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BOUNDARY INTEGRAL EQUATIONS FOR PLANE PROBLEMS – REMARK TO THE FORMULATION FOR EXTERIOR REGIONS

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Dedicated to Professor Barna Szabó on his Sixty Fifth Birthday

Abstract. Assuming linear displacements and constant strains and stresses at infinity we reformulate the equations of direct method for plane problems of elasticity. This makes possible that plane problems on exterior regions can be attacked without replacing the region under consideration by a bounded one.

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1. Introduction

In spite of a great number of publications devoted to plane problems – without being able to achieve completeness we mention only a few articles [1], [2], [3], [4], [5] and the books [6], [7], [8] in which further references can be found – the formulation for exterior regions has the disadvantage that no stresses can be prescribed at infinity. As regards the reasons we cite the paper [4] in which a clear assumption is made on the far field pattern of the displacements. This assumption makes possible to establish an appropriate Betti's formula and to prove uniqueness and existence for the exterior Dirchlet and Neuman problems. On the other hand the assumption made on the far field pattern excludes those problems from the theory for which the displacements are linear while the strains and stresses are constant at infinity. If the direct formulation reproduces this displacement field, then the resulting strain and stress conditions are also constant at infinity. Consequently plane problems for the exterior regions can be attacked without replacing the region by

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a bounded one. The idea to approach the problem in a somewhat unusual manner stems from the thesis [9] which presents a direct formulation in terms of stress functions of order one. They must be linear at infinity to produce a constant stress condition there.

The present paper is an attempt to clarify how the formulation changes if we accept that the displacements are linear at infinity.

2. Preliminaries

Throughout this paper x_1 and x_2 are rectangular Cartesian coordinates, referred to an origin O. Greek subscripts are assumed to have the range (1,2), summation over repeated subscripts is implied. The simply connected exterior region under consideration is denoted by A_e and is bounded by the contour $\mathcal{L}o$. We stipulate that the contour admits a nonsingular parametrization in terms of its arc length s. The outer normal is denoted by n_π. In accordance with the notations introduced $\delta_{\kappa\lambda}$ is the Kronecker symbol, ∂_α stands for the derivatives with respect to x_α and $\epsilon_{3\kappa\lambda}$ is the permutation symbol. Assuming plane strain let u_κ, $e_{\kappa\lambda}$ and $t_{\kappa\lambda}$ be the displacement field and the in plane components of stress and strain respectively. For isotropic materials μ is the shear modulus of elasticity and ν is the Poisson number.

For homogenous and isotropic material the plane strain problem of classical elasticity is governed by the kinematic equations

$$e_{\rho\lambda} = \frac{1}{2}(\partial_\rho u_\lambda + u_\rho \partial_\lambda), \tag{2.1}$$

Hook's law

$$t_{\rho\lambda} = 2\mu \left(e_{\rho\lambda} + \frac{\nu}{1-2\nu}\delta_{\rho\lambda} e_{\varphi\varphi} \right) \tag{2.2}$$

and the equilibrium equations

$$t_{\rho\lambda}\partial_\lambda + b_\rho = 0 \tag{2.3}$$

which should be associated with appropriate boundary conditions not detailed here since they play no role in the investigations. The basic equation for u_λ takes the form

$$\mathcal{D}_{\rho\lambda} u_\lambda + \frac{b_\rho}{\mu} = 0 \tag{2.4}$$

where

$$\mathcal{D}_{\rho\lambda} = \triangle\delta_{\rho\lambda} + \frac{1}{1-2\nu}\partial_\rho\partial_\lambda \qquad\qquad \triangle = \partial_\sigma\partial_\sigma\ . \tag{2.5}$$

Let $Q(\xi_1, \xi_2)$ and $M(x_1, x_2)$ be two points in the plane of strain (the source point and the point of effect). We shall assume temporarily that the point Q is fixed. The distance between Q and M is R, the position vector of M relative to Q is r_κ. The small circle as a subscript (for instance $M_\circ$ or $Q_\circ$) indicates that the corresponding points, i.e., Q or M are taken on the contour.

The well known singular fundamental solutions for the basic equation (2.4) are given by the formulas

$$U_{\lambda\kappa}(M,Q) = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{1}{R}\delta_{\kappa\lambda} + \frac{r_\kappa r_\lambda}{R^2} - \frac{7-8\nu}{2}\delta_{\kappa\lambda}\right] \tag{2.6}$$

and

$$T_{\lambda\kappa}(M,Q) = \frac{1}{4\pi(1-\nu)}\frac{1}{R^2}\left[(1-2\nu)(n_\lambda r_\kappa - n_\kappa r_\lambda - n_\sigma r_\sigma\,\delta_{\kappa\lambda}) - 2\frac{n_\sigma r_\sigma\, r_\kappa r_\lambda}{R^2}\right] \tag{2.7}$$

where

$$\mathfrak{u}_\lambda(M) = U_{\lambda\kappa}(M,Q)e_\kappa(Q) \qquad \text{and} \qquad t_\lambda(M) = T_{\lambda\kappa}(M,Q)e_\kappa(Q)$$

are the displacement vector and stress vector on a surface element with outward normal $n_\lambda = n_\lambda(M)$ at the point M due to the force $e_\kappa = e_\kappa(Q)$ at Q.

3. Basic formulas for exterior regions

Figure 1 represents a triple connected region A'_e bounded by the contours $\mathcal{L}_0$, $\mathcal{L}_\varepsilon$ and the circle $\mathcal{L}_R$ with radius ${}_eR$ and center at O. Here $\mathcal{L}_\varepsilon$ is the contour of the neighborhood A_ε of Q with radius R_ε while ${}_eR$ is sufficiently large so that the region bounded by $\mathcal{L}_R$ covers both $\mathcal{L}_0$, and $\mathcal{L}_\varepsilon$. If ${}_eR \to \infty$ and $R_\varepsilon \to 0$ then clearly $A'_e \to A_e$.

Let $u_\kappa(M)$ and $g_\kappa(M)$ be sufficiently smooth – continuously differentiable at least twice – otherwise arbitrary displacement fields on A_e. The stresses obtained from these displacement fields are denoted by $t_{\lambda\kappa}\left[u_\rho(M)\right]$ and $t_{\lambda\kappa}\left[g_\rho(M)\right]$ respectively. Equation

$$\begin{aligned}
&\int_{A'_e - A_\varepsilon}\left[u_\lambda(M)\left(\mu\overset{M}{\mathcal{D}}_{\lambda\sigma}g_\sigma(M)\right) - g_\lambda(M)\left(\mu\overset{M}{\mathcal{D}}_{\lambda\sigma}u_\sigma(M)\right)\right]dA_M = \\
&\quad = \oint_{\mathcal{L}_o}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) - g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right]ds_{M_\circ} \\
&\quad + \oint_{\mathcal{L}_\varepsilon}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) - g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right]ds_{M_\circ} \\
&\quad + \oint_{\mathcal{L}_R}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) - g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right]ds_{M_\circ}\,,
\end{aligned} \tag{3.1}$$

in which M over the letter denotes derivatives taken with respect to the point M and $n_\kappa(M_\circ)$ is the outward normal, is the primal Somigliana identity applied to the triple connected region A'_e. Let $g_\lambda(Q) = \mathfrak{u}_\kappa(Q) = U_{\lambda\kappa}(M,Q)e_\kappa(Q)$, which is an elastic state, non singular in A'_e. We regard $u_\lambda(M)$ as other elastic state in the region A_e. Further we assume that $u_\lambda(M)$ has the far field pattern (asymptotic behavior)

$$\tilde{u}_\kappa(M) = c_\kappa + \varepsilon_{3\rho\kappa}x_\rho\omega + e_{\kappa\beta}(\infty)x_\beta \tag{3.2}$$

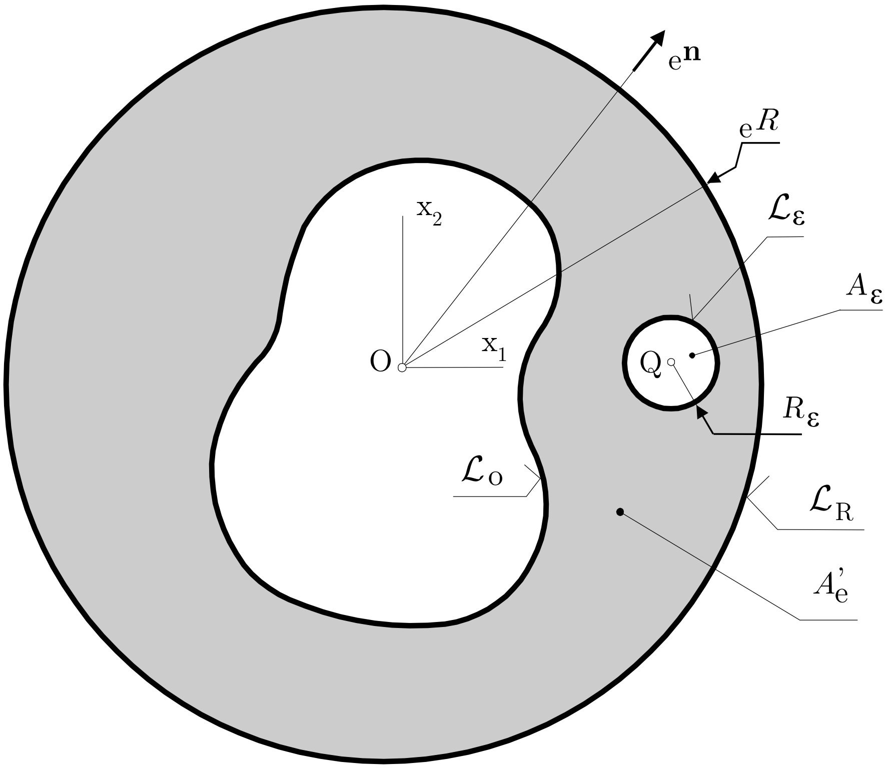

Figure 1: Triple connected region

if x_β or which is the same M tends to infinity, where c_κ is a translation, ω is a rotation in finite, $c_\kappa + \varepsilon_{3\rho\kappa}x_\rho\omega$ is the corresponding rigid body motion, $e_{\kappa\beta}(\infty)$ is a constant strain tensor at infinity and $e_{\kappa\beta}(\infty)x_\beta$ is the corresponding displacement field. The stresses induced by the strains $e_{\kappa\beta}(\infty)$ can be obtained by the Hook law:

$$t_{\rho\lambda}(\infty) = 2\mu\left(e_{\rho\lambda}(\infty) + \frac{\nu}{1-2\nu}\delta_{\rho\lambda}e_{\varphi\varphi}(\infty)\right) \tag{3.3}$$

Upon substitution of the above quantities into the Somigliana identity we have

$$\begin{aligned}
&\int_{A'_e - A_\varepsilon}\left[u_\lambda(M)\left(\mu\overset{M}{\mathcal{D}}_{\lambda\sigma}U_{\sigma\kappa}(M,Q)\right) - \left(\mu\overset{M}{\mathcal{L}}_{\lambda\sigma}u_\sigma(M)\right)U_{\lambda\kappa}(M,Q)\right]dA_M\, e_\kappa(Q) = \\
&\quad = \oint_{\mathcal{L}_o}\left[u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right]\, ds_{M_\circ}\, e_\kappa(Q) + \\
&\quad + \oint_{\mathcal{L}_\varepsilon}\left[u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right]\, ds_{M_\circ}\, e_\kappa(Q) \\
&\quad + \oint_{\mathcal{L}_R}\left[u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right]\, ds_{M_\circ}\, e_\kappa(Q)
\end{aligned} \tag{3.4}$$

since $t_{\lambda\kappa}\,[u_\rho(M_\circ)]\,n_\kappa(M_\circ) = t_\lambda(M_\circ)$ is the stress on the contour and obviously

$$t_{\lambda\kappa}\,[g_\rho(M_\circ)]\,n_\kappa(M_\circ) = T_{\lambda\kappa}(M_\circ, Q)e_\kappa(Q).$$

In the sequel we shall assume that there are no body forces. This assumption has no effect on the result we hope to come to.

It is clear that one can omit $e_\kappa(Q)$. As regards the equation we obtain by omitting $e_\kappa(Q)$ our aim is to clarify what the limit is if $R_\varepsilon \longrightarrow 0$ and ${}_eR \longrightarrow \infty$. The left side vanishes under the conditions detailed above and as is known (see [6], [7])

$$\oint_{\mathcal{L}_o} \cdots + \lim_{R_\varepsilon \longrightarrow 0} \oint_{\mathcal{L}_\varepsilon} \cdots = u_\kappa(Q) + \oint_{\mathcal{L}_o} [u_\lambda(\overset{o}{M})T_{\lambda\kappa}(M_\circ, Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q)]\, ds_{M_\circ}$$

Consequently

$$u_\kappa(Q) = \lim_{{}_eR \longrightarrow \infty} \oint_{\mathcal{L}_R} [t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q)]\; ds_{M_\circ} + \\ + \oint_{\mathcal{L}_o} [t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q)]\; ds_{M_\circ}\,. \quad (3.5)$$

In order to establish the first Somigliana formula for the exterior region one has to find the limit of the first integral on the right side.

4. Modified Somigliana formulas for exterior regions

In this section our main objective is to prove that

$$I_\kappa = \lim_{{}_eR \longrightarrow \infty} \oint_{\mathcal{L}_R} [t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q)]\; ds_{M_\circ} = \\ = c_\kappa + \varepsilon_{3\rho\kappa}\xi_\rho\omega + e_{\kappa\beta}(\infty)\xi_\beta = \tilde{u}_\kappa(Q) \quad (4.1)$$

Taking the relation

$$x_\beta(\overset{o}{M}) = {}_eRn_\beta(\overset{o}{M})$$

into account we substitute $t_{\lambda\rho}(\infty)n_\rho(M_\circ)$ for $t_\lambda(M_\circ)$ and $c_\lambda + \varepsilon_{3\rho\lambda}x_\rho\omega + e_{\kappa\beta}(\infty)x_\beta$ for $u_\lambda(M_\circ)$. In this way we get

$$I_\kappa = \overset{(1)}{I}_\kappa + \overset{(2)}{I}_\kappa + \overset{(3)}{I}_\kappa + \overset{(4)}{I}_\kappa = - \lim_{{}_eR \longrightarrow \infty} \oint_{\mathcal{L}_R} c_\lambda T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} \\ - \lim_{{}_eR \longrightarrow \infty} \varepsilon_{3\rho\lambda\,e}R\omega \oint_{\mathcal{L}_R} n_\rho(\overset{o}{M})T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} + \\ + \lim_{{}_eR \longrightarrow \infty} t_{\lambda\rho}(\infty) \oint_{\mathcal{L}_R} n_\rho(M_\circ)U_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} \\ - \lim_{{}_eR \longrightarrow \infty} e_{\kappa\beta}(\infty)_eR \oint_{\mathcal{L}_R} n_\beta(\overset{o}{M})T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ}\,. \quad (4.2)$$

REMARK 1.: In accordance with (3.2) the stresses and strains are taken as constant quantities which are therefore independent of the arc coordinate s.

Since

$$\oint_{\mathcal{L}_R} T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} = -\delta_{\kappa\lambda} \quad \text{and} \quad \varepsilon_{3\rho\lambda} \oint_{\mathcal{L}_R} r_\rho T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} = 0$$

(the former is the moment about the origin of the stresses due to a unit force at Q) one can write

$$\begin{aligned} \overset{(1)}{I}_\kappa + \overset{(2)}{I}_\kappa = c_\kappa - \lim_{_eR \longrightarrow \infty} \varepsilon_{3\rho\lambda}\omega \oint_{\mathcal{L}_R} (\xi_\rho + r_\rho) T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} + \\ = c_\kappa - \lim_{_eR \longrightarrow \infty} \left[\varepsilon_{3\rho\lambda}\omega\xi_\rho \oint_{\mathcal{L}_R} T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} + \varepsilon_{3\rho\lambda}\omega \oint_{\mathcal{L}_R} r_\rho T_{\lambda\kappa}(M_\circ, Q)\, ds_{M_\circ} \right] \\ = c_\kappa + \varepsilon_{3\rho\kappa}\omega\xi_\rho \end{aligned} \tag{4.3}$$

which clearly shows that (4.3) reproduces the rigid body motion.

Determination of the limits $\overset{(3)}{I}_\kappa$ and $\overset{(4)}{I}_\kappa$ requires long formal transformations. For this reason we confine ourselves to the line of thought and the results of the most important steps.

First we have to expand $U_{\lambda\kappa}$ and $T_{\lambda\kappa}$ into series in terms of $_eR$ to the power $1, 0, -1, -2$ etc. This transformation is based on the relations:

$$n_\alpha(\overset{o}{M}) = n_\alpha \qquad r_\alpha(\overset{o}{M}, Q) = x_\alpha(\overset{o}{M}) - \xi_\alpha(Q) = x_\alpha - \xi_\alpha \tag{4.4a}$$

$$\frac{1}{R} = \frac{1}{_eR}\left(1 + \frac{n_\alpha \xi_\alpha}{_eR} - \frac{1}{2}\frac{\xi_\alpha \xi_\alpha}{_eR^2} + \cdots\right) \tag{4.4b}$$

$$\ln\frac{1}{R} = \ln\frac{1}{_eR} + \frac{n_\alpha \xi_\alpha}{_eR} - \frac{1}{2}\frac{\xi_\alpha \xi_\alpha}{_eR^2} + \cdots \tag{4.4c}$$

$$\frac{r_\alpha r_\beta}{_eR^2} \approx n_\alpha n_\beta + 2 n_\alpha n_\beta \frac{n_\psi \xi_\psi}{_eR} - \frac{1}{_eR}\left(\xi_\alpha n_\beta + n_\alpha \xi_\beta\right) + \cdots \tag{4.4d}$$

Making use of the relations (4.4a,...,d) for $\overset{(3)}{I}_{\kappa}$ and $\overset{(4)}{I}_{\kappa}$we have

$$\overset{(3)}{I}_{\kappa} = \overset{(31)}{I}_{\kappa} + \overset{(32)}{I}_{\kappa} + \overset{(33)}{I}_{\kappa} + \overset{(34)}{I}_{\kappa} + \overset{(35)}{I}_{\kappa} =$$

$$= \lim_{{}_eR\longrightarrow\infty} t_{\lambda\rho}(\infty) \oint_{\mathcal{L}_R} n_\rho C_1 (3-4\nu) \left[\ln \frac{1}{{}_eR} + C_2 \delta_{\lambda\kappa}\right] ds_{M_\circ} +$$

$$+ \lim_{{}_eR\longrightarrow\infty} t_{\lambda\rho}(\infty) \oint_{\mathcal{L}_R} n_\rho C_1 (3-4\nu) \frac{n_\alpha \xi_\alpha}{{}_eR} ds_{M_\circ} \delta_{\lambda\kappa} +$$

$$+ \lim_{{}_eR\longrightarrow\infty} t_{\lambda\rho}(\infty) \oint_{\mathcal{L}_R} n_\rho C_1 n_\lambda n_\kappa ds_{M_\circ} +$$

$$+ \lim_{{}_eR\longrightarrow\infty} \lambda\rho(\infty) \oint_{\mathcal{L}_R} n_\rho C_1 2 n_\lambda n_\kappa \frac{n_\alpha \xi_\alpha}{{}_eR} ds_{M_\circ} -$$

$$- \lim_{{}_eR\longrightarrow\infty} t_{\lambda\rho}(\infty) \oint_{\mathcal{L}_R} n_\rho C_1 \frac{1}{{}_eR} (\xi_\lambda n_\kappa + n_\lambda \xi_\kappa) ds_{M_\circ} \tag{4.5}$$

and

$$\overset{(4)}{I}_{\kappa} = \overset{(41)}{I}_{\kappa} + \overset{(42)}{I}_{\kappa} =$$

$$= \lim_{{}_eR\longrightarrow\infty} e_{\kappa\beta}(\infty) 2\mu(1-2\nu) C_1 \oint_{\mathcal{L}_R} n_\beta \left(n_\lambda \xi_\kappa + n_\kappa \xi_\lambda + n_\alpha \xi_\alpha \delta_{\lambda\kappa} +_e R\delta_{\lambda\kappa}\right) \frac{ds_{M_\circ}}{{}_eR}$$

$$+ \lim_{{}_eR\longrightarrow\infty} e_{\kappa\beta}(\infty) 4\mu C_1 \oint_{\mathcal{L}_R} n_\beta \left(n_\lambda n_\kappa + 3 n_\lambda n_\kappa \frac{n_\alpha \xi_\alpha}{{}_eR} - \frac{1}{{}_eR}(\xi_\lambda n_\kappa + n_\lambda \xi_\kappa)\right) ds_{M_\circ} . \tag{4.6}$$

where

$$C_1 = \frac{1}{8\pi\mu(1-\nu)} \qquad \text{and} \qquad C_2 = \frac{7-8\nu}{2}$$

When calculating the integrals $\overset{(3)}{I}_{\kappa}$ and $\overset{(4)}{I}_{\kappa}$ one also has to utilize the followings:

1. The outward unit normal on $\mathcal{L}_R$ is given by

$$n_a = (\sin\varphi, \cos\varphi) \tag{4.7}$$

   where φ is the polar angle.

2. The arc element on $\mathcal{L}_R$ admits the form

$$ds_{\overset{\circ}{M}} = {}_eR d\varphi \, . \tag{4.8}$$

3. As ${}_eR \longrightarrow \infty$ the coefficient(s) of ${}_eR$ always assumes (assume) the form: an expression constant at infinity and multiplied by

$$\int_0^{2\pi} \sin^n \varphi \cos^k \varphi \, d\varphi$$

   where the powers n and k are natural numbers and depend on the term considered but the integral vanishes.

4. The structure of those terms being the coefficients of ${}_eR$ to the power zero is similar but they involve ξ_α and the trigonometric integrals are not necessarily equal to zero.

5. The most important trigonometric integrals one needs are as follows

$$
\begin{gathered}
\int_0^{2\pi} \sin^2 \varphi d\varphi = \int_0^{2\pi} \cos^2 \varphi \, d\varphi = \pi \\
\int_0^{2\pi} \sin^3 \varphi \, d\varphi = \int_0^{2\pi} \sin^2 \varphi \cos \varphi \, d\varphi = \int_0^{2\pi} \sin \varphi \cos^2 \varphi \, d\varphi = \int_0^{2\pi} \cos^3 \varphi \, d\varphi = 0 \\
\int_0^{2\pi} \sin^4 \varphi \, d\varphi = \int_0^{2\pi} \cos^4 \varphi \, d\varphi = \frac{3}{4}\pi \\
\int_0^{2\pi} \sin^3 \varphi \, \cos \varphi d\varphi = \int_0^{2\pi} \sin \varphi \, \cos^3 \varphi d\varphi = 0 \\
\int_0^{2\pi} \sin^2 \varphi \cos^2 \varphi \, d\varphi = \frac{1}{4}\pi
\end{gathered}
\tag{4.9}
$$

After performing the integrations we have

$$
\begin{aligned}
\overset{(31)}{I}_\kappa = \overset{(33)}{I}_\kappa &= 0 \\
\overset{(32)}{I}_\kappa &= C_1\pi\,(3-4\nu)\,[t_{\kappa 1}(\infty)\xi_1 + t_{\kappa 2}(\infty)\xi_2] \\
\overset{(34)}{I}_1 &= C_1\pi\tfrac{1}{2}\,[3t_{11}(\infty)\xi_1 + 2t_{12}(\infty)\xi_2 + t_{22}(\infty)\xi_2] \\
\overset{(34)}{I}_2 &= C_1\pi\tfrac{1}{2}\,[t_{11}(\infty)\xi_2 + 2t_{12}(\infty)\xi_1 + 3t_{22}(\infty)\xi_1] \\
\overset{(35)}{I}_1 &= -C_1\pi\,[2t_{11}(\infty)\xi_1 + t_{12}(\infty)\xi_2 + t_{22}(\infty)\xi_1] \\
\overset{(35)}{I}_2 &= -C_1\pi\,[t_{11}(\infty)\xi_2 + t_{12}(\infty)\xi_1 + 2t_{22}(\infty)\xi_2]
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
\overset{(41)}{I}_\kappa &= C_1\pi 2\mu\,(1-2\nu)\,[e_{11}(\infty) + e_{22}(\infty)]\,\xi_\kappa \\
\overset{(42)}{I}_1 &= -C_1\pi 4\mu\,(1-2\nu)\,[2e_{11}(\infty)\xi_1 + e_{21}(\infty)\xi_2 + e_{21}(\infty)\xi_1] \\
\overset{(42)}{I}_2 &= -C_1\pi 4\mu\,(1-2\nu)\,[e_{11}(\infty)\xi_2 + e_{21}(\infty)\xi_1 + 2e_{21}(\infty)\xi_2]
\end{aligned}
\tag{4.11}
$$

Using (4.10), (4.11) and (3.3) it follows that

$$\overset{(3)}{I}_\kappa + \overset{(4)}{I}_\kappa = e_{\kappa\beta}(\infty)\xi_\beta$$

Neglecting the rigid body motion, i.e., setting $\overset{(1)}{I}_\kappa + \overset{(2)}{I}_\kappa$ to zero we obtain

$$I_\kappa = e_{\kappa\beta}(\infty)\xi_\beta$$

and the first and modified Somigliana formula immediately follows from (3.5) and (4.1)

$$u_\kappa(Q) = e_{\kappa\beta}(\infty)\xi_\beta(Q) + \oint_{\mathcal{L}_o} [t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q)]\ ds_{M_\circ} \qquad Q \in A_e \quad (4.12)$$

If $Q = Q_\circ$ is on $\mathcal{L}_o$ nothing changes concerning the limit of the integral taken on $\mathcal{L}_R$. Consequently

$$C_{\kappa\rho}u_\rho(Q_\circ) = e_{\kappa\beta}(\infty)\xi_\beta(Q_\circ) + \oint_{\mathcal{L}_o} [t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q_\circ) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q_\circ)]\ ds_{M_\circ} \qquad Q = Q_\circ \in \mathcal{L}_o \quad (4.13)$$

where $C_{\kappa\rho} = \delta_{\kappa\rho}/2$ if the contour is smooth at $Q_\circ$. This integral equation is that of the direct method (or the second Somigliana formula for exterior regions).

If Q is inside the contour $\mathcal{L}_o$ – this region is referred to as A_i – then one can obtain easily that

$$0 = e_{\kappa\beta}(\infty)\xi_\beta(Q) + \oint_{\mathcal{L}_o} [t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q)]\ ds_{M_\circ} \qquad Q = Q_\circ \in A_i \quad (4.14)$$

which is the third Somigliana formula for exterior regions. It is not too difficult to show that the stresses at Q are given by

$$t_{\kappa\sigma}(Q) = t_{\kappa\sigma}(\infty) + \oint_{\mathcal{L}_o} {}_i t_\lambda(M_\circ)\, D_{\lambda\kappa\sigma}(M_\circ, Q)\, ds_{M_\circ} - \oint_{\mathcal{L}_o} u_\lambda(M_\circ)\, S_{\lambda\kappa\sigma}(M_\circ, Q)\, ds_{M_\circ} \quad (4.15)$$

where, as it is well known, the twopoint tensors $D_{\lambda\kappa\sigma}(M_\circ, Q)$ and $S_{kls}(M_\circ, Q)$ are as follows

$$D_{\lambda\kappa\sigma}(M_\circ, Q) = -\frac{1}{8\pi(1-\nu)}\frac{1}{R^2}\left[(1-2\nu)(r_\lambda\delta_{\kappa\sigma} - \delta_{\lambda\sigma}r_\kappa - \delta_{\lambda\kappa}r_\sigma) - 2\frac{r_\lambda\, r_\kappa\, r_\sigma}{R^2}\right] \quad (4.16)$$

and

$$\begin{aligned} S_{\lambda\kappa\sigma}(M_\circ, Q) = \frac{\mu}{2\pi(1-\nu)}\frac{1}{R^2}\Big\{ &\frac{2}{R^2}n_\rho r_\rho\left[(1-2\nu)r_\lambda\,\delta_{\kappa\sigma} + \nu(\delta_{\lambda\kappa}r_\sigma + \delta_{\lambda\sigma}r_\kappa)\right] \\ &- 8\, n_\rho r_\rho\,\frac{r_\lambda r_\kappa r_\sigma}{R^4} + \frac{2}{R^2}\nu(n_\kappa r_\lambda r_\sigma + n_\sigma r_\lambda r_\kappa) - (1-4\nu)\, n_\lambda\delta_{\kappa\sigma} \\ &+ \frac{1}{R^2}(1-2\nu)\,(2n_\lambda r_\kappa r_\sigma + n_\kappa\delta_{\lambda\sigma} + n_\sigma\delta_{\lambda\kappa})\Big\} \end{aligned} \quad (4.17)$$

If one recalls formulas (4.12) and (4.15) it turns out that modification of existing codes can be performed in a straightforward manner.

5. Concluding remarks

First we remark that the paper by Constanda [5] gives such an asymptotic expansion for the displacements at infinity which ensures the validity of the Betti formula for exterior regions. Under this condition the total strain energy stored in the region is bounded. In addition uniqueness and existence proofs can be given with ease.

We have modified the Somigliana formulas for exterior regions by assuming that the strains are constants and accordingly the displacements are linear at infinity. Under this condition not the strain energy but the strain energy density is bounded and there is no need to replace the exterior region by a finite one if a constant stress condition is prescribed at infinity. This can be an advantage if one considers an infinite plane with holes or cracks in it subjected to constant stresses at infinity and an attempt is made to determine the stresses in finite. Modification of existing codes can easily be performed.

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