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# ON COMPATIBILITY CONDITIONS FOR MIXED BOUNDARY VALUE PROBLEMS IN MICROPOLAR THEORY OF ELASTICITY

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**Abstract.** The main objective of the present paper is the investigation of the conditions of single-valuedness for multiply-connected micropolar bodies. Assuming mixed boundary value problems it has been shown that the supplementary conditions of single-valuedness including the compatibility conditions in the large are natural boundary conditions of the principle of minimum complementary energy as a variational principle.

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*Keywords*: variational principles, single–valuedness

**Zusammenfassung.** Der erste Teil des Artikels führt den Begriff einer neuen Ausschließungsfunktion ohne Ableitungen ein und gibt die grundlegenden charakteristischen Eigenschaften dieser Funktion an. Der neue Begriff wird im zweiten Teil zur Bestimmung des globalen Minimums einer speziellen Maximumfunktionen benutzt.

## 1. Introduction

For micropolar bodies the problem of the necessary number of stress functions to represent any state of stress and that of the necessary and sufficient number of compatibility conditions have been investigated in papers [1] and [2] in the same manner as in the classical case under the assumption of a simply-connected body and mixed boundary value problems [3,4,5].

By *macro conditions of compatibility* is meant the totality of those additional conditions the strains should meet to be compatible on a multiply–connected body. Depending on

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what the boundary conditions are in the points of a simply-connected and closed curve on the surface of the body the macro conditions of compatibility are separated into two groups. If tractions are imposed in each point of the curve the condition is referred to as a *compatibility condition in the large.* If there exists at least one arc on the curve along which displacements are imposed then the corresponding condition is called *supplementary condition of single–valuedness.*

The paper [6] was devoted to the problem of compatibility conditions on a triple connected body assuming that tractions are imposed and it had been pointed out that the compatibility conditions in the large can be obtained in two distinct ways both from the general dual form of principle of virtual work and from the principle of minimum complementary energy.

In view of the foregoing it seems to be an open question what supplementary conditions of single-valuedness are needed for mixed boundary value problems on multiply connected regions. On the bases of all that has been said the present paper is aimed at investigating the problem of what natural boundary conditions follow from the principle of minimum complementary energy for micropolar bodies under the assumption of a three dimensional and multiply-connected body and a certain class of mixed boundary value problems.

We shall also assume that the linear theory of deformations is valid. When applying Castigliano's principle in addition it will be assumed that the body is linearly elastic.

In section 2 we collect some preliminary results and detail some geometrical considerations. Section 3 is devoted to the problem of how the supplementary conditions of single valuedness can be obtained from the principle of minimum complementary energy. Section 4 is a summary of the results. Finally there is section 5 where some longer transformations are presented.

## 2. Preliminaries

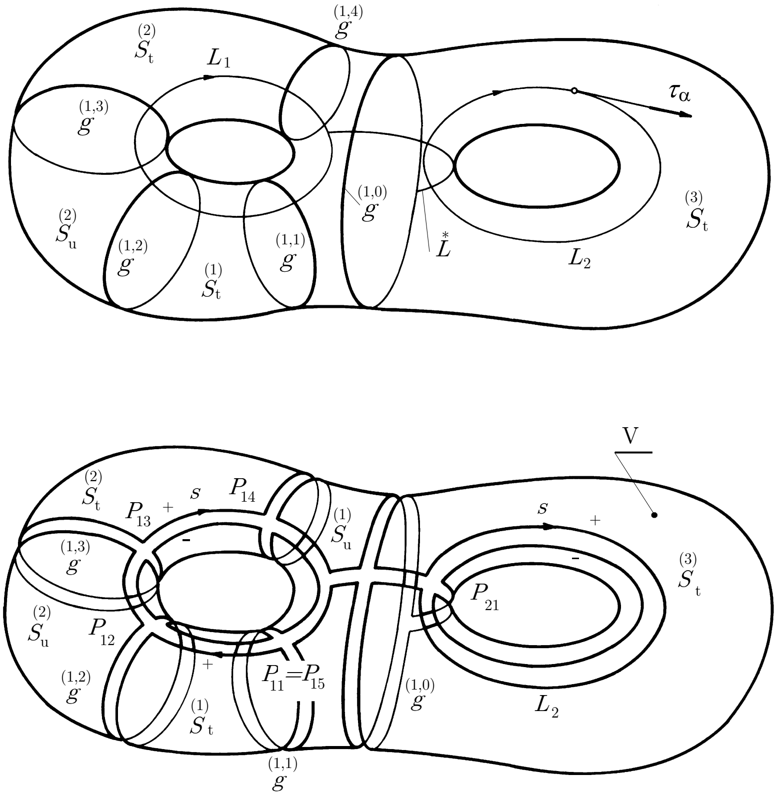
The bounded region of the three dimensional space occupied by the multiply-connected body and the surface of the body are denoted respectively by $V$ and $S$. In principle the surface $S$ of the body may consist of not only one but more closed surfaces, in which case the region is multiply-bordered, though the latter circumstance will play no role in the investigations. The common bounding curve of the parts $S_u$ and $S_t$ of $S$ is denoted by $g$.

The present paper restricts its attention to the triple-connected but single-bordered body represented in Fig.1 which contains some further notational conventions. It is clear from Fig.1 that both the subsurfaces $S_u$, $S_t$ and the curve $g$ consists of more parts, i.e.,

$$S_u = \overset{(1)}{S_u} \cup \overset{(2)}{S_u}; \qquad S_t = \overset{(1)}{S_t} \cup \overset{(2)}{S_t} \cup \overset{(3)}{S_t} \quad \text{and} \quad g = \overset{(1,0)}{g} \cup \overset{(1,1)}{g} \cup \overset{(1,2)}{g} \cup \overset{(1,3)}{g} \cup \overset{(1,4)}{g}$$

In a limit case any of the subsurfaces

$$\overset{(1)}{S_u}, \overset{(2)}{S_u}, \qquad \text{and} \qquad S_u \qquad \text{or} \qquad \overset{(1)}{S_t}, \overset{(2)}{S_t}, \overset{(3)}{S_t} \qquad \text{and} \qquad S_t$$




Figures 1(a) and (b)

may be an empty set. The non-intersecting, simple and closed curves $L_1$ and $L_2$ encircle the first and second holes.

It is essential for the further investigations that $\overset{(1)}{S}_u$ and $\overset{(2)}{S}_u$ are respectively triple and double connected surfaces in the way they are represented in Fig.1. The curves $L_1$ and $L_2$ intersect the curve $g$ and $\overset{*}{L}$ at the points $P_{11}$, $P_{12}$, $P_{13}$, $P_{14}$ and $P_{21}$. Let the parts of $L_1$ be defined by $L_{1j} = P_{1j}\,P_{1,j+1}$ $j = 1,\ldots,4$. When performing integral transformations by making use of Stokes' theorem one must keep in mind that the theorem is applicable under the condition that the surface considered is simply-connected. Fig.2. represents a possibility for cutting up the surface $S$ into simply-connected parts by utilizing the curves $g$, $L_1$, $L_2$ and $\overset{*}{L}$ on $S$.

It is worthy of mention that the restrictions we have made in connection with the body considered are not essential for the body is at least triple-connected and the partition of boundary surface $S$ is sufficiently general.

Indicial notations and three coordinate systems, the $(y^1, y^2, y^3)$ Cartesian, the $(x^1, x^2, x^3)$ curvilinear and the $(\xi^1, \xi^2, \xi^3)$ curvilinear, defined on the surface $S$, are employed throughout this paper. Scalars and tensors, unless the opposite is stated are denoted independently of the coordinate system by the same letter. Distinction is aided by the indication of the arguments $y$, $x$ and $\xi$ that are used to denote the totality of the corresponding coordinates. Volume and surface integrals – except formula (2.7) – are considered, respectively, in the coordinate systems $(x^1, x^2, x^3)$ and $(\xi^1, \xi^2, \xi^3)$. Consequently, in the case of integrals, the arguments are omitted.

In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a semicolon denote covariant differentiation with respect to the corresponding subscripts. Latin and Greek indices range over the integers 1, 2, 3 and 1, 2, respectively. $\epsilon^{klm}$ and $\epsilon_{pqr}$ stand for the permutation tensors; $\delta_k^l$ is the *Kronecker* delta. In the coordinate system $(x^1, x^2, x^3)$ the metric tensors are denoted by $g_{kl}$ and $g^{pq}$.

Let $x^k = x^k(\xi^1, \xi^2)$ be the equation of the surface $S$ where $\xi^1$ and $\xi^2$ are the surface coordinates. Further let $\xi^3$ be the perpendicular distance measured on the outward unit normal $\mathbf{n}$ to the surface $S$. On $S$ $\xi^3 = 0$ .[Base vectors] {Metric tensors} on $S$ are denoted by [$\mathbf{a}^k$ and $\mathbf{a}_k$] {$a_{kl}$ and $a^{kl}$}. In the surface oriented coordinate system $(\xi^1, \xi^2, \xi^3)$

$$\mathbf{n} = \mathbf{a}_3 = \mathbf{a}^3 \,, \qquad n^3 = 1 \qquad \text{and} \qquad n^\eta = 0 \tag{2.1}$$

We shall assume that the relationships $y^k = y^k(x^1, x^2, x^3)$ and $x^k = x^k(\xi^1, \xi^2, \xi^3)$ are both one to one.

Now we shall assemble equations of micropolar elastostatics – in a form suited to our objective – in primal and dual systems as well.

Let $u_k$ and $\varphi^b$ be the displacement field and the rotation field. (For brevity's sake $u_k$ and $\varphi^b$ are referred to as displacements). Further let $\gamma_{kl}$ and $\kappa_{a.}^{\ b}$ be the asymmetric strain tensor and curvature twist tensor, respectively (together strains). By $t^{kl}$ and $\mu^a_{.b}$ we denote the asymmetric stress tensor and couple-stress tensor (together stresses). Displacements and strains are assumed to be small.

In the primal system the three dimensional problems of micropolar elastostatics are governed by the kinematic equations

$$\gamma_{kl}(x) = u_{l;k} + \epsilon_{lks}\varphi^s \,, \qquad \kappa_{a.}^{\ b}(x) = \varphi^b_{.;a} \qquad x \in V \tag{2.2}$$

Hook's law (valid in this form for a centrosymmetric body)

$$t^{kl} = A^{klpq}\gamma_{pq} \,, \qquad \mu^{ab} = B^{abpq}\kappa_{pq} \qquad x \in V \tag{2.3}$$

and the equilibrium equations

$$t^{kl}_{..;k}(x) + b^l = 0 \,, \qquad \mu^a_{.b;a}(x) + \epsilon_{bkl}t^{kl} + c_b = 0 \qquad x \in V \tag{2.4}$$

where $A^{klpq}$ and $B^{abpq}$ are the tensors of elastic coefficients. Field equations (2.2-6) should be supplemented by the boundary conditions

$$u_k = \hat{u}_k \qquad \varphi^b = \hat{\varphi}^b \,, \qquad \xi \in S_u \tag{2.5}$$

$$n_k t^{kl} = \hat{t}^l \qquad n_a\mu^a_{.b} = \hat{\mu}_b \qquad \xi \in S_t \tag{2.6}$$

where $\hat{u}_k$ and $\hat{\varphi}^b$ are the prescribed displacements while $\hat{t}^l$ and $\hat{\mu}_b$ are the prescribed tractions.

Strains $\gamma_{kl}$ and $\kappa_{a.}^{\ b}$ are said to be [compatible] {kinematically admissible} if the kinematic equations (2.2) have sufficiently smooth single-valued solution to the displacements $u_l$ and $\varphi^b$ in $V$ and the solution [does not satisfy other conditions] {satisfies the displacement boundary conditions (2.5)}.

Stresses $t^{kl}$ and $\mu^a_{.b}$ are said to be [equilibrated] {statically admissible} if they satisfy the equilibrium equations (2.4) and [do not meet other conditions] {the stress boundary conditions (2.6)}.

Body forces $b^l$ and body couples $c_b$ can always be given in the form

$$b^l = -\Delta B^l = -g^{pq} B^l_{.;pq} \qquad c_b = -\Delta C^b = -g^{pq} C_{b;pq} \qquad x \in V$$

where $B^l(x)$ and $C_b(x)$, provided that the integrals

$$B^l\left[y^r(Q)\right] = \frac{1}{4\pi}\int_V \frac{b^l\left[y^r(P)\right]}{|y^s(P) - y^s(Q)|} dV_P \quad C_b\left[y^r(Q)\right] = \frac{1}{4\pi}\int_V \frac{c_b\left[y^r(P)\right]}{|y^s(P) - y^s(Q)|} dV_P \quad Q \in V \tag{2.7}$$

have been determined first, are obtained from (2.7) by transformastion.

It can be shown [7,8] that every solution of the equilibrium equations (2.4) admits the following representation found independently of each other by H. Schaefer and D. Carlson

$$t^{kl} = \epsilon^{kpy} \check{F}_{y.;p}^{\ l} + g^{ks} B^l_{.;s}\,, \qquad \mu^a_{.b} = \epsilon^{apy}(\check{H}_{yb;p} + \epsilon_{bpl} \check{F}_{y.}^{\ l}) + g^{al}(\epsilon_{lbs} B^s + C_{b;l}) \qquad x \in V \tag{2.8}$$

where $\check{H}_{yb}$ and $\check{F}_{p.}^{\ l}$ are the stress function tensors whose components will also be referred to as stress functions. The above stress representation involve nine-nine stress functions. It can be shown, however, that every stress condition can be given in terms of six-six stress functions [1].

Let $\beta_{k.}^{\ l}$ and $\alpha_{ab}$ be arbitrary tensor fields on $V$. Furthermore let $r^l(x)$ and $w_b(x)$ $\;x \in V$ be two unknown vector fields. By ${}_{K.}^{\ L}$ and ${}_{AB}$ we denote those subsets of the possible values of index pairs ${}_{k.}^{\ l}$ and ${}_{ab}$ for which the differential equations

$$r^L_{.;K} = \beta_{K.}^{\ L}(x)\,, \qquad w_{B;A} + \epsilon_{BAp} r^p = \alpha_{AB}(x) \qquad x \in V$$

have solutions for the vector fields $r^l(x)$ and $w_b(x)$. It is obvious that the index pairs ${}_{K.}^{\ L}$ and ${}_{AB}$ may have only three-three different values. Let $\{\,{}_{S.}^{\ T}\}\,[{}_{XY}]$ be the supplementary subsets of index pairs the union of which with $\{\,{}_{K.}^{\ L}\}\,[{}_{AB}]$ is the set of index pairs $\{{}_{k.}^{\ l}\}\,[{}_{ab}]$. It is clear, that the index pairs ${}_{S.}^{\ T}$ and ${}_{XY}$ may have only six-six different values.

Since the stress state in terms of the stress functions

$$H_{yb} = \check{H}_{yb} + w_{b;y} + \epsilon_{bys} r^s\,, \qquad F_{y.}^{\ l} = \check{F}_{y.}^{\ l} + r^l_{.;y} \qquad x \in V \tag{2.9}$$

is the same as the stress state in terms of $\check{H}_{yb}$ and $\check{F}_{y.}^{\ l}$ the stress functions $H_{AB}$ and $F_{K.}^{\ L}$ can be set to zero and we shall assume that $H_{AB} \equiv 0$ and $F_{K.}^{\ L} \equiv 0$ independently of what indices are used, i.e., capitalized or not – see paper [1] for further details.

By inverting Hooke's law one obtains

$$\gamma_{kl} = \overset{-1}{A}_{klrs} t^{rs} \qquad \kappa_{ab} = \overset{-1}{B}_{abpq} \mu^{pq} \qquad x \in V\,. \tag{2.10}$$

The strains $\gamma_{kl}$ and $\kappa_{a.}^{\ b}$ are said to be [equilibrated]{statically admissible} if they are calculated from Hooke's law (2.10) by substituting [equilibrated]{statically admissible} stresses $t^{kl}$ , $\mu^{a}_{.b}$.

For the strains $\gamma_{kl}$ and $\kappa_{a.}^{\ b}$ to be [compatible] {kinematically admissible} in a simply-connected region $V$ it is necessary and sufficient that the differential equations of compatibility

$$\mathcal{Y}^{XY}(x) = \epsilon^{Xpk} \kappa_{k\ .;p}^{\ Y} = 0 \qquad \mathcal{D}^{S}_{.T}(x) = \epsilon^{Spq}(\gamma_{qT;p} + \epsilon_{qTb}\kappa_{p.}^{\ b}) = 0 \quad x \in V \tag{2.11}$$

($\mathcal{Y}^{mb}$ and $\mathcal{D}^{m}_{.l}$ are the tensors of incompatibility) and the boundary conditions of compatibility

$$n_3\mathcal{Y}^{3b}(\xi) = n_3\epsilon^{3\pi\chi}\kappa_{\chi.;\pi}^{\ b} = 0, \qquad n_3\mathcal{D}^{3}_{.l}(\xi) = n_3\epsilon^{3\pi\chi}(\gamma_{\chi l;\pi} + \epsilon_{\chi lb}\kappa_{\pi.}^{\ b}) = 0$$

[(2.13) for $\xi \in S$] {(2.14) for $\xi \in S_t$}

and [no further conditions] {the kinematic boundary conditions

$$\kappa_{\eta.}^{\ b} - \hat{\varphi}^{b}_{.;\eta} = 0 \qquad \gamma_{\chi l} - \hat{u}_{l;\chi} - \epsilon_{l\chi b}\hat{\varphi}^{b} = 0 \qquad \xi \in S_u \tag{2.14}$$

} should be fulfilled. It can be shown that the fulfillment of kinematic boundary conditions implies the fulfillment of boundary conditions of compatibility [1].

For the equilibrated stresses

$$t^{kl} = \epsilon^{kpy} F_{y\ .;p}^{\ l} + g^{ks} B^{l}_{.;s} \qquad \mu^{a}_{.b} = \epsilon^{apy}(H_{yb;p} + \epsilon_{bpl} F_{y\ .}^{\ l}) + g^{al}(\epsilon_{lbs}B^{s} + C_{b;l}) \qquad x \in V \tag{2.15}$$

obtained from (2.8) by substituting $H_{yb}$ and $F_{y.}^{\ l}$ for $\check{H}$ and $\check{F}$ to be statically admissible it is sufficient if the traction boundary conditions (2.6) in terms of stress functions $H_{yb}$ and $F_{y.}^{\ l}$ are fulfilled

$$\hat{t}^{l} = n_3 t^{3l} = n_3(\epsilon^{3\eta\pi} F_{\pi.;\eta}^{\ l} + a^{3s} B^{\ l}_{.;s}) \qquad \xi \in S_t \tag{2.16-a}$$

$$\hat{\mu}_b = n_3\mu^{3}_{.b} = n_3\left[\epsilon^{3\pi\eta}(H_{\eta b;\pi} + \epsilon_{b\pi l}F_{\eta.}^{\ l}) + a^{3l}(\epsilon_{lbs}B^{s} + C_{b;l})\right] \qquad \xi \in S_t \tag{2.16-b}$$

For simply-connected micropolar bodies the three dimensional problems of elasticity in the dual system are governed by the dual kinematic equations (2.15) the dual constitutive equations (2.10) and the dual balance equations (2.11).

Field equations (2.15), (2.10) and (2.11) are associated with the boundary condition of compatibility (2.16a,b) and the traction boundary condition (2.16-a,b) on $S_t$ and the kinematic boundary conditions (2.14a,b) on $S_u$.

## 3. Derivation of Supplementary Conditions

For simply-connected domains all the conditions the strains $\gamma_{kl}$, $\kappa_{b.}^{a}$ should meet in order to be [compatible] {kinematically admissible} can be derived from the principle of maximum complementary energy [1]. Here and in the sequel an attempt will be made to derive not only the conditions mentioned but also the supplementary conditions of single-valuedness from the principle of maximum complementary energy. Consequently, we shall follow the line of thought presented in [1] with special attention to the line integrals obtained by applying Stoke's theorem on the simply-connected parts of $S$.

For micropolar bodies the total complementary energy functional

$$K = -\frac{1}{2}\int_V (t^{kl}\gamma_{kl} + \mu^a_{.b}\kappa_a^{\ b})\, dV + \int_{S_u} (n_3 t^{3l}\hat{u}_l + n_3\mu^3_{.b}\hat{\varphi}^b)\, dA \tag{3.1}$$

is a function of the statically admissible stresses $t^{kl}$, $\mu^a_{.b}$ and strains $\gamma_{kl}$, $\kappa_a^{\ b}$. In view of the equation

$$\delta(t^{kl}\gamma_{kl} + \mu^a_{.b}\kappa_a^{\ b}) = 2(\delta t^{kl}\gamma_{kl} + \delta\mu^a_{.b}\kappa_a^{\ b}) = 2(t^{kl}\delta\gamma_{kl} + \mu^a_{.b}\delta\kappa_a^{\ b})$$

extremum condition for the complementary energy functional (3.1) assumes the form

$$-\delta K = \int_V (\gamma_{kl}\delta t^{kl} + \kappa_a^{\ b}\delta\mu^a_{.b})\, dV - \int_{S_u} (n_3\delta t^{3l}\hat{u}_l + n_3\delta\mu^3_{.b}\hat{\varphi}^b)\, dA \; = \; 0 \tag{3.2}$$

Since the stresses are statically addmissible the variations of stresses $\delta t^{kl}$, $\delta\mu^a_{.b}$ can not be taken at will but should meet the preconditions

$$\delta t^{kl}_{..;k} = 0 \qquad \delta\mu^a_{.b;a}(x) + \epsilon_{bkl}\delta t^{kl} = 0 \qquad x \in V \tag{3.3-a}$$

$$n_3\delta t^{3l} = 0 \qquad n_3\delta\mu^3_{.b} = 0 \qquad \xi \in S_t \tag{3.3-b}$$

which follow from (2.4) and (2.6) by taking into account that the variations of body forces and couples $b^l$, $c_b$ as well as of the prescribed tractions $\hat{t}^l$, $\hat{\mu}_b$ are assumed to be zero. It can readily be shown that variations of stresses $\delta t^{kl}$, $\mu^a_{.b}$ meet the conditions ( 3.3-a) if they are given in terms of variations of stress functions $\delta H_{xy}$, $\delta F_s^{\ t}$ as follows:

$$\delta t^{kl} = \epsilon^{kpy}\delta F_{y.;p}^{\ l} \qquad \delta\mu^a_{.b} = \epsilon^{apy}(\delta H_{yb;p} + \epsilon_{bpl}\delta F_y^{\ l}) \qquad x \in V \tag{3.4}$$

where $\delta F_y^{\ l}$ and $\delta H_{yb}$ are arbitrary on $V$ and $S_u$ but should satisfy the side conditions

$$n_3\delta t^{3l} = n_3\epsilon^{3\eta\pi}\delta F_y^{\ l} = 0 \quad n_3\delta\mu^3_{.b} = n_3\epsilon^{3\pi\eta}(\delta H_{yb;p} + \epsilon_{b\pi l}\delta F_y^{\ l}) = 0 \qquad \xi \in S_t \tag{3.5}$$

obtained from (3.3-b) by substituting equation (3.4) and renaming some dummy indices. Let $\delta w_l$ and $\delta r^b$ be two vector fields defined on $S_t$. If $\delta F_\eta^{\ l}$ and $\delta H_{\eta b}$ meet the conditions

$$\delta F_\eta^{\ l} = \delta r^l_{.;\eta} \qquad \delta H_{\eta b} = \delta w_{b;\eta} + \epsilon_{b\eta m}\delta r^m \qquad \xi \in S_t \tag{3.6}$$

then equations (3.3-b) hold. Substitution of expressions ( 3.5) for the variations of stresses in (3.2) yields

$$-\delta K = I_1^{MV} + I_1^{MS_u} = \int_V [\varepsilon^{kmy}\delta F_{y.;m}^{\ l} + \epsilon^{apy}(\delta H_{yb;p} + \epsilon_{bps}\delta F_y^{\ s})\kappa_{a.}^{\ b}]\, dV$$

$$- \int_{S_u} [n_3\epsilon^{3\eta\pi}\delta F_{\pi.;\eta}^{\ l}\hat{u}_l + n_3\epsilon^{3\pi\eta}(\delta H_{\eta b;\pi} + \epsilon_{b\pi s}\delta F_\eta^{\ s})\hat{\varphi}^b]\, dA \,=\, 0 \tag{3.7}$$

By applying Gauss' theorem and a subsequent rearrangement volume integral $I_1^{MV}$ can be manipulated into a more suitable form. If in addition to this one recalls that $\delta H_{AB} \equiv \delta F_{K.}^{\ L} \equiv 0$ $x \in V$ while $\delta H_{AB}$ and $\delta F_{S.}^{\ T}$ $x \in V$ are arbitrary and substitute the tensors of incompatibility $\mathcal{Y}^{XY}$ and $\mathcal{D}_{.T}^{S}$ where possible then $I_1^{MV}$ takes the form

$$I_1^{MV} = \; I_2^{MV} + I_2^{MS_u} + I_1^{MS_t} = \int_V (\mathcal{Y}^{XY}\delta H_{XY} + \mathcal{D}_{.T}^{S}\delta F_{S.}^{\ T})\, dV$$

$$- \int_{S_u} (n_3\epsilon^{3\chi\eta}\delta F_\eta^{\ l}\gamma_{\chi l} + n_3\epsilon^{3\pi\eta}\delta H_{\eta b}\kappa_{\pi.}^{\ b})dA - \int_{S_t} (n_3\epsilon^{3\chi\eta}\delta F_{\eta.}^{\ l}\gamma_{\chi l} + n_3\epsilon^{3\pi\eta}\delta H_{\eta b}\kappa_{\pi.}^{\ b})dA \tag{3.8}$$

Integral $I_1^{MS_u}$ can be transformed further by making use of equation (A.2) whose application requires, however, the substitution of $\hat{u}_l$, $\hat{\varphi}^b$, $\delta F_\eta^{\ l}$, $\delta H_{\eta b}$, $S_u$ and $g$ for $u_l$, $\varphi^b$, $F_{\eta.}^{\ l}$, $H_{\eta b}$, $S_o$ and $g_o$. Recalling that the sign of line integral turn to the opposite on $g$ we have

$$I_1^{MS_u} = \; I_3^{MS_u} + I_1^{MG} = \int_{S_u} [n_3\epsilon^{3\chi\eta}(\hat{u}_{l;\chi} + \epsilon_{l\chi b}\hat{\varphi}^b)\delta F_\eta^{\ l} + n_3\epsilon^{3\pi\eta}\hat{\varphi}_{.;\eta}^{b}\delta H_{\eta b}]\, dA$$

$$+ \int_g (\tau^\eta\hat{u}_l\delta F_\eta^{\ l} + \tau^\eta\hat{\varphi}^b\delta H_{\eta b})\, ds \tag{3.9}$$

Substitution of side conditions (3.6) into $I_1^{MS_t}$ yields

$$I_1^{MS_t} = -\int_{S_t} [n_3\epsilon^{3\chi\eta}\gamma_{\chi l}\delta r_{.;\eta}^{l} + n_3\epsilon^{3\pi\eta}\kappa_{\pi.}^{\ b}(\delta w_{b;\eta} + \epsilon_{b\eta s}\delta r^s)]dA\,. \tag{3.10}$$

Now one should observe that apart from of its sign the above integral coincides with the surface integral in the left hand side of (A.3) if in the latter $\delta r^l$, $\delta w_b$ and $S_u$ are substituted for $r^l$, $w_b$ and $S_o$. When using the right hand side of (A.3) to transform further equation (3.10) one has to keep in mind that (a) the union of $g$ and $L$ corresponds to $g_o$, (b) by applying Stokes'theorem one goes twice along the curve $L$, (c) $\delta r^l$ and $\delta w_b$ are assumed to have jumps

$$\left[\delta r^l\right] = \delta r^{l\,+} - \delta r^{l\,-} \qquad [\delta w_b] = \delta w_b^+ - \delta w_b^-\,, \qquad \xi \in L$$

(d) by rearranging the terms in the surface integral resulted it becomes possible to substitute the boundary conditions of compatibility (2.13).

On the bases of all that has been said we obtain

$$I_1^{MS_t} = \; I_2^{MS_t} + I_2^{MG} + I_1^{ML} = -\int_{S_t} (n_3\mathcal{D}_{.l}^{3}\delta r^l + n_3\mathcal{Y}^{3l}\delta w_l)\, dA$$

$$+ \int_g (\tau^\eta\gamma_{\eta l}\delta r^l + \tau^\eta\kappa_{\eta.}^{\ b}\delta w_b)\, dA + \int_L (\tau^\eta\gamma_{\eta l}\left[\delta r^l\right] + \tau^\eta\kappa_{\eta.}^{\ b}\left[\delta w_b\right])\, dA\,. \tag{3.11}$$

Comparing (3.7), (3.8), (3.9) and (3.11) we have

$$-\delta K = I_2^{MV} + I_2^{MS_u} + I_3^{MS_u} + I_2^{MS_t} + +I_1^{MG} + I_2^{MG} + I_1^{ML}$$

or

$$-\delta K = \int_V (\mathcal{Y}^{XY}\delta H_{XY} + \mathcal{D}^{S}_{.T}\delta F^{T}_{S.})\,dV - \int_{S_t}(n_3\mathcal{D}^3_{.l}\delta r^l + n_3\mathcal{Y}^{3l}\delta w_l)\,dA$$
$$-\int_{S_u}[n_3\epsilon^{3\chi\eta}(\gamma_{\chi l} - \hat{u}_{l;\chi} - \epsilon_{l\chi b}\hat{\varphi}^b)\delta F^{\;l}_{\eta.} + n_3\epsilon^{3\pi\eta}(\kappa_{\pi.}^{\;b} - \hat{\varphi}^b_{.;\pi})\delta H_{\eta b}]\,dA + I_1^{MG} + I_2^{MG} + I_1^{ML} = 0$$

from which it follows – $\delta H_{XY}$, $\delta F^T_{S.}$ are arbitrary on V and $S_u$; $\delta r^l$ and $\delta w_l$ are arbitrary on $S_t$; volume, surface and line integrals are independent – the fulfillment of differential equations of compatibility (2.11,b), boundary conditions of compatibility (2.13a,b) and kinematic boundary conditions (2.14). Observe that the equations mentioned are those the strains $\gamma_{kl}$, $\kappa_{a.}^{\;b}$ should satisfy in order to be kinematically admissible on a simply-connected domain V.

As regards the line integrals $I_1^{MG}+I_2^{MG}$ we substitute conditions (3.6) – the stress functions are assumed to be continuous on S – and perform partial integrations keeping in mind that $\delta w_b$ and $\delta r^l$ have a jump at $P_{1i}$ and $P_{1,i+1}$ $(i = 1,3)$. After a subsequent rearrangement we obtain

$$I_1^{MG} + I_2^{MG} = I_3^{MG} + \Sigma_1^M = \int_g [\tau^\eta(\gamma_{\chi l} - \hat{u}_{l;\chi} - \epsilon_{l\chi b}\hat{\varphi}^b)\delta r^l + \tau^\eta(\kappa_{\pi.}^{\;b} - \hat{\varphi}^b_{.;\pi})\delta w_b]\,dA$$
$$+\sum_{i=1,3}\left\{\hat{\varphi}^b\,[\delta w_b]\big|_{P_{1i}} - \hat{\varphi}^b\,[\delta w_b]\big|_{P_{1,i+1}} + \hat{u}_l\,\left[\delta r^l\right]\big|_{P_{1i}} - \hat{u}_l\,\left[\delta r^l\right]\big|_{P_{1,i+1}}\right\} \tag{3.12}$$

Vanishing of the line integral $I_3^{MG}$ for arbitrary $\delta w_b$ and $\delta r^l$ leads to the fulfillment of the continuity conditions

$$\tau^\eta(\gamma_{\chi l} - \hat{u}_{l;\chi} - \epsilon_{l\chi b}\hat{\varphi}^b) = 0 \qquad \tau^\eta(\kappa_{\pi.}^{\;b} - \hat{\varphi}^b_{.;\pi}) = 0 \qquad \xi \in g$$

Investigation of condition

$$I_1^{ML} + \Sigma_1^M = 0 \tag{3.13}$$

requires some preparations. Let us assume that the jumps $\left[\delta r^l\right]$, $[\delta w_b]$ have the following forms

$$[\delta w_b] = \delta \overset{(i1)}{c_b} + \epsilon_{svb}\delta\overset{(1i)}{C^s}[R^v - R^v(P_{1,i+1})] \quad [\delta r^s] = \delta\overset{(1i)}{C^s} \quad i = 1,3 \quad \xi \in L_{1i} \tag{3.14}$$

$$[\delta w_b] = \delta \overset{(21)}{c_b} + \epsilon_{svb}\delta\overset{(21)}{C^s}[R^v - R^v(P_{21})] \quad [\delta r^s] = \delta\overset{(1i)}{C^s} \qquad \xi \in L_2 \tag{3.15}$$

in which the vectors $\delta\overset{(i1)}{c_b}$, $\delta\overset{(1i)}{C^s}$, $\delta\overset{(21)}{c_b}$ and $\delta\overset{(21)}{C^s}$ are arbitrary constants, $R^v$ and $R^v(P)$ are the position vectors of the points $\xi$ and $P$ in the local basis $\xi$.

The point in the above choice is that there are no stress functions due to the representations of vector fields $[\delta w_b]$ and $[\delta r^s]$ in the form

$$[\delta w_b] = \delta c_b + \epsilon_{svb}\delta C^s[R^v - R^v(P)]\,, \qquad [\delta r^s] = \delta C^s \qquad\qquad \xi \in S$$

where $\delta c_b$ and $\delta C^s$ are arbitrary constants and $P$ is an arbitrary but fixed point. Really, it can readily be shown that $[\delta w_b]_{;\chi} + \epsilon_{l\kappa s}[\delta r^s] \equiv 0$, $[\delta r^s]_{;\beta} \equiv 0$ if $\xi \in S$ To obtain the final form of (3.13) we substitute $I_1^{ML}$ from (3.11) and $\Sigma_1^M$ from (3.12) which makes possible the substitution of representations (3.14) and (3.15). Then we gather the coefficients of

$$\delta \overset{(21)}{c_b}, \delta \overset{(21)}{C^s} \qquad \text{and} \qquad \delta \overset{(i1)}{c_b}, \delta \overset{(1i)}{C^s}$$

and make some rearrangements. Finally we have

$$\begin{aligned}
I_1^{ML} + \Sigma_1^M = &\oint_{L_2} \tau^\eta \kappa_{\eta.}^{\ b} \mathbf{a}_b\, ds \,\cdot\, \mathbf{a}^l \delta \overset{(21)}{c_l} + \\
&\oint_{L_2} \frac{d\xi^\chi}{ds}\left[\gamma_{\chi l} + \epsilon_{blv}(R^v(s) - R^v(P_{21}))\kappa_{\chi.}^{\ b}\right] \mathbf{a}^l\, ds \cdot\, \mathbf{a}_k\, \delta \overset{(21)}{C^k} + \\
&\sum_{i=1,3} \left( -\hat{\varphi}^b \mathbf{a}_b\big|_{P_{1,i+1}} + \hat{\varphi}^b \mathbf{a}_b\big|_{P_{1i}} + \int_{L_{1i}} \tau^\eta \kappa_{\eta.}^{\ b} \mathbf{a}_b\, ds \right) \cdot\, \mathbf{a}^l\, \delta \overset{(i1)}{c_l} + \\
&\sum_{i=1,3} \Big[ -\hat{u}_l \mathbf{a}^l\big|_{P_{1,i+1}} + \hat{u}_l \mathbf{a}^l\big|_{P_{1i}} + \epsilon_{lvk}\hat{\varphi}^k(R^v(P_{1,i+1}) - R^v(P_{1i}))\mathbf{a}^l\big|_{P_{1i}} + \\
&\int_{L_{1i}} \frac{d\xi^\chi}{ds}\left[\gamma_{\chi l} + \epsilon_{blv}(R^v(s) - R^v(P_{1i}))\kappa_{\chi.}^{\ b}\right] \mathbf{a}^l\, ds \cdot\, \mathbf{a}_k\, \delta \overset{(1i)}{C^k} \Big]
\end{aligned}$$

from which with regard to the arbitrariness of $\delta \overset{(21)}{c_l}$, $\delta \overset{(21)}{C^k}$, $\delta \overset{(i1)}{c_l}$ and $\delta \overset{(1i)}{C^k}$ it follows the compatibility conditions in the large

$$\oint_{L_2} \tau^\eta \kappa_{\eta.}^{\ b} \mathbf{a}_b\, ds \;=\; 0, \quad \oint_{L_2} \frac{d\xi^\chi}{ds}\left[\gamma_{\chi l} + \epsilon_{blv}(R^v(s) - R^v(P_{21}))\kappa_{\chi.}^{\ b}\right] \mathbf{a}^l\, ds \;=\; 0 \tag{3.16}$$

and the supplementary conditions of single-valuedness

$$-\hat{\varphi}^b \mathbf{a}_b\big|_{P_{1,i+1}} + \hat{\varphi}^b \mathbf{a}_b\big|_{P_{1i}} + \int_{L_{1i}} \tau^\eta \kappa_{\eta.}^{\ b} \mathbf{a}_b\, ds \;=\; 0 \qquad i = 1,3 \tag{3.17}$$

and

$$\begin{aligned}
&-\hat{u}_l \mathbf{a}^l\big|_{P_{1,i+1}} + \hat{u}_l \mathbf{a}^l\big|_{P_{1,i+1}} + \epsilon_{lvk}\varphi^k(R^v(P_{1,i+1}) - R^v(P_{1i}))\mathbf{a}^l\big|_{P_{1i}} \\
&+ \int_{L_{1i}} \frac{d\xi^\chi}{ds}\left[\gamma_{\chi l} + \epsilon_{blv}(R^v(s) - R^v(P_{1i}))\kappa_{\chi.}^{\ b}\right] \mathbf{a}^l\, ds \;=\; 0. \quad i = 1,3
\end{aligned} \tag{3.18}$$

It is worthy of mention that the supplementary conditions of single valuedness can also be derived from a geometrical line of thought in the same manner as in the classical case [9].

## 4. Concluding Remarks

In accordance with the aims detailed in section 1 the present paper has studied the question what further conditions the strain fields should meet in addition to the usual ones in order to be kinematically admissible on multiply connected regions providing three dimensional and mixed boundary value problems. It has been proved that both the compatibility conditions in the large and the supplementary conditions of single-valuedness are natural boundary conditions that follow from the principle of maximum complementary energy. The significance of this conclusion is inherent in the applications. In other words application of direct methods – finite element method for instance – to find approximate solutions from the extremum condition do not require that the admissible fields should satisfy these conditions in advance.

## 5. Apendix

Let $S_o$ be an arbitrary open surface closed by the directed boundary curve $g_o$. Further let the positive direction on $g_o$ be taken so that $\tau_\alpha, n_3$ and $\nu_\alpha$ – $\nu_\alpha$ is the normal to the boundary curve $g_o$ that lies in the tangent plane – form a right hand triad. Let $b_{\alpha.}^{\ l}(\xi)$ and $c_l(\xi)$ be surface tensors. Applying the Stokes theorem it can be shown that

$$\int_{S_o} n_3\, \epsilon^{3\eta\alpha}\, b_{\alpha\,.|\eta}^{\ l}\, c_l\, dA = \oint_{g_o} b_{\alpha}^{\ l}\, c_l\, \tau^\alpha\, ds - \int_{S_o} n_3\, \epsilon^{3\eta\alpha}\, b_{\alpha.}^{\ l}\, c_{l|\eta}\, dA \tag{A.1}$$

The above rule is that of partial integration. Making use of Stokes' theorem (A.1) and performing partial integrations it can readily be shown that

$$\begin{aligned}\int_{S_o} [n_3\epsilon^{3\eta\pi} F_{\pi.;\eta}^{\ l} u_l + n_3\epsilon^{3\eta\pi}(H_{\eta b;\pi} + \epsilon_{b\pi s} F_{\eta.}^{\ s})\varphi^b]\, dA = \oint_{g_o} (\tau^\eta u_l F_{\eta.}^{\ l} + \tau^\eta \varphi^b H_{\eta b})\, ds \\ - \int_{S_o} [n_3\epsilon^{3\chi\eta}(u_{l;\chi} + \epsilon_{l\chi b}\varphi^b) F_{\eta.}^{\ l} + n_3\epsilon^{3\pi\eta}\varphi^b_{.;\eta} H_{\eta b}]\, dA\end{aligned} \tag{A.2}$$

In the same way it can be shown that

$$\begin{aligned}\int_{S_o} [n_3\epsilon^{3\chi\eta}\gamma_{\chi l} r^l_{.;\eta} + n_3\epsilon^{3\pi\eta}\kappa_{\pi.}^{\ b}(w_{b;\eta} + \epsilon_{b\eta s} r^s)]\, dA = \oint_{g_o} (\tau^\eta \gamma_{\eta l}\, r^l + \tau^\eta \kappa_{\eta.}^{\ b} w_b) ds \\ + \int_{S_o} [n_3\epsilon^{3\pi\chi}(\gamma_{\chi l;\pi} + \epsilon_{\chi lb}\kappa_{\pi.}^{\ b}) r^l + n_3\epsilon^{3\eta\pi}\kappa_{\pi.;\eta}^{\ b} w^b]\, dA\end{aligned} \tag{A.3}$$

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