Publ. Univ. of Miskolc, Series D. Natural Sciences. Vol 38. Mathematics (1998) pp. 117 – 132

NATURAL FREQUENCIES OF A CIRCULAR ARCH – COMPUTATIONS BY THE USE OF GREEN FUNCTIONS

GY. SZEIDL, K. KELEMEN[1] and Á. SZEIDL [2]

Abstract. A defintion is given for the Green matrix function of a degenerate system of ordinary differential equations associated with homegenous linear boundary conditions. In the knowledge of the Green matrix function self adjoint eigenvalue problems can be replaced by a system of homogenous Fredholm integral equations with cross symmetric kernel. The latter eigenvalue problem can replaced by an algebraic one. Solutions are presented for the free vibrations of circular arches.

Mathematical Subject Classification: 34B27, 45C05
Keywords: Natural frequencies, Green function matrix, eigenvalue problem

1. Introduction

There is a classical definition – see for instance [1] – for the Green function of ordinary linear differential equations with homogenous boundary conditions. The concept has been generalized for a class of degenerate systems of linear differential equations by keeping up at the same time the structure of the definition in [2].

In the knowledge of the Green function self adjoint eigenvalue problems governed by an ordinary differential equation can be reduced to an eigenvalue problem for a Fredholm integral equation with symmetric kernel [3]. This approach provides advantages when discretizing the problem one replaces it with an algebraic eigenvalue problem. In addition

[1]Dr. György SZEIDL associate professor, and Katalin KELEMEN Ph.D. student
Department of Mechanics, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
[2]Ádám SZEIDL student
Eötvös Loránd University of Budapest, Hungary
Received: September 15, 1998

it also remains valid for eigenvalue problems described by a degenerate and self adjoint system of differential equations with appropriate boundary conditions.

There are a lot of works on eigenvalue problems associated with the free vibration and stability of circular arches. Without trying to achieve completeness we should mention the book by Federhoffer [4], the papers [5], [6] and the thesis [2]. Shear deformation and higher order deformations can also be taken into account. In this respect the reader is referred to [7] and the papers cited in it.

2. The Green matrix function

Consider the degenerate system of differential equations

$$\mathbf{K}(\mathbf{y}) = \sum_{\nu=0}^{n} \overset{\nu}{\mathbf{P}}(x)\mathbf{y}^{(\nu)}(x) = \begin{bmatrix} 0 & 0 \\ 0 & \overset{n}{\mathbf{P}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}^{(n)} + \cdots + \begin{bmatrix} 0 & 0 \\ 0 & \overset{k+1}{\mathbf{P}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}^{(k+1)}$$

$$+ \begin{bmatrix} \overset{k}{\mathbf{P}}_{11} & \overset{k}{\mathbf{P}}_{12} \\ 0 & \overset{k}{\mathbf{P}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}^{(k)} + \cdots + \begin{bmatrix} \overset{s}{\mathbf{P}}_{11} & \overset{s}{\mathbf{P}}_{12} \\ \overset{s}{\mathbf{P}}_{21} & \overset{s}{\mathbf{P}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}^{(s)} + \cdots = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \tag{2.1}$$

where $n > k > s > 0$, l is the number of unknown functions (or which is the same the size of $\mathbf{y}$), j is the size of $\mathbf{y}_2$ and the matrices $\overset{\nu}{\mathbf{P}}$ and $\mathbf{r}$ are continuous for $x \in [a, b]$; $a < b$. We shall assume that

– $\overset{n}{\mathbf{P}}_{22}$ and $\overset{k}{\mathbf{P}}_{11}$ are invertible if $x \in [a, b]$
– the system of ODS (2.1) is associated with linear homogenous boundary conditions

$$\mathbf{U}_\mu(\mathbf{y}) = \sum_{\nu=0}^{n-1} \left[\mathbf{A}_{\nu\mu}\mathbf{y}^{(\nu)}(a) + \mathbf{B}_{\nu\mu}\mathbf{y}^{(\nu)}(b)\right] =$$

$$= \sum_{\nu=0}^{n-1} \left\{ \begin{bmatrix} \overset{11}{\mathbf{A}}_{\nu\mu} & \overset{12}{\mathbf{A}}_{\nu\mu} \\ \overset{21}{\mathbf{A}}_{\nu\mu} & \overset{22}{\mathbf{A}}_{\nu\mu} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1(a) \\ \mathbf{y}_2(a) \end{bmatrix}^{(v)} + \begin{bmatrix} \overset{11}{\mathbf{B}}_{\nu\mu} & \overset{12}{\mathbf{B}}_{\nu\mu} \\ \overset{21}{\mathbf{B}}_{\nu\mu} & \overset{22}{\mathbf{B}}_{\nu\mu} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1(b) \\ \mathbf{y}_2(b) \end{bmatrix}^{(v)} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.2}$$

where $\mu = 1, \ldots, n$ and for $\nu \le k$ the constant matrices $\mathbf{A}_{\nu\mu}$ and $\mathbf{B}_{\nu\mu}$ fulfill the conditions

$$\overset{11}{\mathbf{A}}_{\nu\mu} = \overset{21}{\mathbf{A}}_{\nu\mu} = \overset{11}{\mathbf{B}}_{\nu\mu} = \overset{21}{\mathbf{B}}_{\nu\mu} = 0$$

– $\bar{l} = nl - [(l - j)k + nj]$ rows are identically zero in the hypermatrix

$$\mathbf{P}_f = \begin{bmatrix} \mathbf{A}_{01} & \ldots & \mathbf{A}_{n-1,1} & \mathbf{B}_{01} & \ldots & \mathbf{B}_{n-1,1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{A}_{0n} & \ldots & \mathbf{A}_{n-1,n} & \mathbf{B}_{0n} & \ldots & \mathbf{B}_{n-1,n} \end{bmatrix} \tag{2.3}$$

By introducing appropriate new variables the system of ODE (2.1) can be replaced by $(l - j)k + nj$ differential equations of order one and one can construct the corresponding

Green function. In the present case however there is no need for this transformation since the definition is based on the original equation. The second condition expresses that $\mathbf{y}_1^{(\nu)}$ can not appear in the boundary conditions if $\nu \geq k$. The third condition is a restriction on the number of boundary conditions.

Solution to the boundary value problem (2.1), (2.2) is sought in the form

$$\mathbf{y}(x) = \int_a^b \mathbf{G}(x,\xi)\mathbf{r}(\xi)d\xi \tag{2.4}$$

where $\mathbf{G}(x,\xi)$ is the Green matrix function defined by the following properties:

1. The Green matrix function is a continuous function of x and ξ in each of the triangles $a \leq x \leq \xi \leq b$ and $a \leq \xi \leq x \leq b$. The functions

   $$(\mathbf{G}_{11}(x,\xi), \mathbf{G}_{12}(x,\xi)) \;\; [\mathbf{G}_{21}(x,\xi), \mathbf{G}_{22}(x,\xi)]$$

   are (k times) [n times] differentiable with respect to x and the derivatives

   $$\begin{aligned} \frac{\partial^\nu \mathbf{G}(x,\xi)}{\partial x^\nu} &= \mathbf{G}^{(\nu)}(x,\xi) \qquad (\nu = 1,2,\ldots,k) \\ \frac{\partial^\nu \mathbf{G}_{2i}(x,\xi)}{\partial x^\nu} &= \mathbf{G}_{2i}^{(\nu)}(x,\xi) \qquad (\nu = 1,2,\ldots,n;\; i = 1,2) \end{aligned}$$

   are continuous functions of x and ξ.
2. Let ξ be fixed in $[a,b]$. Though the derivatives

   $$\begin{array}{llll} \mathbf{G}_{11}^{(\nu)}(x,\xi) & (\nu = 1,2,\ldots,k-2); & \mathbf{G}_{12}^{(\nu)}(x,\xi) & (\nu = 1,2,\ldots,k-1) \\ \mathbf{G}_{21}^{(\nu)}(x,\xi) & (\nu = 1,2,\ldots,n-1) & \mathbf{G}_{22}^{(\nu)}(x,\xi) & (\nu = 1,2,\ldots,n-2) \end{array}$$

   are continuous for $x = \xi$, the higher derivatives $\mathbf{G}_{11}^{(k-1)}(x,\xi)$ and $\mathbf{G}_{22}^{(n-1)}(x,\xi)$ have a jump on the diagonal, i.e.,

   $$\begin{aligned} \lim_{\varepsilon\to 0}\left[\mathbf{G}_{11}^{(k-1)}(\xi+\varepsilon,\xi) - \mathbf{G}_{11}^{(k-1)}(\xi-\varepsilon,\xi)\right] &= \overset{k}{\mathbf{P}}{}_{11}^{-1}(\xi), \\ \lim_{\varepsilon\to 0}\left[\mathbf{G}_{22}^{(n-1)}(\xi+\varepsilon,\xi) - \mathbf{G}_{22}^{(n-1)}(\xi-\varepsilon,\xi)\right] &= \overset{n}{\mathbf{P}}{}_{22}^{-1}(\xi). \end{aligned}$$

3. Let $\boldsymbol{\alpha}$ be an arbitrary otherwise constant vector. For a fixed $\xi \in [a,b]$ the vector $\mathbf{G}(x,\xi)\boldsymbol{\alpha}$ as a function of x ($x \neq \xi$) should satisfy the homogenous differential equation

   $$\mathbf{K}\left[\mathbf{G}(x,\xi)\boldsymbol{\alpha}\right] = 0$$

4. The vector $\mathbf{G}(x,\xi)\boldsymbol{\alpha}$ as a function of x should satisfy the boundary conditions

   $$\mathbf{U}_\mu\left[\mathbf{G}(x,\xi)\boldsymbol{\alpha}\right] = 0 \quad \mu = 1,\ldots,n$$

If there exists the Green matrix function defined above for the BVP (2.1), (2.2) then the vector (2.4) satisfies the differential equation (2.1) and the boundary conditions (2.2).

The part of our statement concerning the boundary conditions follows immediately form the comparison of the fourth property to the formula (2.4). As regards the second part of our statement substitute the representation (2.4) into (2.1) and utilize that the matrices $\mathbf{G}_{11}^{(k-1)}$ and $\mathbf{G}_{22}^{(n-1)}$ are discontinuous if $x = \xi$. In this way we may write

$$\begin{aligned}
\mathbf{K}(\mathbf{y}) &= \sum_{\nu=0}^{n} \overset{\nu}{\mathbf{P}}(x)\mathbf{y}^{(\nu)}(x) = \begin{bmatrix} 0 & 0 \\ 0 & \overset{n}{\mathbf{P}}_{22}(x) \end{bmatrix} \int_a^b \begin{bmatrix} 0 & 0 \\ \mathbf{G}_{21}^{(n)}(x,\xi) & \mathbf{G}_{22}^{(n)}(x,\xi) \end{bmatrix} \begin{bmatrix} \mathbf{r}_1(\xi) \\ \mathbf{r}_2(\xi) \end{bmatrix} d\xi + \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & \overset{n}{\mathbf{P}}_{22}(x)\left\{\mathbf{G}_{22}^{(n-1)}(x,x-0) - \mathbf{G}_{22}^{(n-1)}(x,x+0)\right\} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1(x) \\ \mathbf{r}_2(x) \end{bmatrix} + \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & \overset{k+1}{\mathbf{P}}_{22}(x) \end{bmatrix} \int_a^b \begin{bmatrix} 0 & 0 \\ \mathbf{G}_{21}^{(k+1)}(x,\xi) & \mathbf{G}_{22}^{(k+1)}(x,\xi) \end{bmatrix} \begin{bmatrix} \mathbf{r}_1(\xi) \\ \mathbf{r}_2(\xi) \end{bmatrix} d\xi + \\
&+ \begin{bmatrix} \overset{k}{\mathbf{P}}_{11}(x) & \overset{k}{\mathbf{P}}_{12}(x) \\ \overset{k}{\mathbf{P}}_{22}(x) & \overset{k}{\mathbf{P}}_{22}(x) \end{bmatrix} \int_a^b \begin{bmatrix} \mathbf{G}_{11}^{(k)}(x,\xi) & \mathbf{G}_{12}^{(k)}(x,\xi) \\ \mathbf{G}_{21}^{(k)}(x,\xi) & \mathbf{G}_{22}^{(k)}(x,\xi) \end{bmatrix} \begin{bmatrix} \mathbf{r}_1(\xi) \\ \mathbf{r}_2(\xi) \end{bmatrix} d\xi + \\
&+ \begin{bmatrix} \overset{n}{\mathbf{P}}_{11}(x)\left\{\mathbf{G}_{11}^{(k-1)}(x,x-0) - \mathbf{G}_{11}^{(k-1)}(x,x+0)\right\} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1(x) \\ \mathbf{r}_2(x) \end{bmatrix} + \dots
\end{aligned}$$

Taking into account the first property of the definition and the fact that due to the second property the sum of integrals vanishes substitute the value of the jump. This transformation results in the right side of (2.1), i.e., the representation (2.4) really satisfies the differential equation (2.1).

Let $\mathbf{r}(\xi) = \mathbf{e}(\eta)\delta(\xi - \eta)$ where $\mathbf{e}$ is a constant at $\eta \in [a, b]$. It follows from (2.4) that

$$\mathbf{y}(x) = \mathbf{G}(x, \eta)\mathbf{e}$$

In other words the columns of $\mathbf{G}(x, \eta)$ are the solutions due to the unit discontinuities $\delta(\xi - \eta)\mathbf{e}_i$ in which $\mathbf{e}_i$ is the i-th unit column matrix.

3. Existence of the Green matrix function

The general solution of the differential equation $\mathbf{K}(\mathbf{y}) = 0$ is of he form

$$\mathbf{y} = \left[\sum_{i=1}^{n} \underset{(l\times l)}{\mathbf{Y}_i} \underset{(l\times l)}{\mathbf{C}_i}\right] \underset{(l\times 1)}{\mathbf{e}} \tag{3.5}$$

where $\mathbf{C}_i$ is a constant non-singular matrix and $\mathbf{e}$ is a constant vector.

For the sake of distinguishing the columns in $\mathbf{Y}_i$ they are denoted by $\mathbf{y}_{i\nu}$ $\nu = 1,\dots,l$. Obviously $\bar{l}$ columns are identically zero. Consider the hypermatrix

$$\mathbf{D} = \begin{bmatrix} \mathbf{U}_1[\mathbf{y}_{11}] & \dots & \mathbf{U}_1[\mathbf{y}_{1l}] & \dots & \mathbf{U}_1[\mathbf{y}_{i\nu}] & \dots & \mathbf{U}_1[\mathbf{y}_{nl}] \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{U}_n[\mathbf{y}_{11}] & \dots & \mathbf{U}_n[\mathbf{y}_{1l}] & \dots & \mathbf{U}_n[\mathbf{y}_{i\nu}] & \dots & \mathbf{U}_n[\mathbf{y}_{nl}] \end{bmatrix} . \tag{3.6}$$

Taking into account what has been said about the boundary conditions – here we think of the structure of the matrix $\mathbf{P}_{\underline{f}}$ – and recalling that $\bar{l}$ columns are identically zero one can come to the conclusion that $\bar{l}$ rows and columns are identically zero in the hypermatrix $\mathbf{D}$. Denoting by red($\mathbf{D}$) the matrix obtained by removing the zero rows and columns one can establish the following statement:

If $\det[\text{red}(\mathbf{D})] \neq 0$ then there exists a uniquely determined Green matrix function which meets the properties 1. to 4. of the definition. In addition the solution given by (2.4) is the only solution to the boundary value problem (2.1), (2.2) for arbitrary right side $\mathbf{r}$.

The proof of our statement is similar to that given in [1].

With regard to the third property of the definition it can readily be seen that $\mathbf{G}(x,\xi)$ takes the form

$$\mathbf{G}(x,\xi) = \textstyle\sum_{i=1}^{n} \mathbf{Y}_i(x)\,[\mathbf{A}_i(\xi) + \mathbf{B}_i(\xi)] \qquad x \leq \xi \tag{3.7a}$$
$$\mathbf{G}(x,\xi) = \textstyle\sum_{i=1}^{n} \mathbf{Y}_i(x)\,[\mathbf{A}_i(\xi) - \mathbf{B}_i(\xi)] \qquad x \geq \xi \tag{3.7b}$$

where $\mathbf{A}_i(\xi)$ and $\mathbf{B}_i(\xi)$ are $l \times l$ matrices. After partitioning the matrices $\mathbf{Y}_i$ and $\mathbf{B}_i$

$$\begin{matrix} l-j\,\{ \\ j\,\{ \end{matrix} \underbrace{\begin{bmatrix} \mathbf{Y}_{i1} \\ \mathbf{Y}_{i2} \end{bmatrix}}_{l\times l} \qquad \underbrace{\begin{bmatrix} \overbrace{\mathbf{B}_{i1}}^{l-j} & \overbrace{\mathbf{B}_{i2}}^{j} \end{bmatrix}}_{l\times l}$$

from the second property of the definition we obtain the system of equations

$$\begin{bmatrix} \sum_i \mathbf{Y}_{i1}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i1}\mathbf{B}_{i2} \\ \sum_i \mathbf{Y}_{i2}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i2}\mathbf{B}_{i2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.8a}$$

$$\begin{bmatrix} \sum_i \mathbf{Y}_{i1}^{(1)}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i1}^{(1)}\mathbf{B}_{i2} \\ \sum_i \mathbf{Y}_{i2}^{(1)}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i2}^{(1)}\mathbf{B}_{i2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.8b}$$

$$\dots = \dots$$

$$\begin{bmatrix} \sum_i \mathbf{Y}_{i1}^{(k-1)}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i1}^{(k-1)}\mathbf{B}_{i2} \\ \sum_i \mathbf{Y}_{i2}^{(k-1)}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i2}^{(k-1)}\mathbf{B}_{i2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\overset{k}{\mathbf{P}}{}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.8c}$$

$$\begin{bmatrix} \sum_i \mathbf{Y}_{i2}^{(k)}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i2}^{(k)}\mathbf{B}_{i2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{3.8d}$$

$$\dots = \dots$$

$$\begin{bmatrix} \sum_i \mathbf{Y}_{i2}^{(n-1)}\mathbf{B}_{i1} & \sum_i \mathbf{Y}_{i2}^{(n-1)}\mathbf{B}_{i2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}\overset{n}{\mathbf{P}}{}_{22}^{-1} \end{bmatrix} \tag{3.8e}$$

We denote the ν-th column vector of $\mathbf{B}_i$ by $\mathbf{B}_{i\nu}$. Let

$$\mathbf{b}_\nu^T = \left[\mathbf{B}_{1\nu}^T | \dots | \mathbf{B}_{i\nu}^T | \dots | \mathbf{B}_{n\nu}^T\right] \tag{3.9}$$

where the upper index T stands for the transpose of a matrix. The system of equations we have just established has the same coefficient matrix for each unknown vector $\mathbf{b}_\nu$:

$$\mathbf{W}\mathbf{b}_\nu = \mathbf{P}_\nu \tag{3.10}$$

where

$$\mathbf{W} = \begin{bmatrix}
\mathbf{y}_{11\,1} & \cdots & \mathbf{y}_{1l\,1} & \cdots & \mathbf{y}_{i\nu\,1} & \cdots & \mathbf{y}_{n1\,1} & \cdots & \mathbf{y}_{nl\,1} \\
\mathbf{y}_{11\,2} & \cdots & \mathbf{y}_{1l\,2} & \cdots & \mathbf{y}_{i\nu\,2} & \cdots & \mathbf{y}_{n1\,2} & \cdots & \mathbf{y}_{nl\,2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{y}_{11\,1}^{(k-1)} & \cdots & \mathbf{y}_{1l\,1}^{(k-1)} & \cdots & \mathbf{y}_{i\nu\,1}^{(k-1)} & \cdots & \mathbf{y}_{n1\,1}^{(k-1)} & \cdots & \mathbf{y}_{nl\,1}^{(k-1)} \\
\mathbf{y}_{11\,2}^{(k-1)} & \cdots & \mathbf{y}_{1l\,2}^{(k-1)} & \cdots & \mathbf{y}_{i\nu\,2}^{(k-1)} & \cdots & \mathbf{y}_{n1\,2}^{(k-1)} & \cdots & \mathbf{y}_{nl\,2}^{(k-1)} \\
\mathbf{y}_{11\,1}^{(k)} & \cdots & \mathbf{y}_{1l\,1}^{(k)} & \cdots & \mathbf{y}_{i\nu\,1}^{(k)} & \cdots & \mathbf{y}_{n1\,1}^{(k)} & \cdots & \mathbf{y}_{nl\,1}^{(k)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{y}_{11\,1}^{(n-1)} & \cdots & \mathbf{y}_{1l\,1}^{(n-1)} & \cdots & \mathbf{y}_{i\nu\,1}^{(n-1)} & \cdots & \mathbf{y}_{n1\,1}^{(n-1)} & \cdots & \mathbf{y}_{nl\,1}^{(n-1)} \\
\mathbf{y}_{11\,2}^{(n-1)} & \cdots & \mathbf{y}_{1l\,2}^{(n-1)} & \cdots & \mathbf{y}_{i\nu\,2}^{(n-1)} & \cdots & \mathbf{y}_{n1\,2}^{(n-1)} & \cdots & \mathbf{y}_{nl\,2}^{(n-1)}
\end{bmatrix}$$

in which

$$\mathbf{y}_{i\nu} = \begin{bmatrix} \mathbf{y}_{i\nu\,1} \\ \mathbf{y}_{i\nu\,2} \end{bmatrix} \begin{matrix} \} \, l-j \\ \} \, j \end{matrix}$$

and $\mathbf{P}_\nu$ is the ν-th column in the transpose of the matrix

columns in the blocks

$$\begin{array}{ccccccccccc}
l & & l & l-j & j & j & & j & j & & \\
\left[\begin{matrix} 0 \\ 0 \end{matrix}\right. & \begin{matrix} \dots \\ \dots \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} -\frac{1}{2}\left(\overset{k}{\mathbf{P}}{}_{11}^{-1}\right)^T \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} \dots \\ \dots \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ -\frac{1}{2}\left(\overset{n}{\mathbf{P}}{}_{22}^{-1}\right)^T \end{matrix}\right] & \begin{matrix} j \\ l\text{ - }j \end{matrix} \\
1 & & k-1 & k & k+1 & & & n-1 & n & &
\end{array}$$

block number

Observe that $\bar{l}$ columns in the matrix $\mathbf{W}$ are identically zero. Consequently the elements of $\mathbf{b}_\nu$ with the same index are also set to zero. Omitting the zero columns in $\mathbf{W}$ one obtains a square matrix denoted by red$(\mathbf{W})$.

Introducing new variables the system of ODEs $\mathbf{L}(\mathbf{y})$ can be replaced by a system of ordinary differential equations of order one. The new variables can always be chosen in such a way that det [red$(\mathbf{W})$] coincides with the Wronsky determinant of the system of differential equations of order one. Since the Wronsky determinant differs from zero and the vector $\mathbf{P}_\nu$ has at least one non zero element the linear equation system (3.10) is soluble, i.e., there exists a solution different from the trivial one.

In the knowledge of $\mathbf{B}_i$ the functions $\mathbf{A}_i$ can be determined from the fourth property of the definition. Let now $\boldsymbol{\alpha}$ be the ν-th unit vector in the space of size $l \times l$. Then it follows from the fourth property that

$$\mathbf{U}_\mu \left[\sum_{i=1}^{n} \mathbf{Y}_i(x) \mathbf{A}_i(\xi) \boldsymbol{\alpha} \right] = \mp \mathbf{U}_\mu \left[\sum_{i=1}^{n} \mathbf{Y}_i(x) \mathbf{B}_i(\xi) \boldsymbol{\alpha} \right] . \tag{3.11}$$

Let $\mathbf{A}_{i\nu}$ be the ν-th column vector in $\mathbf{A}_i$. Further let

$$\mathbf{a}_\nu^T = \left[\mathbf{A}_{1\nu}^T | \ldots | \mathbf{A}_{i\nu}^T | \ldots | \mathbf{A}_{n\nu}^T \right] .$$

Due to the choice of $\boldsymbol{\alpha}$ from (3.11) one obtains the equation system

$$\mathbf{U}_\mu \left[\mathbf{Y}_1(x) | \ldots | \mathbf{Y}_n(x) \right] \mathbf{a}_\nu(\xi) = \mp \mathbf{U}_\mu \left[\mathbf{Y}_1(x) | \ldots | \mathbf{Y}_n(x) \right] \mathbf{b}_\nu(\xi) \quad \mu = 0, \ldots, n-1$$

Taking into account the linearity of $\mathbf{U}_\mu$ and the notational convention we made for the column vectors of $\mathbf{Y}_i$ it can readily be seen, that the coefficient matrix of the equation system above coincide with the matrix $\mathbf{D}$. For the same reasons as before the elements of $\mathbf{a}_\nu$ with the same index as those elements of $\mathbf{b}_\nu$ which have been regarded zero are also set to zero. Taking into account the third property of the definition one can come to the conclusion that $\bar{l}$ equations are identically zero. It follows from all that has been said that the elements of $\mathbf{a}_\nu$ being not identically zero are obtainable from an equation system with the determinant $\det[\mathrm{red}(\mathbf{D})]$. The later quantity is not equal to zero, consequently the equation system is soluble, i.e., there exists the Green matrix function.

4. Eigenvalue problems for degenerate systems of ODEs

Let the system of differential equations read

$$\mathbf{K}[\mathbf{y}] = \lambda \mathbf{y} \tag{4.13}$$

where $\mathbf{K}[\mathbf{y}]$ is given by (2.1) and λ is a parameter (the eigenvalue sought). The system of ODEs (4.13) is associated with the linear homogeneous boundary conditions (2.1). We shall assume that the boundary conditions are independent of λ.

The vectors

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \begin{matrix} \} \, l-j \\ \} \, j \end{matrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \begin{matrix} \} \, l-j \\ \} \, j \end{matrix}$$

are said to be comparison vectors if they are different from zero, satisfy the boundary conditions, and ($\mathbf{u}_1$ and $\mathbf{v}_1$) [$\mathbf{u}_2$ and $\mathbf{v}_2$] posses continuous derivatives of order (k) [n].

The eigenvalue problem (4.13), (2.2) is self adjoint if the product

$$(\mathbf{u},\mathbf{v})_M = \int_a^b \mathbf{u}^T\mathbf{K}[\mathbf{v}]dx \tag{4.14}$$

is commutative, i.e., $(\mathbf{u},\mathbf{v})_M = (\mathbf{v},\mathbf{u})_M$ over the set of comparison vectors and it is full definite if $(\mathbf{u},\mathbf{u})_M > 0$ for any comparison vector $\mathbf{u}$.

If the eigenvalue problem (4.13), (2.2) is self adjoint then the Green matrix function of the boundary value problem (2.1), (2.2) is cross symmetric:

$$\mathbf{G}(x,\xi) = \mathbf{G}^T(\xi,x)$$

The proof of this statement is left to the reader – see [1] for the line of thought. Recalling (2.4) the eigenvalue problem (4.13), (2.2) can be replaced by the integral equation

$$\mathbf{y}(x) = \lambda \int_a^b \mathbf{G}(x,\xi)\mathbf{y}(\xi)d\xi \tag{4.15}$$

Numerical solution of the eigenvalue problem (4.15) can be sought by quadrature methods [8]. Consider the integral formula

$$J(\boldsymbol{\phi}) = \int_a^b \boldsymbol{\phi}(x)dx \equiv \sum_{j=0}^{n} w_j\boldsymbol{\phi}(x_j) \qquad x_j \in [a,b] \tag{4.16}$$

where $\boldsymbol{\phi}(x)$ is a vector and the weights w_j are known. Making use of the latter equation we obtain from (4.15) that

$$\sum_{j=0}^{n} w_j\mathbf{G}(x,x_j)\tilde{\mathbf{y}}(x_j) = \tilde{\kappa}\tilde{\mathbf{y}}(x) \qquad \tilde{\kappa} = 1/\tilde{\lambda} \qquad x \in [a,b] \tag{4.17}$$

the solution of which yields an approximate eigenvalue $\tilde{\lambda} = 1/\tilde{\kappa}$ and an approximate eigenfunction $\tilde{\mathbf{y}}(x)$. After setting x to x_i $(i = 0,1,2,\dots,n)$ we have

$$\sum_{j=0}^{n} w_j\mathbf{G}(x_i,x_j)\tilde{\mathbf{y}}(x_j) = \tilde{\kappa}\tilde{\mathbf{y}}(x) \qquad \tilde{\kappa} = 1/\tilde{\lambda} \qquad x \in [a,b] \tag{4.18}$$

or

$$\mathcal{G}\mathcal{D}\tilde{\mathcal{Y}} = \tilde{\kappa}\tilde{\mathcal{Y}} \tag{4.19}$$

where $\mathcal{G} = [\mathbf{G}(x_i,x_j)]$ is symmetric if the problem is self adjoint,

$$\mathcal{D} = \text{diag}(\underbrace{w_0,\dots,w_0}_{l}|\dots|\underbrace{w_n,\dots,w_n}_{l}) \quad \text{and} \quad \tilde{\mathcal{Y}}^T = [\tilde{\mathbf{y}}^T(x_0)|\tilde{\mathbf{y}}^T(x_1)|\dots|\tilde{\mathbf{y}}^T(x_n)].$$

After solving the generalized algebraic eigenvalue problem (4.19) we have the approximate eigenvalues $\tilde{\lambda}_r$ and eigenvectors $\mathcal{Y}_r$ while the corresponding eigenfunction is obtained by back substitution into (4.17):

$$\tilde{\mathbf{y}}_r(x) = \tilde{\lambda}_r \sum_{j=0}^{n} w_j \mathbf{G}(x, x_j)\tilde{\mathbf{y}}_r(x_j) \tag{4.20}$$

Divide the interval $[a, b]$ into equidistant subintervals of length h and apply the integration formula to each subinterval. By repeating the line of thought leading to (4.19) one can show that the algebraic eigenvalue problem obtained is of the same structure as (4.19).

It is also possible to consider the integral equation (4.15) as if it were a boundary integral equation and to apply isoparametric approximation on the subintervals, i.e., on the elements. If this is the case one can approximate the eigenfunction on the e-th element (the e-th subinterval which is mapped onto the interval $\eta \in [-1, 1]$ and is denoted by $\mathfrak{L}_e$) by

$$\overset{e}{\mathbf{y}} = [\mathbf{N}_1(\eta)|\mathbf{N}_2(\eta)|\mathbf{N}_3(\eta)] \begin{bmatrix} \overset{e}{\mathbf{y}}_1 \\ \overset{e}{\mathbf{y}}_2 \\ \overset{e}{\mathbf{y}}_3 \end{bmatrix} \tag{4.21}$$

where quadratic local approximation is assumed, $\mathbf{N}_i = \mathrm{diag}(N_i)$, $N_1 = 0.5\eta(\eta - 1)$, $N_2 = 1 - \eta^2$, $N_3 = 0.5\eta(\eta + 1)$, $\overset{e}{\mathbf{y}}_i$ is the value of the eigenfunction $\mathbf{y}(x)$ at the left endpoint, the midpoint and the right endpoint of the element, respectively. Substituting the equation (4.21) into (4.15) we have

$$\tilde{\mathbf{y}}(x) = \tilde{\lambda} \sum_{e=1}^{n_{be}} \int_{\mathfrak{L}_e} \mathbf{G}(x, \eta)[\mathbf{N}_1(\eta)|\mathbf{N}_2(\eta)|\mathbf{N}_3(\eta)] d\eta \begin{bmatrix} \overset{e}{\mathbf{y}}_1 \\ \overset{e}{\mathbf{y}}_2 \\ \overset{e}{\mathbf{y}}_3 \end{bmatrix} \tag{4.22}$$

in which n_{be} is the number of elements (subintervals). Using equation (4.22) as a point of departure and repeating the line of thought leading to (4.19) one shall find again an algebraic eigenvalue problem.

5. In plane vibrations of circular arches

Let R be the radius. It is assumed that the arch is symmetric with respect to the plane of its center line. The cross sectional area and the second moment of inertia with respect to the centroidal axis y are denoted by A and $I = I_y$ respectively. The angle coordinate φ changes in the interval $[-\vartheta, \vartheta]$, the central angle ϑ subtended by the arch is equal to 2ϑ while U and V are the amplitudes of the tangential and normal displacements on the center line. The Young modulus of elasticity is denoted by E. In plane vibrations of circular arches [2] is governed by the system of differential equations

$$\mathbf{K}[\mathbf{y}] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}^{(4)} + \begin{bmatrix} -\tilde{m} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}^{(2)} +$$

$$+ \begin{bmatrix} 0 & -\tilde{m} \\ \tilde{m} & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} = \lambda \begin{bmatrix} U \\ W \end{bmatrix} = \lambda \mathbf{y} \tag{5.23}$$

where $(\dots)^{(n)} = d^{(n)}/d\varphi^{(n)}$,

$$m = \frac{AR^2}{I}, \qquad \tilde{m} = m - 1 \qquad \text{and} \qquad \lambda = \frac{R^4}{IE} q\alpha^2 \tag{5.24}$$

in which q is the mass per unit length and α is the circular frequency of the free vibrations. Depending on the supports applied the system of ODS (5.24) is associated with the following boundary conditions:

Simple supported arch:

$$U|_{-\vartheta} = 0, \quad W|_{-\vartheta} = 0, \quad W^{(2)}\big|_{-\vartheta} = 0; \qquad U|_{\vartheta} = 0, \quad W|_{\vartheta} = 0, \quad W^{(2)}\big|_{\vartheta} = 0 \tag{5.25}$$

Fixed arch:

$$U|_{-\vartheta} = 0, \quad W|_{-\vartheta} = 0, \quad W^{(1)}\big|_{-\vartheta} = 0; \qquad U|_{\vartheta} = 0, \quad W|_{\vartheta} = 0, \quad W^{(1)}\big|_{\vartheta} = 0 \tag{5.26}$$

One can show by using the definitions given in Section 4 that each of the eigenvalue problems (5.24),(5.25) and (5.24), (5.26) is self adjoint and positive definite if $E > 0$. Under this condition the eigenvalues are strictly positive.

It is worthy of mention the paper [9] by Lin in which the Green function is constructed for curved Thimosenko beams by reducing the problem to an ordinary differential equation.

It follows from the equations (3.7a,b) that the corresponding Green function matrix assumes the form

$$\underbrace{\mathbf{G}(\varphi, \psi)}_{(2\times 2)} = \sum_{i=1}^{4} \mathbf{Y}_i(\varphi) \left[\mathbf{A}_i(\psi) \pm \mathbf{B}_i(\psi)\right] \tag{5.27}$$

where the sign is {positive}[negative] if $\{\varphi \leq \psi\}[\varphi \geq \psi]$,

$$\mathbf{Y}_1 = \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix} \qquad \mathbf{Y}_2 = \begin{bmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \end{bmatrix}$$
$$\mathbf{Y}_3 = \begin{bmatrix} -\sin\varphi + \varphi\cos\varphi & (\tilde{m}+1)\varphi \\ \varphi\sin\varphi & -\tilde{m} \end{bmatrix} \qquad \mathbf{Y}_4 = \begin{bmatrix} -\cos\varphi - \varphi\sin\varphi & 1 \\ \varphi\cos\varphi & 0 \end{bmatrix} \tag{5.28}$$

$$\mathbf{A}_i = \begin{bmatrix} \overset{i}{A}_{11} & \overset{i}{A}_{12} \\ \overset{i}{A}_{21} & \overset{i}{A}_{22} \end{bmatrix} \qquad \mathbf{B}_i = \begin{bmatrix} \overset{i}{B}_{11} & \overset{i}{B}_{12} \\ \overset{i}{B}_{21} & \overset{i}{B}_{22} \end{bmatrix} \qquad i = 1, \dots, 4 \tag{5.29}$$

As we have already seen in Section 3 – see the equations (3.8a,...,3.8e) – the second property of the definition yields the equation system

$$\begin{bmatrix} \cos\psi & -\sin\psi & -\sin\psi+\psi\cos\psi & (1+\tilde m)\psi & -\cos\psi-\psi\sin\psi & 1 \\ \sin\psi & \cos\psi & \psi\sin\psi & -\tilde m & \psi\cos\psi & 0 \\ -\sin\psi & -\cos\psi & -\psi\sin\psi & 1+\tilde m & -\psi\cos\psi & 0 \\ \cos\psi & -\sin\psi & \psi\cos\psi+\sin\psi & 0 & -\psi\sin\psi+\cos\psi & 0 \\ -\sin\psi & -\cos\psi & -\psi\sin\psi+2\cos\psi & 0 & -\psi\cos\psi-2\sin\psi & 0 \\ -\cos\psi & \sin\psi & -\psi\cos\psi-3\sin\psi & 0 & \psi\sin\psi-3\cos\psi & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{B}_{11} & \overset{1}{B}_{12} \\ \overset{2}{B}_{11} & \overset{2}{B}_{12} \\ \overset{3}{B}_{11} & \overset{3}{B}_{12} \\ \overset{3}{B}_{21} & \overset{3}{B}_{22} \\ \overset{4}{B}_{11} & \overset{4}{B}_{12} \\ \overset{4}{B}_{21} & \overset{4}{B}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{2\tilde m} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\dfrac{1}{2} \end{bmatrix} \tag{5.30}$$

from which

$$\begin{aligned} \overset{1}{B}_{11} &= \frac{1}{2}\sin\psi - \frac{1}{4}\psi\cos\psi & \overset{1}{B}_{12} &= -\frac{1}{4}\cos\psi - \frac{1}{4}\psi\sin\psi \\ \overset{2}{B}_{11} &= \frac{1}{4}\psi\sin\psi + \frac{1}{2}\cos\psi & \overset{2}{B}_{12} &= \frac{1}{4}\sin\psi - \frac{1}{4}\psi\cos\psi \\ \overset{3}{B}_{11} &= \frac{1}{4}\cos\psi & \overset{3}{B}_{12} &= \frac{1}{4}\sin\psi \\ \overset{3}{B}_{21} &= \frac{1}{2\tilde m} & \overset{3}{B}_{22} &= 0 \\ \overset{4}{B}_{11} &= -\frac{1}{4}\sin\psi & \overset{4}{B}_{12} &= \frac{1}{4}\cos\psi \\ \overset{4}{B}_{21} &= -\frac{1}{2}\left(1+\tilde m\right)\frac{\psi}{\tilde m} & \overset{4}{B}_{21} &= \frac{1}{2} \end{aligned} \tag{5.31}$$

Observe that the functions $\overset{1}{B}_{11}(\psi),\dots,\overset{4}{B}_{21}(\psi);\psi\in[-\vartheta,\vartheta]$ are independent of the boundary conditions. For the sake of simplicity we introduce the following notations

$$a=\overset{1}{B}_{1i},\ b=\overset{2}{B}_{1i},\ c=\overset{3}{B}_{1i},\ d=\overset{3}{B}_{2i},\ e=\overset{4}{B}_{1i},\ f=\overset{4}{B}_{2i} \qquad i=1,2$$

Taking into account the boundary conditions (5.25), (5.26) and without entering into details we can establish the following linear equation systems for the functions

$$\overset{1}{A}_{11}(\psi),\dots,\overset{4}{A}_{21}(\psi);\psi\in[-\vartheta,\vartheta]$$

Simply supported arch:

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \sin\vartheta-\vartheta\cos\vartheta & -(1+\tilde m)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & -\sin\vartheta+\vartheta\cos\vartheta & (1+\tilde m)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -\tilde m & -\vartheta\cos\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -\tilde m & \vartheta\cos\vartheta & 0 \\ \sin\vartheta & -\cos\vartheta & -\vartheta\sin\vartheta+2\cos\vartheta & 0 & \vartheta\cos\vartheta+2\sin\vartheta & 0 \\ -\sin\vartheta & -\cos\vartheta & -\vartheta\sin\vartheta+2\cos\vartheta & 0 & -\vartheta\cos\vartheta-2\sin\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta-b\sin\vartheta-c(\sin\vartheta-\vartheta\cos\vartheta)+d(1+\tilde m)\vartheta+e(\cos\vartheta+\vartheta\sin\vartheta)-f \\ a\cos\vartheta-b\sin\vartheta-c(\sin\vartheta-\vartheta\cos\vartheta)+d(1+\tilde m)\vartheta-e(\cos\vartheta+\vartheta\sin\vartheta)+f \\ a\sin\vartheta-b\cos\vartheta-c\vartheta\sin\vartheta+d\tilde m+e\vartheta\cos\vartheta \\ a\sin\vartheta+b\cos\vartheta+c\vartheta\sin\vartheta-d\tilde m+e\vartheta\cos\vartheta \\ -a\sin\vartheta+b\cos\vartheta+c(\vartheta\sin\vartheta-2\cos\vartheta)-e(\vartheta\cos\vartheta+2\sin\vartheta) \\ -a\sin\vartheta-b\cos\vartheta+c(-\vartheta\sin\vartheta+2\cos\vartheta)-e(\vartheta\cos\vartheta+2\sin\vartheta) \end{bmatrix} \qquad (5.32)$$

Solving the equations (5.32) we have

$$\overset{1}{A}_{1i} = \frac{\overset{2}{B}_{1i}}{\sin\vartheta}\cos\vartheta+\frac{\overset{3}{B}_{1i}\vartheta}{\sin^2\vartheta}-\frac{\overset{3}{B}_{2i}\tilde m}{2\sin^2\vartheta}(\vartheta\cos\vartheta+2\sin\vartheta) \qquad (5.33a)$$

$$\overset{2}{A}_{1i} = \frac{1}{C}\left[\begin{matrix} \overset{1}{B}_{1i}\left(2(1+\tilde m)\vartheta\cos\vartheta\sin\vartheta-\tilde m+3\tilde m\cos^2\vartheta\right)+ \\ \overset{4}{B}_{1i}\left(3\vartheta^2\tilde m+2\vartheta^2-2\tilde m\right)+\overset{4}{B}_{2i}\tilde m\,(2\cos\vartheta-\vartheta\sin\vartheta) \end{matrix}\right] \qquad (5.33b)$$

$$\overset{3}{A}_{1i} = \frac{1}{C}\left[\overset{1}{B}_{1i}\tilde m-\overset{4}{B}_{1i}\left(\tilde m+2\vartheta(1+\tilde m)\cos\vartheta\sin\vartheta-3\tilde m\sin^2\vartheta\right)+\overset{4}{B}_{2i}\tilde m\cos\vartheta\right] \qquad (5.33c)$$

$$\overset{3}{A}_{2i} = \frac{2}{C}\left[\overset{1}{B}_{1i}\cos\vartheta+\overset{4}{B}_{1i}(\vartheta\sin\vartheta-\cos\vartheta)+\overset{4}{B}_{2i}\cos^2\vartheta\right] \qquad (5.33d)$$

$$\overset{4}{A}_{1i} = -\frac{1}{2\sin\vartheta}\left[2\overset{3}{B}_{1i}\cos\vartheta-\overset{3}{B}_{2i}\tilde m\right] \qquad (5.33e)$$

$$\overset{4}{A}_{2i} = -\frac{\overset{2}{B}_{1i}}{\sin\vartheta}-\frac{1}{\sin^2\vartheta}\left[\vartheta\cos\vartheta+\sin\vartheta\right]\overset{3}{B}_{1i} +\frac{1}{2\sin^2\vartheta}\left[3\tilde m\cos\vartheta\sin\vartheta-2(1+\tilde m)\vartheta\cos^2\vartheta+2\vartheta+3\vartheta\tilde m\right]\overset{3}{B}_{2i} \qquad (5.33f)$$

where

$$C=-3\tilde m\sin\vartheta\cos\vartheta+2(1+\tilde m)\vartheta\cos^2\vartheta+\vartheta\tilde m$$

Fixed arch:

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \sin\vartheta-\vartheta\cos\vartheta & -(1+\tilde m)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & -\sin\vartheta+\vartheta\cos\vartheta & (1+\tilde m)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -\tilde m & -\vartheta\cos\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -\tilde m & \vartheta\cos\vartheta & 0 \\ \cos\vartheta & \sin\vartheta & -\sin\vartheta-\vartheta\cos\vartheta & 0 & \cos\vartheta-\vartheta\sin\vartheta & 0 \\ \cos\vartheta & -\sin\vartheta & \sin\vartheta+\vartheta\cos\vartheta & 0 & \cos\vartheta-\vartheta\sin\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta-b\sin\vartheta-c(\sin\vartheta-\vartheta\cos\vartheta)+d(1+\tilde m)\vartheta+e(\cos\vartheta+\vartheta\sin\vartheta)-f \\ a\cos\vartheta-b\sin\vartheta-c(\sin\vartheta-\vartheta\cos\vartheta)+d(1+\tilde m)\vartheta-e(\cos\vartheta+\vartheta\sin\vartheta)+f \\ a\sin\vartheta-b\cos\vartheta-c\vartheta\sin\vartheta+d\tilde m+e\vartheta\cos\vartheta \\ a\sin\vartheta+b\cos\vartheta+c\vartheta\sin\vartheta-d\tilde m+e\vartheta\cos\vartheta \\ -a\cos\vartheta-b\sin\vartheta+c(\sin\vartheta+\vartheta\cos\vartheta)-e(\cos\vartheta-\vartheta\sin\vartheta) \\ a\cos\vartheta-b\sin\vartheta+c(\sin\vartheta+\vartheta\cos\vartheta)+e(\cos\vartheta-\vartheta\sin\vartheta) \end{bmatrix} \tag{5.34}$$

Solving the equation system (5.34) we have

$$\overset{1}{A}_{1i} = -\frac{1}{\vartheta-\sin\vartheta\cos\vartheta}\left[\overset{2}{B}_{1i}\cos^2\vartheta-\overset{3}{B}_{1i}\vartheta^2+\overset{3}{B}_{2i}\tilde m\left(\vartheta\sin\vartheta-\cos\vartheta\right)\right] \tag{5.35a}$$

$$\overset{2}{A}_{1i} = \frac{1}{C}\Big[\overset{1}{B}_{1i}\left[(1+\tilde m)\,\vartheta\sin\vartheta+2\tilde m\cos\vartheta\right]\sin\vartheta+\overset{4}{B}_{1i}\left[\vartheta^3\left(1+\tilde m\right)-2\vartheta\tilde m\right]+$$
$$+\overset{4}{B}_{2i}\tilde m\left(\sin\vartheta+\vartheta\cos\vartheta\right)\Big] \tag{5.35b}$$

$$\overset{3}{A}_{1i} = \frac{1}{C}\left[\overset{1}{B}_{1i}\vartheta\left(1+\tilde m\right)+\overset{4}{B}_{1i}\left[(1+\tilde m)\,\vartheta\cos\vartheta-2\tilde m\sin\vartheta\right]\cos\vartheta+\overset{4}{B}_{2i}\tilde m\sin\vartheta\right] \tag{5.35c}$$

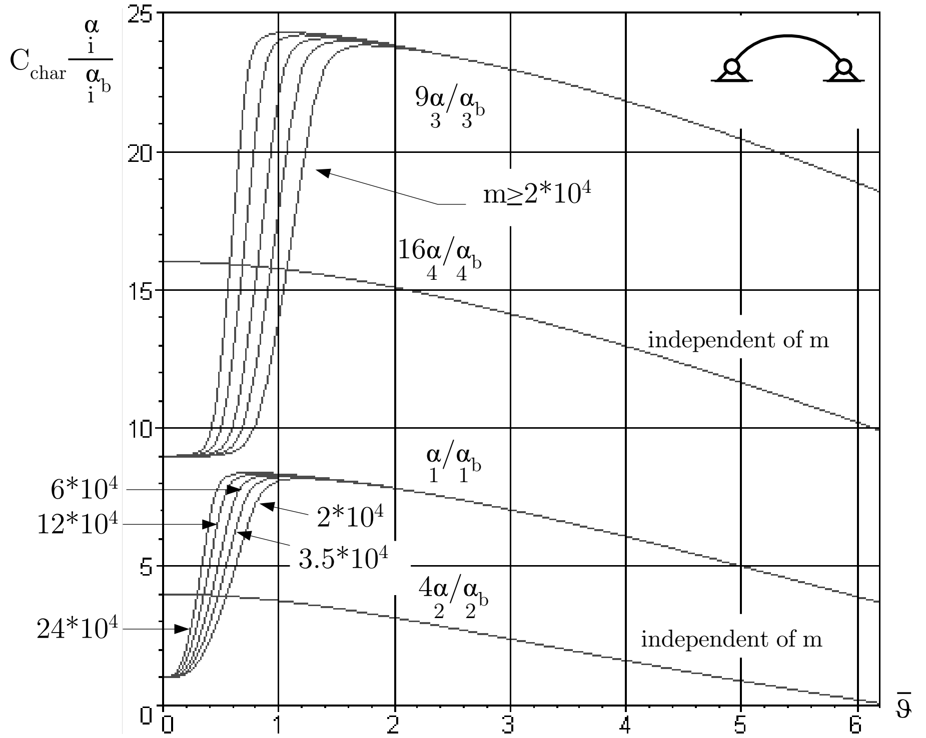
$$\overset{3}{A}_{2i} = \frac{2}{C}\left[\overset{1}{B}_{1i}\sin\vartheta-\overset{4}{B}_{1i}\vartheta\cos\vartheta+\frac{1}{2}\overset{4}{B}_{2i}\left(\vartheta+\sin\vartheta\cos\vartheta\right)\right] \tag{5.35d}$$

$$\overset{4}{A}_{1i} = \frac{1}{\vartheta-\sin\vartheta\cos\vartheta}\left[\overset{2}{B}_{1i}-\overset{3}{B}_{1i}\sin^2\vartheta-\overset{3}{B}_{2i}\tilde m\cos\vartheta\right] \tag{5.35e}$$

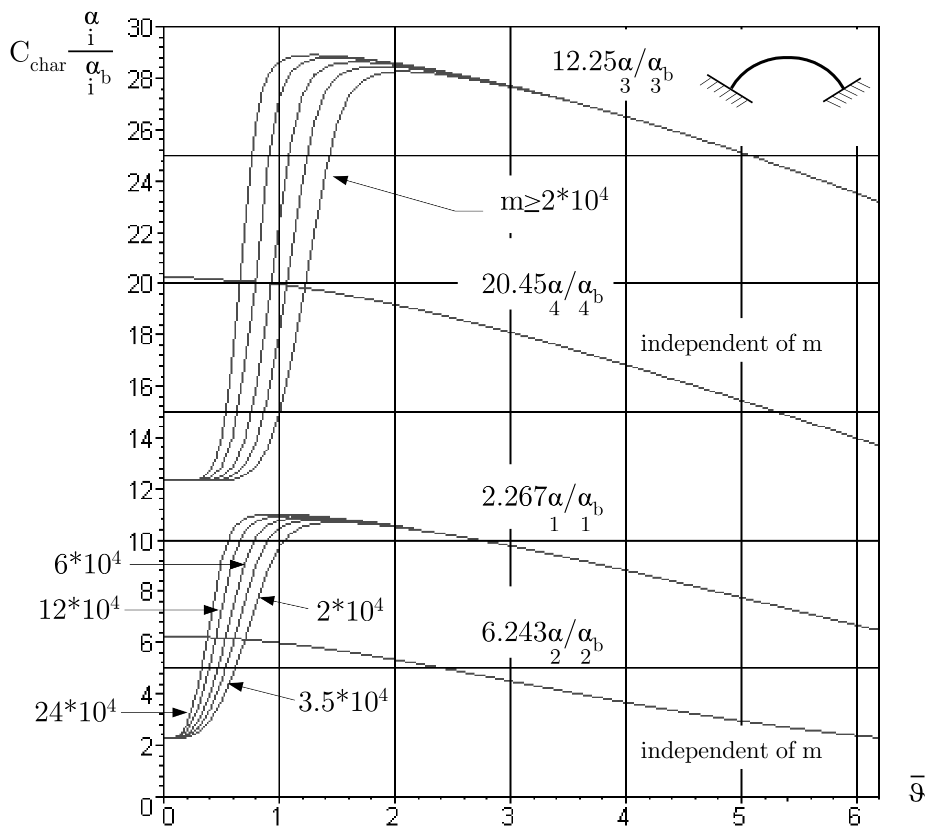
$$\overset{4}{A}_{2i} = \frac{1}{\vartheta-\sin\vartheta\cos\vartheta}\Big[2\overset{2}{B}_{1i}\cos\vartheta-2\overset{3}{B}_{1i}\vartheta\sin\vartheta-$$
$$-\overset{3}{B}_{2i}\left[2m\cos^2\vartheta-(1+m)\left(\vartheta^2-\vartheta\sin\vartheta\cos\vartheta\right)\right]\Big] \tag{5.35f}$$

where

$$C=\vartheta(\vartheta+\cos\vartheta\sin\vartheta)(1+\tilde m)-2\tilde m\sin^2\vartheta$$



$C_{char}\frac{\alpha_i}{\alpha_i^b}$
$9\alpha_3/\alpha_3^b$
m≥2*10^4
$16\alpha_4/\alpha_4^b$
independent of m
α_1/α_1^b
6*10^4
12*10^4
2*10^4
3.5*10^4
$4\alpha_2/\alpha_2^b$
24*10^4
independent of m
$\bar{\vartheta}$



$C_{char}\frac{\alpha_i}{\alpha_i^b}$
$12.25\alpha_3/\alpha_3^b$
m≥2*10^4
$20.45\alpha_4/\alpha_4^b$
independent of m
$2.267\alpha_1/\alpha_1^b$
6*10^4
12*10^4
2*10^4
$6.243\alpha_2/\alpha_2^b$
3.5*10^4
24*10^4
independent of m
$\bar{\vartheta}$

Figure 1.

In the knowledge of the functions $\overset{1}{B}_{11}(\psi), \ldots, \overset{4}{B}_{21}(\psi), \overset{1}{A}_{11}(\psi), \ldots, \overset{4}{A}_{21}(\psi);\ \psi \in [-\vartheta, \vartheta]$ we can substitute in the formula (5.27) from which we get the Green function matrix. Then the natural frequencies are obtained by solving the eigenvalue problem

$$\mathbf{y}(\varphi) = \lambda \int_a^b \mathbf{G}(\varphi, \psi)\mathbf{y}(\psi)d\psi \tag{5.36}$$

Two numerical procedures were used for the computation of the natural frequencies:

First we applied the repeated trapezium rule to obtain an algebraic eigenvalue problem. The latter was solved by QZ algorithm. We remark that the solution found for the approximate eigenvalues tends to the exact value as $h \to 0$ provided that the eigenvalue is simple and the corresponding theorems, which are valid for scalar integral equations [8], remain valid for our case.

Second we used the boundary integral equation approach to find an algebraic eigenvalue problem which was solved again by QZ algorithm. The natural frequencies we computed were the same for three to four digits. The results obtained are shown in Figures 1. to 2. The variable along the longitudinal axis is the central angle $\bar{\vartheta}$.

It is well known that the i-th eigenvalue of the differential equation governing the free vibrations of a beam (transverse vibrations) is related to the corresponding circular frequency by

$$\sqrt{\underset{i}{\lambda}_b} = \frac{q}{IE} l_b^2 \underset{i}{\alpha}_b = \underset{i}{C}_{char}\pi^2 \qquad i = 1, 2, 3, \ldots$$

where q, I_b and E are characteristic values for the beam while the value of $\underset{i}{C}_{char}$ depends on how the beam is supported:

Support	$i = 1$	$i = 2$	$i = 3$	$i = 4$
Simply supported	1.000	4.000	9.00	16.00
Fixed	2.266	6.243	12.23	20.25

The pair of the above relation for circular arches is

$$\sqrt{\underset{i}{\lambda}} = \frac{q}{IE} R^2 \underset{i}{\alpha} \qquad i = 1, 2, 3, \ldots \tag{5.37}$$

Assume that we are comparing a beam and a circular arch for which q/IE and the lengths are the same (i.e. $l_b = R\bar{\vartheta}$) and the supports are also the same. Then

$$\underset{i}{C}_{char} \frac{\underset{i}{\alpha}}{\underset{i}{\alpha}_b} = \underset{i}{C}_{char}\bar{\vartheta}^2 \sqrt{\frac{\underset{i}{\lambda}}{\underset{i}{\lambda}_b}} = \frac{\bar{\vartheta}^2}{\pi^2}\sqrt{\underset{i}{\lambda}} \tag{5.38}$$

Figures 1 represents the quotient (5.38) provided that

$$\lim_{\bar{\vartheta}\to 0} \underset{i}{\alpha}/\underset{i}{\alpha}_b = 1 \qquad i = 1, 2, 3, 4 \tag{5.39}$$

in which l is the same for both cases.

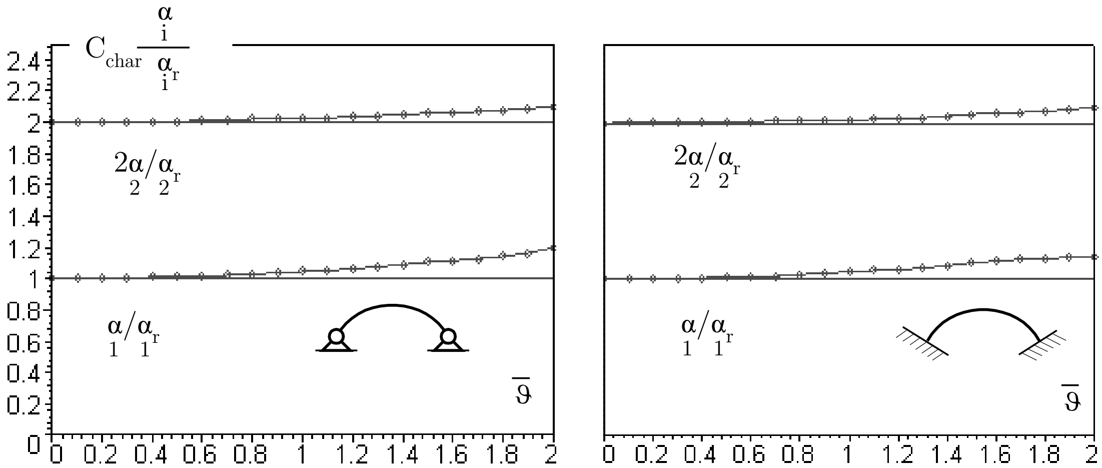


Figure 2.

It is also known that the i-th eigenvalue of the differential equation governing the free vibrations of a rod fixed at its ends (longitudinal vibrations) is related to the corresponding eigenvalue by

$$\sqrt{\underset{i}{\lambda}_r} = \sqrt{\frac{\rho}{E}}\underset{i}{\alpha}_r = \frac{\underset{i}{C}_{char}}{l_r}\pi \qquad i = 1, 2, 3, \ldots \tag{5.40}$$

where ρ is the mass per unit volume and $\underset{i}{C}_{char} = i, \; (i = 1, 2, 3, \ldots)$. Comparing (5.37) and (5.40) by taking into account (5.24) and the equation $q = \rho A$ and assuming again the same length, material and cross section we have

$$\underset{i}{C}_{char}\frac{\underset{i}{\alpha}}{\underset{i}{\alpha}_r} = \frac{1}{\sqrt{\tilde{m}}}\frac{\bar{\vartheta}}{\pi}\sqrt{\underset{i}{\lambda}} \tag{5.41}$$

Figure 2 represents the quotient (5.41) provided that

$$\lim_{\bar{\vartheta}\to 0} \underset{i}{\alpha}/\underset{i}{\alpha}_r = 1 \qquad i = 1, 2 \tag{5.42}$$

in which l is the same for both cases. We should also remark that the numbering reflects the magnitude for small central angles $\bar{\vartheta}$ only. The natural frequencies we computed are in very good agreement with those obtained by using a different method – see [2] for details.

6. Concluding remarks

In the paper a definition is presented for the Green function matrix of a class of degenerate system of ordinary differential equations. The existence of the Green function matrix has also been proved. Using the Green function matrix self adjoint eigenvalue problems governed by degenerate systems of differential equations and homogenous linear boundary conditions can be replaced by an eigenvalue problem for a system of Fredholm integral equations with the Green function matrix as kernel.

We have determined the Green function matrix for simply supported and fixed circular arches. In the knowledge of the Green function matrix the self adjoint eigenvalue problem giving the natural frequencies of the free vibrations of the two circular arches has been replaced by an eigenvalue problem described by a system of Fredholm integral equations. The latter is reduced to an algebraic eigenvalue problem and the first eigenvalues are computed by using the QZ algorithm.

Acknowledgement: The authors are grateful to Barna SZABÓ – Washington Univwersity, St. Louis, USA – and Miklós RONTÓ – University of Miskolc, Miskolc, Hungary – for their critical remarks.
The support provided within the framework of the Hungarian-German Intergovernment Cooperation Programme – project No. Tét D-27/96 – is gratefully acknowledged.

REFERENCES

[1] COLLATZ, L.: *Eigenwertaufgaben mit technischen Anwendungen.* Izdatelstvo Nauka, Moscow, 1968, (in Russian).

[2] SZEIDL, GY.: *Effect of Change in Length on the Natural Frequencies and Stability of Circular Arcs*, Ph.D Thesis, Department of Mechanics, University of Miskolc, Hungary, 1975, (in Hungarian).

[3] COLLATZ, L.: *The Numerical Treatment of Differential Equations*, Sprimger Verlag, Berlin – Göttingen – Heidelberg, 1960.

[4] FEDERHOFFER, K.: *Über den Einfluss der Achsendehnung der Rotationsträgheit und der Schubkraft auf die Frequenzen der Biegungsschwingungen eines Kreisringes.* Sitz-Ber. Akad. Wiss. Austria, Series IIa. Vol 144., 1935.

[5] VELETSOS, A.: *In Plane Vibration of Circular Arches*, Journal of the Engineering Mechanics Division, ASCE, Vol 98., (1972), 1311–1318.

[6] LAURA, P. A. A. and FILIPICH, C. P.: *First and Second Natural Frequences of Hinged and Clamped Circular Arcs: a Discussion of a Classical Paper*, Journal of Sound and Vibration, Vol 125. (1988), 393–396.

[7] MATSUNAGA, H.: *Effects of Higher Order Deformations on In-Plane Vibration and Stability of Thick Circular Rings*, Acta Mechanica, Vol. 124, (1997), 47–61.

[8] BAKER, C. T. H.: *The Numerical Treatment of Integral Equations – Monographs on Numerical Analysis edited by L. Fox and J. Walsh.* Clarendon Press, Oxford, 1977.

[9] LIN, S. M.: *Exact Solutions for Extensible Circular Curved Timoshenko Beams with Nonhomogenous Elastic Boundary Conditions*, Acta Mechanica, Vol. 130, (1998), 67–90.