed then the corresponding condition is called supplementary condition of single-valuedness.

1.4 In view of the foregoing it seems to be an open question what supplementary conditions of single-valuedness are needed for mixed boundary value problems on multiply connected regions. On the bases of all that has been said the present paper is aimed at investigating the problem of what natural boundary
conditions follow from the principle of minimum complementary energy assuming a three-dimensional and multiply-connected body and a certain class of mixed boundary value problems. We shall also assume that the linear theory of deformations is valid. When applying Castigliano’s principle in addition it will be assumed that the material of the body is linearly elastic.

1.5 In section 2 we collect some preliminary results and derive the supplementary conditions of single-valuedness from geometrical considerations concentrating attention on the classical case. Section 3 is devoted to the problem of how the supplementary conditions of single valuedness can be obtained from the principle of minimum complementary energy. Section 4 is a summary of the results. Finally there is section 5 where some fundamental mathematical relations being used in this paper and some longer transformations are presented.

2 Derivation of the Supplementary Conditions from Geometrical Considerations

2.1 The bounded region of the three-dimensional space occupied by the multiply connected body and the surface of the body are denoted respectively by \( V \) and \( S \). In principle the surface \( S \) of the body may consist of not only one but more closed surfaces, in which case the region is multiply-bordered, though the last circumstance will play no role in the investigations. The surface \( S \) is divided into parts \( S_u \) and \( S_t \) whose common bounding curve is denoted by \( g \).

The present paper restricts its attention to the triple-connected but single-bordered body represented in Figure 1(a) which contains some further notational conventions. It is clear from Figure 1(a) that both the subsurfaces \( S_u \), \( S_t \) and the curve \( g \) consists of more parts, i.e.,

\[
S_u = S_u^{(1)} \cup S_u^{(2)}; \quad S_t = S_t^{(1)} \cup S_t^{(2)} \cup S_t^{(3)} \quad \text{and} \quad g = g^{(1,0)} \cup g^{(1,1)} \cup g^{(1,2)} \cup g^{(1,3)} \cup g^{(1,4)}
\]

In a limit case any of the subsurfaces

\[
S_u^{(1)}, S_u^{(2)}, \quad \text{and} \quad S_u \quad \text{or} \quad S_t^{(1)}, S_t^{(1)} \quad \text{and} \quad S_t
\]
may be an empty set.

The non-intersecting, simple and closed curves \( L_1 \) and \( L_2 \) encircle the first and second holes.

It is essential for the further investigations that \( S_u \) and \( S_u \) are respectively triple and double connected surfaces in the way they are represented in Figure 1(b). The curves \( L_1 \) and \( L_2 \) intersect the curve \( g \) and \( g^* \) in the points

\[
P_{11}, P_{12}, P_{13}, P_{14} \quad \text{and} \quad P_{21}
\]

Let the parts of \( L_1 \) be defined by

\[
L_{1j} = P_{1j} P_{1,j+1} \quad j = 1, \ldots, 4.
\]

When performing integral transformations by making use of Stokes’ theorem one must keep in mind that the theorem is applicable under the condition that the surface considered is simple-connected. Figure 1(b) represents a possibility for cutting up the surface \( S \) into simple-connected parts by utilizing the curves \( g, L_1, L_2 \) and \( g^* \).

It is worthy of mention that the restrictions we have made in connection with the body considered are not essential for the body is at least triple-connected and the partition of the boundary surface \( S \) is sufficiently general.

2.2 Indicial notations and two coordinate systems: the \((x^1, x^2, x^3)\) curvilinear and the \((\xi^1, \xi^2, \xi^3)\) curvilinear (defined on the surface \( S \)) are employed throughout this paper. Scalars and tensors, unless the opposite is stated are denoted independently of the coordinate system by the same letter. Distinction is aided by the indication of the arguments \( x \) and \( \xi \) used to denote the totality of the corresponding coordinates.

When referring to some formulas of the paper by Kozák and Szeidl (1996a) the equation number will be followed by the roman number \( i \).

Volume and surface integrals are considered, respectively, in the coordinate systems \((x^1, x^2, x^3)\) and \((\xi^1, \xi^2, \xi^3)\). Consequently, in the case of integrals, arguments are omitted.

In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a semicolon denote covariant differentiation with respect to the corresponding subscripts. Latin and Greek indices range over the integers 1, 2, 3 and 1, 2, respectively. \( \epsilon^{kln} \) and \( \epsilon_{pqrs} \) stand for the permutation tensors; \( \delta^i_k \) is the Kronecker delta. In the system of coordinates \((x^1, x^2, x^3)\) \( g_k^l \) and \( g^k_l \) are the covariant and contravariant base vectors. The corresponding metric tensors are denoted by \( g_{kl} \) and \( g^{kl} \).

A pair of subscripts is enclosed by parentheses ( ) to indicate the symmetric part of a tensor of order two and by brackets [ ] to indicate the skew part. Covariant derivative is denoted by a Latin subscript preceded by a semicolon – see (A.4,i) for details.

2.3 Calculations carried out on the surface \( S \) can be better understood by introducing a suitable surface oriented coordinate system. Let \( x^k = x^k(\xi^1, \xi^2) \) be the equation of the surface \( S \) where \( \xi^1 \) and \( \xi^2 \) are the surface coordinates. Let \( \xi^3 \) be the perpendicular distance measured on the outward unit normal \( n \) to the surface \( S \). On \( S \) \( \xi^3 = 0 \). [Base vectors] [Metric tensors] on \( S \) are denoted by \([a_k^l \text{ and } a_k] \) \([a_{kl} \text{ and } a^{kl}] \).

In the surface oriented coordinate system \((\xi^1, \xi^2, \xi^3)\)

\[
n = a_3 = a^3, \quad n^3 = 1 \quad \text{and} \quad n^i = 0.
\]

(2.1)

If \( |\xi^1|/(\min\{|R_1|, |R_2|\}) < 1 \) in which \( R_1 \) and \( R_2 \) are the principal radii of curvature on \( S \) then the relationship \( x^k = x^k(\xi^1, \xi^2, \xi^3) \) is always one-to-one. The tensor of curvature is denoted by \( b_{\alpha\beta} \) – see (A.6,i). [Covariant derivative on the surface \( S \)] [The surface covariant derivative] is denoted by a Greek subscript preceded by \([\text{a semicolon (or one short vertical line) – see (A.8,i)]} \) \( \{\text{two short vertical lines – see (A.9,i)}\} \).

It will be assumed that the vector and tensor fields involved in the investigations are sufficiently smooth.

2.4 Now we shall assemble equations of elastostatics – in a form suited to our objective – in primal and dual systems as well. In accordance with this, paragraphs 2.5 and 2.6 provide a brief overview of the equations of the primal system while the following part of the present section turns its attention to the equations and boundary conditions of the dual system including the missing conditions of single-valuedness.
Let $u_{k}$, $e_{kl}$ and $t^{kl}$ be the displacement field, strain tensor and stress tensor (or displacements, strains and stresses for short). Body forces will be denoted by $\dot{b}$.

2.5 In the primal system the three-dimensional problems under consideration are governed by the kinematic equations

$$e_{kl} = \frac{1}{2}(u_{k;l} + u_{l;k}), \quad x \in V$$

(2.2)

Hook’s law

$$t^{kl} = C^{kldr} e_{rs}, \quad x \in V$$

(2.3)

and the equilibrium equations

$$t^{kl}_{;k} + \dot{b} = 0 \quad x \in V$$

(2.4)

where $C^{kldr}$ is the tensor of elastic coefficients.

Field equations (2.2), (2.3) and (2.4) are to be supplemented by boundary conditions. It will be assumed that [tractions] {displacements} are prescribed on $[S_t] \{S_u\}$. Then

$$u_t = \hat{u}_t \quad \xi \in S_u$$

(2.5)

is the displacement boundary condition and

$$n_3 t^3 = \hat{b} \quad \xi \in S_t$$

(2.6)

is the traction boundary condition in which $\hat{u}_t$ and $\hat{b}$ are the pre-assigned displacement and traction components.

The strains $e_{kl}$ are said to be [compatible] {kinematically admissible} if the kinematic equations (2.2) have a sufficiently smooth solution to the displacements $u_t$ and the solution [meets no other conditions] {satisfies the displacement boundary condition (2.5)}.

Stresses $t^{kl}$ are said to be [equilibrated] {statically admissible} if they satisfy the equilibrium equation (2.4) and [meet no other conditions] {and the traction boundary condition (2.6)}.

Every solution of equilibrium equations (2.2) admits the following representation found in its final form by Schaefer (1953) (cf. e.g. Gurtin, 1972):

$$t^{kl} = \varepsilon^{pgk} \varepsilon^{ldr} \tilde{H}_{ydk;lr} + g^{pk} B_{;q}^{kl} + g^{ql} B_{;pq}^{k} - g^{pl} B_{;pk}^{l} \quad x \in V$$

(2.7)

where $\tilde{H}_{ydl} = \tilde{H}_{ydk}$ is the stress function tensor (whose components will also be referred to as stress functions) and the vector field $B^l$ is given by (2.8), i.e., it is a particular solution of the Poisson equation (2.7). For this reason we shall assume that the vector field $B^l(x)$ is known.

Because of its role in the further investigations an important property of the above stress representations should be mentioned here. To begin with one has to introduce some notations.

The index pairs which range over a subset of the nine possible values will be capitalized.

Let $\alpha_{ab}$ be a sufficiently smooth otherwise arbitrary symmetric tensor field in $V$. Furthermore let $w_l(x)$ be an unknown vector field on $V$. By $\alpha_{AB}$ we denote those subsets of the possible values of the index pairs $ab$ for which the differential equation

$$\frac{1}{2}(w_{A:B} + w_{B:A}) = \alpha_{AB}(x) \quad x \in V$$

always have a solution for the vector field $w_l(x)$. It is obvious that the index pairs $\alpha_{AB}$ may have only three different values.

Let $RS$ be the supplementary subset of the index pairs whose union with $\alpha_{AB}$ is the set of index pairs $ab$. Obviously, the index pairs $\alpha_{RS}$ may have six distinct values. Because of the symmetry, however, the corresponding tensor components $\alpha_{RS}$ represent three distinct functions only.

The stress functions

$$\mathcal{H}_{ab} = \tilde{H}_{ab} + w_{(a;b)} \quad x \in V$$

(2.8)

lead to the same stress state as $\tilde{H}_{ab}$ since

$$\varepsilon^{pgk} \varepsilon^{ldr} w_{(y;dl);kr} \equiv 0 \quad x \in V$$
It immediately follows from this that by solving the differential equations

\[ w_{(A,B)} = -\bar{H}_{ab} \quad x \in V \]  

for \( w(x) \) and substituting the result into equations (2.8) we find that the three stress functions identified by the indices \( AB \) will vanish. In other words: any stress state can be given in terms of the stress functions \( \mathcal{H}_{RS} \), or what is the same thing, in terms of three stress functions. This result is due to Finzi (1934).

**NOTE 1.** In the sequel it will be assumed, that \( \mathcal{H}_{AB} = 0 \quad x \in V \) independently of the circumstance what subscripts are employed, i.e., capitalized or not.

2.6 By inverting Hook’s law we obtain the equation

\[ e_{kl} = \frac{-1}{C} C_{klrs} \varepsilon^r \quad x \in V \]  

(2.10)

in which the fourth order tensor \( \frac{-1}{C} C_{klrs} \) follows from the condition \( \frac{-1}{C} C_{klrs} C^{rspq} = \delta^p_k \delta^q_l \). The strains \( e_{kl} \) are said to be [equilibrated] (statistically admissible) if they are calculated from Hook’s law (2.10) by substituting [equilibrated] (statistically admissible) stresses \( \varepsilon^r \).

The tensor of incompatibility \( \mathcal{Y}^{ab} \) is defined by the equation

\[ \mathcal{Y}^{ab} = \varepsilon^{akm} \varepsilon^{blp} e_{klmp} \quad x \in V \]

The following results are those of Kozák (1980a,b).

For strains \( e_{kl} \) to be [compatible] (kinematically admissible) in a single connected region \( V \) it is necessary and sufficient that the differential equations of compatibility

\[ \mathcal{Y}^{RS} = \varepsilon^{Rkm} \varepsilon^{Spq} e_{klmp} = 0, \quad x \in V \]  

(2.11)

the boundary conditions of compatibility

\[ n_{\alpha} \mathcal{Y}^{ab} = n_{\alpha} \mathcal{Y}^{3b} = n_{\alpha} \varepsilon^{3km} \varepsilon^{dp} e_{klmp} = n_{\alpha} \varepsilon^{3pq} \varepsilon^{dp} e_{pqpc} = 0 \]  

(2.12)

and [no further conditions] (the kinematic boundary conditions)

\[ e_{\lambda \kappa} - \hat{u}_{(\lambda;\kappa)} = 0 \quad \xi \in S \]  

(2.13a)

\[ (e_{3\kappa} - u_{3\kappa})_{(\lambda)} + b_{(\lambda)}^N (e_{\lambda \kappa} - u_{\lambda \kappa}) - (e_{\lambda \kappa 3} - e_{3 \lambda \kappa}) = 0 \]  

(2.13b)

should be fulfilled. From equations (2.13a,b) there follows the fulfillment of equation

\[ e_{\lambda \kappa}(\partial + e_{\lambda \kappa}(\partial - (u_{\lambda \kappa})_{(\lambda)} - u_{3\kappa} b_{\theta \kappa} = 0 \]  

(2.13c)

For proof see the paragraphs 3.9 and 5.16 in the paper by Kozák–Szeidl (1996a).

It can also be shown that the fulfillment of kinematic boundary conditions (2.13a,b) on \( S \) automatically assures the fulfillment of boundary conditions of compatibility – see the paragraphs 3.7, 5.14 and 5.14 in the paper cited above.

2.7 For the equilibrated stresses

\[ t^{pl} = \varepsilon^{pjk} \varepsilon^{lqr} \mathcal{H}_{yqdkr} + g^{pq} B^l_{rq} + g^{lq} B^p_{rq} - g^{pl} B^k_{rk} \quad x \in V \]  

(2.14)

obtained from (2.7) by substituting \( \mathcal{H}_{yd} \) for \( \bar{H}_{yd} \) to be statically admissible it is sufficient if the traction boundary condition

\[ t^l = n_p (\varepsilon^{pjk} \varepsilon^{lqr} \mathcal{H}_{yqdkr} + g^{pq} B^l_{rq} + g^{lq} B^p_{rq} - g^{pl} B^k_{rk}) \]

\[ = n_p (\varepsilon^{pjk} \varepsilon^{lqr} \mathcal{H}_{yqdkr} + a^{pq} B^l_{rq} + a^{lq} B^p_{rq} - a^{pl} B^k_{rk}) \]  

(2.15)

is fulfilled.

2.8 For single-connected bodies the three dimensional problems of elasticity in the dual system are governed by the dual kinematic equation (2.14), the constitutive equation (2.10) and the dual balance equation (2.11).
Observe that we have as many unknowns — the dual configuration variable $H_{RS}$, the first intermediate variable $t^{kl}$ and the second intermediate variable $e_{kl}$ — as there are field equations, viz. (2.14), (2.10) and (2.11); both numbers are 15.

Field equations (2.14), (2.10) and (2.11) are associated with the boundary conditions (2.12) of compatibility and the traction boundary condition (2.15) on $S_t$ and with the strain boundary conditions (2.13a,b) on $S_u$.

The terminology used in the present paragraph was proposed by Tonti (1972).

\[ \tau^\alpha a_\alpha = \frac{dR}{ds} = \frac{d(R(s) - R(P))}{ds} = \frac{\partial R}{\partial \xi^\alpha} \frac{d\xi^\alpha}{ds} a_\alpha \quad s \in L_1 \] (2.16)

With the rotation field
\[ \omega^l = \frac{1}{2} t^{pd} u_{dp} \quad x \in V \] (2.17)

and equations (2.2) one can write
\[ \omega^3 = \frac{1}{2} e^{3\theta} u_{\partial 3} \quad \xi \in S \] (2.18a)

and
\[ \omega^\lambda = e^{3\lambda \theta} \left( \frac{1}{2}(u_{\partial 3} + u_{3\partial}) - u_{3\partial} \right) = e^{3\lambda \theta} (e_{\partial 3} - u_{3\partial}) \quad \xi \in S \] (2.18b)

NOTE 2: Assume for clarity that $u_k$ is a displacement field given on $S$. Equation (2.18a) reflects that $u_k(\xi)$ uniquely determines the rotation $\omega^3$. On the contrary equation (2.18b) involves the derivative $u_{\lambda 3}$ taken along the normal to the surface $S$ consequently $\omega^\lambda$ is only partly determined by $u_k(\xi)$.

2.10 By making use of the kinematic equation (2.2) from (2.17) it follows
\[ \omega_{;y}^l = \frac{1}{2} t^{pd} (u_{dy} + u_{dy}) = t^{pd} c_{ydp} \quad x \in V \] (2.19)

which, with regard to (2.16) and the notations of Figure 3. leads to
\[ \frac{\partial \omega^3 a_\alpha}{\partial s} = \tau^\alpha e^{3\theta} c_{ydp} a_t \quad \xi \in S \] (2.20)

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and consequently

\[(\omega^\lambda a_\lambda + \omega^2 a_3)|_P = (\omega^\lambda a_\lambda + \omega^2 a_3)|_{P_{11}} + \int_{P_{11}}^P \tau^\eta e^{ipd} e_{\eta p}\, a_\eta \, ds.\]  

(2.21)

A comparison of equations (2.2) and (2.17) yields

\[u_{k,\eta} = \epsilon_{k\eta} - \epsilon_{k\gamma} \omega^\gamma \quad \gamma \in V\]

from which utilizing again the notations of Figure 3, and substituting the equations (2.16) and (2.19) one obtains

\[\frac{\partial u_k a^l}{\partial s} = \frac{\partial}{\partial s} \left\{(R^\nu(s) - R^\nu(P)) \epsilon_{kl}\omega^\nu a^k(s)\right\} + [\tau^\eta e_{\eta k} + \tau^\eta \epsilon_{krt} e^{ipd}(R^\nu(s) - R^\nu(P)) a^k(s) \right\} \quad s \in \mathcal{L}_1 \]  

(2.22)

or after integration

\[u_k a^k|_P = u_k a^k|_{P_{11}} + \epsilon_{krt} \omega^r (R^\nu(s) - R^\nu(P)) a^k|_{P_{11}} + \int_{P_{11}}^P \tau^\eta e_{\eta k} + \epsilon_{krt} \omega^r (R^\nu(s) - R^\nu(P)) e_{\eta p} a^k ds \]

(2.23)

Equations (2.21) and (2.23) coincide with Cesaro’s formulae. Outline of their derivation is presented herein for the sake of those further transformations in which same partial results of the derivation will be utilized.

![Figure 3](image)

Figure 3.

Returning to the three-dimensional elasticity problem considered on the triple-connected region \(V\) — see Figure 1 — for the strains \(\epsilon_{kl}\) to be compatible it is also necessary, in addition to the fulfillment of field equation (2.11), that the rotation field \(\omega^\nu\) and the displacement \(u_k\) should be single-valued on all closed simple curves which encircle the two holes. It can readily be shown that the fulfillment of the supplementary conditions of single-valuedness on one-one single curve encircling the holes — one can chose \(\mathcal{L}_1\) and \(\mathcal{L}_2\) for instance — is sufficient for the rotation \(\omega^\nu\) and the displacements \(u_k\) to be single-valued along all the other curves that encircle the holes provided that the differential equations of compatibility hold.

In the light of this circumstance we confine ourselves to the curves \(\mathcal{L}_1\) and \(\mathcal{L}_2\) during further investigations. Observing that \(\mathcal{L}_2\) lies inside \(S_t\) for the single-valuedness of the rotation \(\omega^\nu\) there follows from equation (2.21) the compatibility conditions in the large

\[\int_{\mathcal{L}_2} \tau^\eta e^{ipd} e_{\eta p} a_\eta \, ds = \int_{\mathcal{L}_2} \tau^\eta e^{ip\omega_3} e_{\eta p} a_3 \, ds + \int_{\mathcal{L}_2} \tau^\eta e^{ip\omega_3} (e_{\eta 3|0} - e_{\eta 0|3}) a_\lambda \, ds = 0 \]  

(2.24)

Using exactly analogous reasoning from equation (2.23) one obtains the compatibility condition in the large

\[\int_{\mathcal{L}_2} \tau^\eta e_{\eta k} + \epsilon_{krt} \omega^r (R^\nu(s) - R^\nu(P_{21})) e_{\eta p} a^k \, ds = 0 \]

(2.25)

for the displacements \(u_k\) to be single-valued along \(\mathcal{L}_2\).

As regards the curve \(\mathcal{L}_1\) its parts \([\mathcal{L}_{11}\) and \(\mathcal{L}_{13}\) \{\mathcal{L}_{12}\ and \(\mathcal{L}_{13}\)\} lie wholly in \(S_t\) \{in \(S_n\)\}. In view of this fact the problem raised in paragraph 1.4 can be reworded in the following way: What is the effect on the compatibility conditions in the large of the circumstance that any simple closed curve encircling the first hole on \(S\) passes through the boundary part \(S_n\) twice. Solution to the problem posed is sought by means of a geometrical line of thought. Assume that the curve \(\mathcal{L}_1\) is divided into two parts as shown
in Figure 3. On arc \( L_{ij} \) \((i = 1, 3)\) drawn in heavy line the tractions are given while displacements are prescribed on the parts of arc \( P_{i,j+1}, P_{i,j} \) lying in the neighborhood of the endpoints. For the rotation \( \omega^j \) to be continuous on the closed curve \( L_1 \) being divided into two parts in two ways it is necessary that the continuity conditions

\[
\omega^j a_i|_{P_{i,1}} + \omega^j a_i|_{P_{i,1}} = 0 \quad \text{and} \quad \omega^j a_i|_{P_{i,1}} + \omega^j a_i|_{P_{i,1}} = 0 \quad (2.26)
\]

be satisfied. The first term in the left hand sides can be calculated from equation (2.21) by an appropriate change of limits. As regards the second term one has to utilize equations (2.18a,b) by taking into consideration the circumstance that the displacements are known if

\[
s \in (s(P_{ij}) - \epsilon, s(P_{ij})) \quad \text{or} \quad s \in (s(P_{i,j+1}), s(P_{i,j+1}) + \epsilon) \quad j = 1, 3 \quad (2.27)
\]

in which \( \epsilon \) is a small positive otherwise arbitrary quantity. In this way one obtains

\[
\left[ \frac{1}{2} \varepsilon_{i3} \xi_{\theta i} + \phi_{i3} (\xi_{\theta i} - \xi_{i3}) \right]_{P_{i,1}} + \int_{L_1} \tau^g \varepsilon_{i3} \phi_{i3} \partial a_i \partial s + \int_{L_1} \tau^g \varepsilon_{i3} \phi_{i3} \partial a_i \partial s = 0 \quad i = 1, 3 \quad (2.28)
\]

One should notice, that the rotation is continuous along the whole curve \( L_1 \) if the condition (2.28) is fulfilled.

Repeating the line of thought leading to equation (2.28) we find that the displacements should meet the continuity conditions

\[
u_k a_i|_{P_{i,1}} + u_k a_i|_{P_{i,1}} = 0 \quad \text{and} \quad u_k a_i|_{P_{i,1}} + u_k a_i|_{P_{i,1}} = 0 \quad (2.29)
\]

In order to obtain a more suitable form the above continuity conditions will be transformed further in three steps:

(a) First one has to take into consideration again that the displacements are prescribed if \( s \) meets the conditions (2.27); this circumstance affects the second term in the left hand sides.

(b) Secondly Cesaro’s formula (2.23) should be used to determine the first difference (term) in the left hand sides.

(c) Finally one has to substitute equations (2.18a,b) so as to determine the rotations \( \omega^j \) and \( \omega^k \) involved in the Cesaro formula at the points \( P_{11} \) and \( P_{13} \).

After all these manipulations one obtains

\[
u_k a_i|_{P_{i,1}} + u_k a_i|_{P_{i,1}} = \frac{1}{2} \varepsilon_{i3} \xi_{\theta i} (\xi_{\theta i} - \xi_{i3}) \partial a_i \partial s + \int_{L_1} \tau^g \varepsilon_{i3} \phi_{i3} \partial a_i \partial s = 0 \quad i = 1, 3 \quad (2.30)
\]

NOTE 3.: Conditions of single-valuedness (2.28) and (2.30) reduce formally to the compatibility conditions (2.27) and (2.25) in the large if moving in clockwise direction the point \( P_{1,i+1} \) reaches the point \( P_{1i} \) or, what is the same thing, if the arc \( L_{1i} \) coincides with the closed curve \( L_1 \).

2.11 For triple-connected domains the dual equation system (2.14), (2.10) and (2.11), which is associated with the boundary conditions (2.12), (2.15) and (2.13a,b), should be supplemented by the compatibility conditions in the large (2.24), (2.25) and supplementary conditions (2.28) and (2.30) of single-valuedness.

3 Derivation of the Supplementary Conditions from Casigliano’s Principle

3.1 All conditions the strains \( e_{ik} \) should satisfy in order to be [compatible] \( \{ \text{kinematically admissible} \} \) in a simply-connected three dimensional region can be derived from Castiglione’s principle – see the paper by Kozák (1980a) for details. Consequently the present section follows the line of thought of Kozák except in two things:

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1. Stress functions satisfying the side condition (2.15) can be varied provided that there are no stresses due to the variations. In this respect the paper will utilize such a form which does not require the knowledge of derivatives taken along the normal to the surface $S_t$ of the vector field in terms of which the variations causing no stresses can be given.

2. Since now the volume region is multiply-connected, special care will be taken with those terms obtained by applying Stoke’s theorem on the simple-connected parts of $S$ in order to find a proper form for the jumps of stress functions and to ensure in this way the fulfilment of the subsidiary conditions of single-valuedness.

3.2 Let $-K$ be the total complementary energy functional. Then the functional

$$K = -\frac{1}{2} \int_V t^{kl} e_{kl} \, dA + \int_{S_u} n_3 t^{3\ell} \dot{u}_\ell \, dA \quad (3.1)$$

as a function of the statically admissible stresses $t_{kl}$ and strains $e_{kl}$ has a strict maximum at the actual solution.

The necessary condition for extremum of functional (3.1) is the vanishing of the first variation

$$\delta K = -\int_V e_{kl} \delta t^{kl} \, dV + \int_{S_u} n_3 \delta t^{3\ell} \dot{u}_\ell \, dA = 0 \quad (3.2)$$

obtained from (3.1) by making use of the fact that

$$\frac{1}{2} \delta \left( t^{kl} e_{kl} \right) = e_{kl} \delta t^{kl}.$$

Paragraphs 3.3 to 3.8 are devoted to the long formal transformations aimed at casting the extremum condition $-\delta K = 0$ into an appropriate form. In order to clarify the nature of the various steps a short outline of the transformations is presented below.

(a) In paragraph 3.3 the variations of stresses are given in such a form that the side conditions are satisfied both in $V$ and on $S_t$.

(b) paragraph 3.4 is devoted to the integral transformations. When applying Stoke’s theorem we must keep in mind that the surface $S$ has been cut up and the integrals have to be taken on simple-connected subsurfaces. Consequently line integrals are also obtained. From these some line integrals cancel each other if the following conditions hold:

1. All the variables in the integrals are continuous on the part of the boundary curve regarded.
2. We go twice along this part of the boundary curve. (When applying Stokes’ theorem on $S_u$ for instance we go along the arc $P_{12}, P_{13}$ twice.)

It is worthy of mention here that except $\delta w_1$ all the other variables are continuous on the entire boundary $S$. As regards $\delta w_1$ it is continuous on the simple-connected subsurfaces

$$\begin{array}{c}
S_1 \quad \text{and} \quad S_t \quad \text{and} \quad S_t \\
(1) \quad (2) \quad \text{and} \quad (3)
\end{array}$$

but has a jump on the arc

$$\mathcal{L} = \mathcal{L}_{11} \cup \mathcal{L}_{13} \cup \mathcal{L}_2.$$  

The first integral to contain $\delta w_1$ is the one resulting from equation (3.14) after substituting the side conditions (3.6a,b). In the subsequent steps special care should be taken with those integrals performed along $\mathcal{L}_{11}, \mathcal{L}_{13},$ and $\mathcal{L}_2$.

(c) paragraph 3.5 contains the final form of the extremum condition in respect of the volume and surface integrals.

(d) paragraphs 3.6 to 3.8 are devoted to the analysis of the line integrals taken along $g$ and $L$.

3.3 Being statically admissible the stresses $t^{kl}$ are to be varied under the subsidiary conditions (2.4) and (2.6). Consequently the variations $\delta t^{kl}$ can not be taken at will but should meet the conditions

$$\delta t^{kl}_{x,k} = 0 \quad x \in V \quad \text{and} \quad n_3 \delta t^{3\ell} = 0 \quad \xi \in S_t \quad (3.3)$$

which follow from equations (2.4) and (2.6) by taking into consideration that the variations of body forces $b_l$ and prescribed tractions $\vec{l}$ are equal to zero. It is also assumed that the variation $\delta \vec{B}^l$ is equal to zero.
It follows from equations (2.14) and (2.15) that the variations of stresses satisfying (3.3) can be given in terms of variations of stress functions:

\[ \delta \varepsilon^{kl} = \varepsilon^{km} \varepsilon^{lp} \delta H_{r,s;mp} \quad x \in V \]  

(3.4)

where \( \delta H_{r,s} \) is arbitrary on \( V \) and \( S_u \) but should satisfy the side condition

\[ \delta I^l = n_3 e^{3n_s e^{lp}} \delta H_{n,p} \quad \xi \in S_t \]

(3.5)

Let \( \delta w_l \) be a vector field defined on \( S_t \). If \( \delta H_{n,p} \) satisfies the conditions

\[ \delta H_{n,p} = \delta w_{(l,k)} \quad \xi \in S_t \]

(3.6a)

\[ (\delta H_{\kappa\lambda} - \delta w_{\lambda|k})_{|\lambda} + b_{\kappa}^l (\delta H_{\kappa\lambda} - \delta w_{\kappa|l})_{|\lambda} - (\delta H_{\kappa\lambda;3} - \delta H_{\lambda\kappa;3}) = 0 \quad \xi \in S_t \]

(3.6b)

then condition (3.5) holds. The proof of this statement is very simple if one notices that equations (3.5) and (3.6a,b) coincide with the compatibility boundary condition (2.12) and kinematic boundary conditions (2.13a,b) provided that \( e_{n,p} \) and \( w_l \) are substituted for \( \delta H_{n,p} \) and \( \delta w_l \) in equations (3.5) and (3.6a,b), respectively. Then recalling the assertion that the fulfillment of the strain boundary conditions (2.13a,b) implies the fulfillment of the boundary conditions of compatibility one can come to the conclusion that the original statement is true.

**NOTE 4.** The relatively long and cumbersome proof of the assertion mentioned is published in a paper by Kozak and Szeidl (1996), but originally it appeared in a thesis written by Kozak (1980d).

**NOTE 5.** The static-kinematic analogy, which makes possible the use of the proof cited above, was found by Kozak and Szeidl (1996b). This analogy also involves the equation

\[ \delta H_{\kappa\lambda||\kappa} + \delta H_{\lambda\kappa||\kappa} - \delta w_{\lambda|\kappa} - \delta w_{\kappa|\lambda} = 0 \quad \xi \in S_t \]

(3.6c)

which is the dual counterpart of (2.13c).

**NOTE 6.** Kozak (1980) assumes that

\[ \delta H_{kl} = \delta w_{(k,l)} \quad \xi \in S_t \]

(3.7)

which, contrary to conditions (3.6a,b), involves the derivatives \( \delta w_{k;3} \) taken along the normal to the surface \( S_t \). It has also been shown — see the NOTE 11 in the paper by Kozak and Szeidl (1996b) — that equations (3.7) imply equations (3.6a,b) but the opposite statement is not true. Consequently condition (3.7) is less rigorous than conditions (3.6a,b).

**NOTE 7.** Conditions (3.6a,b) are given in terms of \( \delta w_l (\xi) \), therefore, as already mentioned they say nothing about \( \delta w_{\lambda;3} \) on \( S_t \). With regard to the assumed continuity of \( \delta H_{kl} \) on curve \( g \) which separates \( S_u \) and \( S_t \) one may write by using equations (3.7) that

\[ \delta H_{\kappa;3} = \frac{1}{2} (\delta w_{\lambda;3} + \delta w_{3;\lambda}) \quad \xi \in g \]

(3.8a)

On the bases of the latter equation it will be assumed that \( \delta H_{\lambda;3} \) can freely be varied on \( g \) or, what is the same thing, it can be given in terms of \( \delta w_{\lambda;3} \) which is considered to be arbitrary.

**NOTE 8.** With regard to conditions (3.6a,b) and NOTE 7 one has three functions, i.e., \( \delta w_l \) which can be varied freely on \( S_t \) and five functions, i.e., \( \delta w_l \) and \( \delta w_{\lambda;3} \) which can be varied freely on \( g \).

**NOTE 9.** Without any loss of generality one may assume that

\[ \delta H_{3;3} = \delta w_{3;3} \quad \xi \in g \]

(3.8b)

3.4 Upon substitution of condition (3.5) into variation (3.2) one obtains

\[ -\delta K = I^V_4 + I^S_4 = \int_V e^{krm} e^{lsp} \delta H_{r,s;mp} e_{kl} dV - \int_{S_u} e^{3n_s e^{lp}} \delta H_{n,p} e_{kl} dA = 0 \]

(3.9)
Since the surface integral in equation (3.9) coincides with the left hand side of equation (5.1) if in the latter \( \tilde{u}, \delta \mathcal{H}_{kl}, S_u \) and \( g \) are substituted for \( u, \mathcal{H}_{kl}, S_0 \) and \( g_0 \), respectively, one can write

\[
I_{1}^{S_u} = I_{2}^{S_u} + I_{1}^{G} = - \int_{S_u} n_3 \epsilon^{\kappa \delta \lambda} e^{\lambda \delta \beta} \left\{ \delta \mathcal{H}_{\kappa \lambda \rho \gamma (3)} + \left[ (\hat{\tilde{u}}_{\kappa \lambda})_{\| \kappa} + b_{\lambda \gamma} \hat{\tilde{u}}_{\lambda \gamma} + b_{\lambda \gamma} \hat{\tilde{u}}_{\lambda \gamma} \right] \delta \mathcal{H}_{\kappa \lambda \rho \gamma} \right\} dA
+ \left[ (\hat{\tilde{u}}_{\kappa \lambda})_{\| \kappa} + \hat{\tilde{u}}_{\kappa \lambda \rho \kappa} \delta \mathcal{H}_{\kappa \lambda \rho \gamma} + b_{\lambda \gamma} \hat{\tilde{u}}_{\lambda \gamma} \right] dA
- \int_{g} n_3 \epsilon^{\kappa \delta \lambda} \tau^{\delta \lambda} (\delta \mathcal{H}_{\kappa \lambda \rho \gamma} - \hat{\tilde{u}}_{\kappa \lambda \rho \gamma} \hat{\tilde{u}}_{\kappa \lambda \rho \gamma}) dA \tag{3.10}
\]

for the surface \( S_u \) lies on the right hand side with respect to the positive direction chosen on curve \( g \) — see Figure 1(b).

By applying Gauss’ theorem twice and renaming dummy indices one obtains for \( I_{1}^{V} \)

\[
I_{1}^{V} = I_{2}^{V} + I_{1}^{S} = \int_{V} \epsilon^{k m n} e^{l p q} e_{k l m p} \delta \mathcal{H}_{n q} dV + \int_{S} n_3 \epsilon^{\kappa \delta \lambda} e^{l p q} (\epsilon_{k \lambda \| \kappa} + \epsilon_{k \lambda \ lambda}) dA \tag{3.11}
\]

As regards the surface integral it is worth decomposing those sums involving \( e^{l p q} \). After some manipulations we have

\[
I_{1}^{V} = I_{1}^{S_d} + I_{3}^{S_u} = \int_{S} n_3 \epsilon^{\kappa \delta \lambda} e^{l p q} \left\{ \epsilon_{k \lambda \| \kappa} + \epsilon_{k \lambda \ lambda} \right\} dA
- \int_{g} n_3 \epsilon^{\kappa \delta \lambda} \tau^{\delta \lambda} (\delta \mathcal{H}_{\kappa \lambda \rho \gamma} + \hat{\tilde{u}}_{\kappa \lambda \rho \gamma} \hat{\tilde{u}}_{\kappa \lambda \rho \gamma}) dA \tag{3.12}
\]

Integral \( I_{3}^{S_u} \) can be transformed into a more suitable form by making use of equation (5.2). First the dummy indices \( \rho, \vartheta \) in equation (5.2) should be renamed \( \kappa, \lambda \) and vice versa, i.e., \( \kappa, \lambda \) are to be renamed \( \rho, \vartheta \). Then the desired result can readily be achieved if one follows the way leading to equation (3.10), i.e., by substituting \( \tilde{u}, \delta \mathcal{H}_{kl}, S_u \) and \( g \) for \( u, \mathcal{H}_{kl}, S_0 \) and \( g_0 \) and taking into consideration that the positive description on \( g \) is the one which leaves \( S_u \) on the right; the latter convention affects the sign of the line integrals. After renaming some dummy indices in the line integrals in order to factor out the vector \( \tau^{\delta \lambda} \) one has

\[
I_{3}^{S_u} = I_{4}^{S_u} + I_{2}^{S_u} = \int_{S_u} n_3 \epsilon^{\kappa \delta \lambda} \left\{ \epsilon_{\kappa \lambda \| \kappa} + \epsilon_{\kappa \lambda \ lambda} \right\} (\delta \mathcal{H}_{\kappa \lambda \rho \gamma} + \hat{\tilde{u}}_{\kappa \lambda \rho \gamma} \hat{\tilde{u}}_{\kappa \lambda \rho \gamma}) dA
\]

where \( U \) is not upper index but denotes the limit \( \lim_{\epsilon \to 0} = \epsilon_{k l m p} \).

Now we intend to manipulate integral \( I_{1}^{S_d} \) into a form which makes possible the direct substitution of side conditions (3.6a,b,c). To achieve this goal transformation (5.3) should be applied. The procedure is the same as above except three things: (a) There is no need to rename dummy indices in the surface integral; (b) \( S_d \) should be substituted for \( S_0 \), and (c) because of that the sign of the line integrals will remain unchanged.

Finally we have

\[
I_{1}^{S_d} = - \int_{S_d} n_3 \epsilon^{\kappa \delta \lambda} \left\{ \delta \mathcal{H}_{\kappa \lambda \rho \gamma} + \hat{\tilde{u}}_{\kappa \lambda \rho \gamma} \right\} dA
\]

(3.14)

from which by substituting from conditions (3.6a,b,c) and renaming dummy indices in the line integral it follows

\[
I_{1}^{S_d} = I_{2}^{S_d} + I_{1}^{G} = - \int_{S_d} n_3 \epsilon^{\kappa \delta \lambda} \left\{ \delta \mathcal{H}_{\kappa \lambda \rho \gamma} + \hat{\tilde{u}}_{\kappa \lambda \rho \gamma} \right\} dA
\]

(3.15)
Here $T$ stands for the limit $\lim_{t \to \theta} e_{kl}^t$. In the next step integral $I_{2t}^{St}$ will be transformed further into a final form in respect to the surface integral in the resulting equation.

It is worthy of mention that $I_{2t}^{St}$ coincides with the surface integral on the right hand side of equation (5.1) if in the latter $\delta w_1$, $e_{kl}$, and $S_t$ are substituted for $u_t$, $H_{kl}$, and $S_w$. In addition to this, special care should be taken with the line integrals on the boundary of $S_t$.

Recalling that the union of curves $L_{11}$, $L_{13}$, and $L_2$ is $L$ we should also remember that we go twice along $L$ when we apply Stokes’ theorem. Let the jump of $\delta w_t$ be denoted by $[\delta w_t]$. Its definition is as follows

$$[\delta w_t] = \delta w_t^+ - \delta w_t^- \quad \xi \in L$$

(3.16)

where the [positive] [negative] sign denotes the value of $\delta w_t$ taken on the [positive] [negative] side of $L$ – see Figure 1(b) for details. On the bases of all that has been said one has

$$I_{2t}^{St} = I_{3t}^{St} + I_{1t}^{G\delta w} + I_{1t}^{[w]} = \int_{S_t} \epsilon^{3np} \epsilon^{ldp} \epsilon_{npd,pe} \delta w_t dA$$

$$+ \int_{S_n} \eta_3 \epsilon^{\eta_3} \epsilon_3 \left( \delta w_{\eta_3} \cdot e_{\eta_3}^T - \delta w_{\eta_3} \cdot e_{\eta_3}^T \right) dA - \int_{S} \eta_3 \epsilon^{\eta_3} \epsilon_3 \left( \delta w_{\eta_3} \cdot e_{\eta_3}^T - \delta w_{\eta_3} \cdot e_{\eta_3}^T \right) dA$$

(3.17)

3.5 Comparing the left hand sides of equations (3.9), (3.11), (3.12), (3.13), (3.14) and (3.15) we may write

$$-\delta K = I_{1t}^{G} + I_{2t}^{4u} + I_{3t}^{G} + I_{1t}^{1G} + I_{2t}^{G} + I_{3t}^{G} + I_{1t}^{G\delta w} + I_{1t}^{[w]} = 0$$

(3.18)

or after substituting the right hand sides

$$-\delta K = \int_{V} \epsilon^{Rkm} \epsilon^{Si} \epsilon_{kl,mp} \delta H_{RSd} dV$$

$$+ \int_{S_n} \eta_3 \epsilon^{\eta_3} \epsilon_3 \left( \left( e_{\kappa \lambda} - \tilde{u}_{\lambda(\kappa)} \right) \delta H_{\eta \rho \delta \lambda \beta} \right)$$

$$+ [\left( e_{\kappa \lambda} - \tilde{u}_{\lambda(\kappa)} \right) \cdot \eta + b_{\kappa \lambda}^T \left( e_{\kappa \lambda} - \tilde{u}_{\lambda(\kappa)} \right) - \left( e_{\kappa \lambda, \beta} - e_{\lambda, \kappa, \beta} \right) - b_{\kappa \lambda}^T \left( e_{\kappa \lambda} - \tilde{u}_{\lambda(\kappa)} \right) \delta H_{\eta \rho \delta \lambda \beta}$$

$$+ \left( e_{\kappa \lambda, \alpha} + e_{\lambda, \kappa, \alpha} \right) \cdot \eta - \tilde{u}_{\lambda(\kappa)} b_{\alpha \delta} \delta H_{\eta \beta \alpha \delta} - b_{\kappa \lambda} \left( e_{\kappa \lambda} - \tilde{u}_{\lambda(\kappa)} \right) \delta H_{\eta \beta \alpha \delta} \right] dA$$

$$+ \int_{S_t} \epsilon^{3np} \epsilon^{ldp} \epsilon_{npd,pe} \delta w_t dA + I_{1t}^{G} + I_{1t}^{[w]} = 0$$

(3.19a)

where

$$I_{1t}^{G} = I_{1t}^{G} + I_{2t}^{G} + I_{3t}^{G} + I_{1t}^{G\delta w}$$

(3.19b)

Since in equations (3.19a,b) no conditions for

$$\delta H_{RSd}, \delta H_{\eta \rho \delta \lambda \beta}, \delta H_{\eta \beta \alpha \delta}, \delta H_{33}, \text{and } \delta w_t$$

are set down, they are arbitrary. Consequently from the vanishing of volume and surface integrals in equations (3.19a,b) there follows the fulfilment of differential equations (2.11) of compatibility, the fulfilment of compatibility boundary conditions (2.12) and the fulfilment of kinematic boundary conditions (2.13a,b).

In the latter case one has to recall that on one hand the condition (2.13c) is not independent of conditions (2.13a,b) and on the other hand the coefficient of $\delta H_{33}$ coincides with conditions (2.13a,b). Among others this is one of the reasons why the corresponding stress functions $H_{33}$ and $H_{33}$ can be set to zero. It is worthy of mention that the Euler equation and natural boundary conditions obtained from the extremum condition $-\delta K = 0$ are those the strains $e_{kl}$ should satisfy in order to be kinematically admissible on a single-connected volume region.

3.6 Analysis of line integrals involved in equation (3.19b) requires some preparations. Let the jump $[\delta w_t]$ be

$$[\delta w_t] = \delta (11) c^t_b + \epsilon_{\kappa \lambda \beta} \delta C \cdot [R^t - R^{t}(P_{1,i+1})]$$

$$i = 1, 3 \quad \xi \in L_{1i}$$

(3.20)

$$[\delta w_t] = \delta (21) c^t_b + \epsilon_{\kappa \lambda \beta} \delta C \cdot [R^t - R^{t}(P_{21})]$$

$$\xi \in L_2$$

(3.21)
where $c^{(I)}$, $c^{(II)}$, $C^\alpha$, $C^\beta$, and $c_{\alpha\beta}$ are arbitrary constants.

NOTE 10.: Function
\[ [\delta w_b] = \delta c_b + c_{\alpha\beta} \delta C^\alpha [R^\alpha - R^\beta(P)], \quad \xi \in S \]
in which $\delta c_b$ and $\delta C^\alpha$ are arbitrary constants and $P$ is an arbitrary but fixed point, causes no 'stress functions' since the gradient of $[\delta w_b]$ is skew-symmetric:
\[ [\delta w_{b,\lambda}] = c_{\alpha\beta} \delta C^\alpha \quad [\delta w_{\lambda,\rho}] = c_{\alpha\beta} \delta C^\alpha \quad \xi \in S \tag{3.22} \]

3.7 As regards the line integral $I^l_1$ the necessary transformations are presented in the Appendix – see Section 5. Assuming continuity of strains on $S$ it is plain that
\[ e^U_{kt} = e^T_{kt} \quad \xi \in g \tag{3.23} \]

By using the equation
\[ \delta w_{(3)\lambda} - \delta w_{(3)\rho} = \delta w_{(\lambda)\beta} + \delta w_{(\beta)\rho} = \delta w_{\lambda\beta} \quad \xi \in g \]
and omitting the distinguishing $U$ and $T$ from (5.13) and (5.14) one obtains
\[ I^l_1 = \int_{\partial} \left\{ \tau^\eta e^{\lambda\alpha}(e_{\partial\eta\alpha} - e_{\partial\eta\beta}) - \frac{d}{ds} \left[ \frac{1}{2} \delta \lambda \alpha [e_{\theta\beta} - \hat{u}_{(\theta\beta)}] \right] \right\} \delta w_{\lambda\beta} ds \]
\[ + \int_{\partial} \left\{ \tau^\eta e^{\lambda\rho}(e_{\partial\eta\rho} - \frac{d}{ds} \left[ \frac{1}{2} \delta \lambda \rho [e_{\theta\beta} - \hat{u}_{(\theta\beta)}] \right] \right\} \delta w_{\lambda\beta} ds \]
\[ + \int_{\partial} \tau^\eta e^{\lambda\alpha}(e_{\partial\eta\alpha} - \hat{u}_{(\partial\eta\alpha)}\delta w_{\lambda\beta} ds + \delta w_{\lambda\beta} + \delta w_{\lambda\beta} + \delta w_{\lambda\beta} \tag{3.24} \]

Vanishing of the line integrals for arbitrary $\delta w_{\lambda\beta}$, $\delta w_{\lambda\beta}$ and $\delta w_{\lambda\beta}$ yields the continuity conditions
\[ \tau^\eta e^{\lambda\alpha}(e_{\partial\eta\alpha} - e_{\partial\eta\beta}) = \frac{d}{ds} \left[ \frac{1}{2} \delta \lambda \beta [e_{\theta\beta} - \hat{u}_{(\theta\beta)}] \right] \quad \xi \in g \tag{3.25a} \]
\[ \tau^\eta e^{\lambda\rho}(e_{\partial\eta\rho} - \hat{u}_{(\partial\eta\beta)}) = \frac{d}{ds} \left[ \frac{1}{2} \delta \lambda \rho [e_{\theta\beta} - \hat{u}_{(\theta\beta)}] \right] \quad \xi \in g \tag{3.25b} \]
\[ \tau^\eta (e_{\partial\eta\beta} - \hat{u}_{(\partial\eta\beta)}) = 0 \quad \xi \in g \tag{3.25c} \]

If we had not omitted the distinguishing $U$ and $T$ then with respect to equations (2.18a,b) and (2.20) continuity conditions (3.25a,b) would assume the forms
\[ \left( \frac{d\omega^\alpha}{ds} \right)^U = \left( \frac{d\omega^\alpha}{ds} \right)^T \quad \text{and} \quad \left( \frac{d\omega^\beta}{ds} \right)^U = \left( \frac{d\omega^\beta}{ds} \right)^T \quad \xi \in g \tag{3.26} \]

NOTE 11.: Fulfillment of conditions (3.25a,b,c) is sufficient for the determination of $\omega^\beta$, $\hat{u}_{\partial\beta}$ and $\hat{u}_{\partial\beta}$ in terms of $e_{\eta\partial\alpha}$, $e_{\partial\eta\beta}$ and $e_{\partial\eta\beta}$. After integrating equation (3.25b) we obtain $\omega^\beta$. With $\omega^\beta$ and equation (3.25c) it follows
\[ \tau^\eta (e_{\eta\beta} - \hat{u}_{\eta\beta} + \hat{u}_{(\eta\beta)}) = 0 \quad \xi \in g \]

or what is the same thing
\[ \frac{d\hat{u}_{\eta\beta}}{ds} = \tau^\eta (e_{\eta\beta} - e_{\eta\beta}\omega^\beta) \quad \xi \in g \]

from which the displacements $\hat{u}_{\eta\beta}$ are obtained immediately by integration with respect to $s$.

As regards the equation (3.25a), it can be solved for $u_{\partial\beta}$ by integration with respect to $s$. If $u_{\partial\beta}$ is known then
\[ \frac{d\hat{u}_{\beta}}{ds} = \tau^\beta \hat{u}_{\beta} \quad \xi \in g \]
from which $\hat{u}_{\beta}$ can be determined by integration.
3.8 The last part of the line integrals consists of those terms whose presence expresses that the volume region is triple connected. Substituting $I_1^{[w]}$, $\Sigma^1$, $\Sigma^2$, and $\Sigma^3$ from equations (3.17), (5.9),(5.11) and (5.12) and comparing equations (3.19a) and (3.24) we have

$$I_1^{[w]} + \Sigma^1 + \Sigma^2 + \Sigma^3 =$$

$$= \sum_{i=1,3} \left( \int_{L_i} n_i e^{\kappa_3 \rho \lambda} \left[ \delta_{\eta j}\rho \epsilon_{\eta j k} \right] d\sigma + \int_{L_i} \tau^{\eta j} e^{\lambda \delta \rho \lambda} \left[ \epsilon_{\eta j k \delta} \right] d\sigma \right) ds$$

$$+ \int_{L_i} \left[ \frac{1}{2} \epsilon_{\eta j k \delta} \left[ \delta_{\eta j} \right] \right] d\sigma + \int_{L_i} \tau^{\eta j} e^{\lambda \delta \rho \lambda} \left[ \epsilon_{\eta j k \delta} \right] d\sigma$$

$$+ \int_{L_i} \left[ \frac{1}{2} \epsilon_{\eta j k \delta} \left[ \delta_{\eta j} \right] \right] d\sigma$$

$$+ \int_{L_i} \left[ \frac{1}{2} \epsilon_{\eta j k \delta} \left[ \delta_{\eta j} \right] \right] d\sigma$$

To obtain the supplementary conditions of single-valuedness let us substitute equations (3.20) and (3.21) then gather the coefficients of

$$\delta (21)$$

$$\delta (21)$$

$$\delta$$

$$\delta$$

Upon a subsequent rearrangement we have

$$I_1^{[w]} + \Sigma^1 + \Sigma^2 + \Sigma^3 =$$

$$= \left[ \int_{L_i} \tau^{\eta j} e^{\lambda \delta \rho \lambda} \left[ \epsilon_{\eta j k \delta} \right] d\sigma \right] ds$$

$$+ \left[ \int_{L_i} \tau^{\eta j} e^{\lambda \delta \rho \lambda} \left[ \epsilon_{\eta j k \delta} \right] d\sigma \right] ds$$

$$+ \left[ \int_{L_i} \tau^{\eta j} e^{\lambda \delta \rho \lambda} \left[ \epsilon_{\eta j k \delta} \right] d\sigma \right] ds$$

Vanishing of the above expression for arbitrary $\delta (21)$, $\delta (21)$ results in the fulfillment of compatibility conditions (2.24) and (2.25) in the large and the supplementary conditions (2.28) and (2.30) of single valuedness.

4 Concluding Remarks

4.1 In accordance with the aims detailed in Paragraph 1.5 the present paper has studied the question as to what further conditions the strain fields should meet in addition to the usual ones in order to be kinematically admissible on multiply connected regions providing three dimensional and mixed boundary value problems. The mathematical form of the supplementary conditions of single-valuedness has been derived from a geometrical line of thought. Consequently the conditions are independent of the material law.

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4.2 It has also been proved that both the compatibility conditions in the large and the supplementary conditions of single valuedness are natural boundary conditions that follow from the minimum complementary energy. The significance of this conclusion is inherent in the applications. In other words application of direct methods — finite element method for instance — to find approximate solutions from the extremum condition do not require that the admissible fields do satisfy these conditions in advance.

5 Mathematical Transformations

5.1 Let $u_l$ and $H_{kl}$ be sufficiently smooth tensors on $S_o$. Comparing equations (A50,i) and (3.25,i) one can write

$$
\int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \mathcal{L}_{\kappa\mu\nu\rho\tau} u_A \, dA = - \int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left\{ \varepsilon_{\lambda\mu\nu\rho\tau} \mathcal{L}_{\kappa\mu\nu\rho\tau} + \left[ (u_{3|\kappa})_{\lambda\mu} + b_{\lambda}^{\mu} u_{\kappa|\mu} + b_{\mu}^{\lambda} u_{\kappa|\mu} \right] \mathcal{L}_{\kappa\mu\nu} + b_{\kappa}^{\nu} u_{3|\nu} \mathcal{L}_{\lambda\mu\nu} \right\} dA + \oint_{S_o} \tau^{\alpha\beta\gamma} \xi^{A\beta\gamma} \mathcal{L}_{\kappa\mu\nu\rho\tau} u_A \, dA. \quad (5.1)
$$

A detailed proof of the above equation is presented in the paper by Kozák–Szeidl (1996a).

5.2 Using the above notations and comparing equations (A54,i) and (3.26,a,i) one obtains

$$
\int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left\{ \nabla_{\lambda\mu\nu} \mathcal{L}_{\kappa\mu\nu\rho\tau} + \nabla_{\lambda\mu\nu} \left[ (u_{3|\kappa})_{\lambda\mu} + b_{\lambda}^{\mu} u_{\kappa|\mu} + b_{\mu}^{\lambda} u_{\kappa|\mu} \right] \mathcal{L}_{\kappa\mu\nu} + b_{\kappa}^{\nu} u_{3|\nu} \mathcal{L}_{\lambda\mu\nu} \right\} dA =
= \int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left\{ \varepsilon_{\lambda\mu\nu\rho\tau} \mathcal{L}_{\kappa\mu\nu\rho\tau} + \left[ (u_{3|\kappa})_{\lambda\mu} + b_{\lambda}^{\mu} u_{\kappa|\mu} + b_{\mu}^{\lambda} u_{\kappa|\mu} \right] \mathcal{L}_{\kappa\mu\nu} + b_{\kappa}^{\nu} u_{3|\nu} \mathcal{L}_{\lambda\mu\nu} \right\} dA + \oint_{S_o} \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left[ \mathcal{L}_{\kappa\mu\nu\rho\tau} \mathcal{L}_{\delta\gamma\mu\nu} + \left[ (u_{3|\kappa})_{\delta\gamma} + b_{\delta}^{\gamma} u_{\gamma|\kappa} + b_{\gamma}^{\delta} u_{\delta|\kappa} \right] \mathcal{L}_{\delta\gamma\mu} + b_{\kappa}^{\nu} u_{\kappa|\nu} \mathcal{L}_{\lambda\mu\nu} \right] \mathcal{L}_{\lambda\mu\nu} dA.
$$

As regards the transformations leading to (A54,i) the reader is referred to Paragraph 5.18 in the paper by Kozák–Szeidl (1996a). By interchanging the letters $e$ and $H$ in (5.2) the sign of surface integral turns to the opposite. Keeping this in mind we may write

$$
\int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left\{ \nabla_{\lambda\mu\nu} \mathcal{L}_{\kappa\mu\nu\rho\tau} + \nabla_{\lambda\mu\nu} \left[ (u_{3|\kappa})_{\lambda\mu} + b_{\lambda}^{\mu} u_{\kappa|\mu} + b_{\mu}^{\lambda} u_{\kappa|\mu} \right] \mathcal{L}_{\kappa\mu\nu} + b_{\kappa}^{\nu} u_{3|\nu} \mathcal{L}_{\lambda\mu\nu} \right\} dA =
= \int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left\{ \varepsilon_{\lambda\mu\nu\rho\tau} \mathcal{L}_{\kappa\mu\nu\rho\tau} + \left[ (u_{3|\kappa})_{\lambda\mu} + b_{\lambda}^{\mu} u_{\kappa|\mu} + b_{\mu}^{\lambda} u_{\kappa|\mu} \right] \mathcal{L}_{\kappa\mu\nu} + b_{\kappa}^{\nu} u_{3|\nu} \mathcal{L}_{\lambda\mu\nu} \right\} dA + \oint_{S_o} \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left[ \mathcal{L}_{\kappa\mu\nu\rho\tau} \mathcal{L}_{\delta\gamma\mu\nu} + \left[ (u_{3|\kappa})_{\delta\gamma} + b_{\delta}^{\gamma} u_{\gamma|\kappa} + b_{\gamma}^{\delta} u_{\delta|\kappa} \right] \mathcal{L}_{\delta\gamma\mu} + b_{\kappa}^{\nu} u_{\kappa|\nu} \mathcal{L}_{\lambda\mu\nu} \right] \mathcal{L}_{\lambda\mu\nu} dA.
$$

5.3 Transformation of integral $I_1^G$.

By making use of equations (3.10), (3.13), (3.15), (3.17) and (3.19b) we may write

$$
I_1^G = I_1^G + I_2^G + I_3^G + I_4^G = \int_S n_3 \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left[ \nabla_{\lambda\mu\nu} \mathcal{L}_{\kappa\mu\nu\rho\tau} + \nabla_{\lambda\mu\nu} \left[ (u_{3|\kappa})_{\lambda\mu} + b_{\lambda}^{\mu} u_{\kappa|\mu} + b_{\mu}^{\lambda} u_{\kappa|\mu} \right] \mathcal{L}_{\kappa\mu\nu} + b_{\kappa}^{\nu} u_{3|\nu} \mathcal{L}_{\lambda\mu\nu} \right] \mathcal{L}_{\lambda\mu\nu} dA + \oint_{S_o} \varepsilon^{\alpha\beta\gamma} \xi^{A\beta\gamma} \left[ \mathcal{L}_{\kappa\mu\nu\rho\tau} \mathcal{L}_{\delta\gamma\mu\nu} + \left[ (u_{3|\kappa})_{\delta\gamma} + b_{\delta}^{\gamma} u_{\gamma|\kappa} + b_{\gamma}^{\delta} u_{\delta|\kappa} \right] \mathcal{L}_{\delta\gamma\mu} + b_{\kappa}^{\nu} u_{\kappa|\nu} \mathcal{L}_{\lambda\mu\nu} \right] \mathcal{L}_{\lambda\mu\nu} dA.
$$

The above equation clearly shows the structure of those line integrals resulted from the transformations if the volume region under consideration is simply-connected.

Now it is our aim to manipulate integral (5.4) into a more suitable form.
During the transformations we shall utilize, among others, the following equations:

\[ \delta w|_{\alpha|\eta} = -\epsilon_{\phi\xi\eta}\delta\tau_{\phi} \quad \delta w|_{\eta|\theta} = \delta w|_{\alpha|\eta} + \epsilon_{\phi\xi\eta}\delta\tau_{\phi} \quad \xi \in S_t \]  

(5.5)

where \( \delta\tau_{\phi} \) is the third component of the corresponding axial vector.

It follows from equation (3.6b) that

\[ \delta\mathcal{H}_{\alpha\beta\xi} - \delta\mathcal{H}_{3\eta\xi} = \delta\mathcal{H}_{3|\alpha\xi} + b_{\eta\xi}\delta\mathcal{H}_{\alpha\xi} - \delta w_{3|\alpha\xi} - b_{\eta\xi}\delta w_{3|\xi} \quad \xi \in S_t \]

from which by adding \( 0 = b_{\eta\xi}(\delta\mathcal{H}_{33} - \delta w_{33}) \) to the right hand side – see (3.8b) — and taking into consideration the rule (A.9), we obtain

\[
\begin{align*}
\delta\mathcal{H}_{\alpha\beta\xi} - \delta\mathcal{H}_{3\eta\xi} &= \delta\mathcal{H}_{3|\alpha\xi} + b_{\eta\xi}\delta\mathcal{H}_{\alpha\xi} - \delta w_{3|\alpha\xi} - b_{\eta\xi}\delta w_{3|\xi} \\
&= \delta\mathcal{H}_{3|\alpha\xi} - \delta w_{3|\xi}
\end{align*}
\]

(5.6)

In view of (3.6a) it is plain that

\[ \epsilon^{\lambda\xi\eta}\delta\mathcal{H}_{\alpha\beta\eta} = \epsilon^{\lambda\xi\eta}\frac{1}{2}(\delta w|_{\alpha\beta\eta} + \delta w|_{\alpha\beta\xi}) = \epsilon^{\lambda\xi\eta}\frac{1}{2}\delta w|_{\alpha\xi} \quad \xi \in g \]  

(5.7)

Care should be taken to the partial integrations carried out along the curves \( i = 1, \ldots, 4 \) since \( \delta w_i \) has a jump at the points \( P_i \).

Now we shall consider the line integral \( I_G^1 \) — see equation (3.1) or the first line of expression (5.4). After making use of the resolution

\[
-\tau^\eta\epsilon^{\lambda\nu\eta}\delta\mathcal{H}_{\alpha\beta\eta} \tilde{u}_i = \tau^\eta\epsilon^{\lambda\xi\eta}\tilde{u}_i + \tau^\eta\epsilon^{\lambda\xi\eta}(\delta\mathcal{H}_{\alpha\beta\eta} - \delta\mathcal{H}_{3\eta\xi}) \tilde{u}_i \quad \xi \in S_u
\]

(5.8)

one can substitute equations (5.6) and (5.7). Then it becomes possible to carry out partial integrations along the curve \( g \). Finally substitute (3.6a) and (3.8a) for \( \delta\mathcal{H}_{\alpha\xi} \) and \( \delta\mathcal{H}_{3\xi} \). Renaming some dummy indices where necessary we have

\[
I_G^1 = I_{S}^G + \Sigma^1 = \int_g \tau^\eta\epsilon^{\lambda\xi\eta}\frac{1}{2}\delta w|_{\alpha\beta\eta} \tilde{u}_i \quad \xi \in S_u
\]

(5.9)

Observe that the terms denoted by \( \Sigma^1 \) are those one obtains for multiply-connected regions and mixed boundary value problems.

Now we turn our attention to the line integral \( I_G^3 \) — see equations (3.15) and (3.17) or the last line of expression (5.4). Decomposing the sum \( \tau^\eta\epsilon^{\lambda\nu\eta} \delta w_i \) — see equation (5.8) for details — and substituting equations (3.6a), (3.8a) and (5.5) we find after some rearrangement that

\[
I_G^3 = \int_g \tau^\eta\epsilon^{\lambda\nu\eta} \epsilon_{\phi\xi\eta}\delta w_3 \quad \xi \in S_u
\]

(5.10)

A more suitable form of line integral \( I_G^3 \) can be derived if one substitutes equations (5.9) and (5.10) for \( I_G^1 \) and \( I_G^3 \) — see equations (3.6a) and (3.8a) for \( \delta\mathcal{H}_{\alpha\xi} \) and \( \delta\mathcal{H}_{3\xi} \) in \( I_G^3 \) — see the third line in expression (5.4). Next substitute equation (5.5b) for \( \delta w|_{\alpha\eta} \) in the [last] [first] line integral in \( I_G^1 \) (5.9)] [\( I_G^3 \)] After some rearrangement it becomes possible to utilize the transformation

\[
\int_g \tau^\eta\epsilon^{\lambda\nu\eta}(c_{k3} - \tilde{u}_{3|\eta}) \delta w|_{\alpha\eta} ds = -\int_g \frac{d}{ds} \epsilon^{\lambda\nu\eta}(c_{k3} - \tilde{u}_{3|\eta}) \delta w|_{\alpha\eta} ds + \Sigma^2
\]

where

\[
\Sigma^2 = \sum_{i=1,3} \left\{ \epsilon^{\lambda\nu\eta}(c_{k3} - \tilde{u}_{3|\eta}) [\delta w|_{\alpha\eta}]_{P_i} - \epsilon^{\lambda\nu\eta}(c_{k3} - \tilde{u}_{3|\eta}) [\delta w|_{\alpha\eta}]_{P_{i+1}} \right\}
\]

(5.11)

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If in addition to this the right hand side of \( I_G \) is ‘enlarged’ by

\[
0 = - \int_g \frac{d}{ds} \left\{ \frac{1}{2} \epsilon^{\alpha\lambda\sigma} \dot{u}_{[\alpha]} \right\} \delta w_{[\beta]} + \int_g \tau^\alpha \frac{1}{2} \epsilon^{\alpha\lambda\sigma} \dot{u}_{[\alpha]} \delta w_{[\beta]} \, ds + \Sigma^3
\]

in which

\[
\Sigma^3 = \sum_{i=1,3} \left( \frac{1}{2} \epsilon^{\alpha\lambda\sigma} \dot{u}_{[\alpha]} \delta w_{[\beta]} \right)_{|P_{i+1}} - \left( \frac{1}{2} \epsilon^{\alpha\lambda\sigma} \dot{u}_{[\alpha]} \delta w_{[\beta]} \right)_{|P_i}
\]

(5.12)

then, upon a subsequent rearrangement, we have

\[
I_G = I_1^G + I_2^G + I_3^G + I_4^G + \Sigma^1 + \Sigma^2 + \Sigma^3 =
\]

\[
\int_g \left\{ \tau^\alpha \epsilon^{\alpha\beta\gamma} (e_{\alpha\beta\gamma} - e_{3\alpha\beta}) - \frac{d}{ds} \left( \frac{1}{2} \epsilon^{\alpha\beta\gamma} \dot{u}_{[\gamma]} \right) \right\} \delta w_{[\beta]} \, ds
\]

\[
+ \int_g \left[ \tau^\alpha \epsilon^{\alpha\beta\gamma} e_{\beta\gamma|\sigma} - \frac{d}{ds} \left( \frac{1}{2} \epsilon^{\alpha\beta\gamma} \dot{u}_{[\gamma]} \right) \right] \delta w_{[\beta]} \, ds
\]

\[
+ \int_g \left( \delta_{[\beta]} - e_{[\beta]} \right) \delta w_{[\beta]} \, ds + I_G^0 + \Sigma^1 + \Sigma^2 + \Sigma^3
\]

(5.13)

In the above equation

\[
I_G^0 = - \int_g \tau^\alpha \epsilon^{3\beta\gamma} \left( \dot{u}_{[\beta]} \right) \delta w_{[\gamma]} \, ds - \int_g \tau^\alpha \epsilon^{3\beta\gamma} e_{[\beta]} \delta w_{[\gamma]} \, ds
\]

\[
+ \int_g \tau^\alpha \epsilon^{3\beta\gamma} \left( \dot{u}_{[\beta]} \right) \delta w_{[\gamma]} \, ds
\]

\[
+ \int_g \tau^\alpha \epsilon^{3\beta\gamma} \left( \dot{u}_{[\beta]} \right) \delta w_{[\gamma]} \, ds + \int_g \tau^\alpha \epsilon^{3\beta\gamma} e_{[\beta]} \delta w_{[\gamma]} \, ds =
\]

\[
= \int_g \tau^\alpha \epsilon^{3\beta\gamma} (e_{[\beta]} - \dot{u}_{[\beta]}) \delta w_{[\gamma]} \, ds - \int_g \tau^\alpha \epsilon^{3\beta\gamma} (e_{[\beta]} - \dot{u}_{[\beta]}) \delta w_{[\gamma]} \, ds
\]

(5.14)

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Literature


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