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KINEMATIC ADMISSIBILITY OF STRAINS FOR SOME MIXED BOUNDARY VALUE PROBLEMS IN THE DUAL SYSTEM OF MICROPOLAR THEORY OF ELASTICITY

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Abstract. Making use of the principle of minimum complementary energy we have clarified what conditions the strains should meet in order to be kinematically admissible for some mixed boundary value problems of micropolar elasticity in a dual formulation. Emphasis is laid on the question of what form the boundary conditions have since neither the displacements nor the microrotations belong to the set of fundamental variables.

Keywords: Dual system, mixed boundary value problems, micropolar elasticity, strain boundary conditions

1. Introduction

1.1. The necessary and sufficient conditions the strains in the linear theory of micropolar elasticity should meet in order to be kinematically admissible were found by Kozák-Szeidl [1] under the conditions that the boundary surface S of the region V under consideration was divided into two parts S_u and S_t on which tractions (force stresses and couple stresses) and generalized displacements (displacements and microrotations) were imposed.

Similar investigations within the framework of the classical theory have been performed by Kozák [2] who used stress functions of order two when setting up the entire system of variational principles in the dual system of elasticity. The necessary and sufficient conditions of kinematic admissibility for the strains were obtained from the stationary condition of the corresponding functionals.

Bertóti [3] confined himself to the case when the stresses are given in terms of stress functions of order one and under this condition he clarified, among other things, what form the equation system of linear elasticity has including those conditions the strains should meet in order to be kinematically admissible on a simple connected region. Some generalizations of these results are given in [4].

Returning to the micropolar theory, to the author's knowledge the cases when force stresses and microrotations or couple stresses and displacement fields are given on a part of the boundary (tractions are imposed on the other part of the boundary) has not been investigated yet though there are some preliminary results in this respect for the first plane problem – see Szeidl [5] and Iván-Szeidl [6] for details.

1.2. The main objective of the present paper is to clarify the form of the boundary conditions in the dual system if microrotations $\hat{\varphi}^b$ and force stresses $\hat{t}^k$ are imposed on the part $S_{t\varphi}$ of the boundary surface S while tractions (force stresses $\hat{t}^k$ and couple stresses $\hat{\mu}_b$) are prescribed on the other part $S_{t\mu}$ ($S_{t\mu} \cup S_{t\varphi} = S$, $S_{t\mu} \cap S_{t\varphi} = \emptyset$). It is a further aim to derive the unknown boundary conditions if displacements $\hat{u}_l$ and couple stresses $\hat{\mu}_b$ are imposed on $S_{u\mu}$ while tractions (force stresses $\hat{t}^k$ and couple stresses $\hat{\mu}_b$) are given on $S_{t\mu}$ ($S_{t\mu} \cup S_{u\mu} = S$, $S_{t\mu} \cap S_{u\mu} = \emptyset$). By solving the problem posed we shall also clarify all the conditions the strains should meet to be kinematically admissible under the given boundary conditions.

Observe that the problems posed are meaningless in the primal system of micropolar elasticity since the displacement field u_l and the microrotation φ^b (referred to together as displacements) are the configuration variables in this system.

1.3. The paper is organized into six Sections. In Section 2 notations and some preliminary results are presented. Sections 3 and 4 are devoted to the derivation of the missing conditions. Our analysis is based on the principle of minimum complementary energy which, as a variational principle, ensures the fulfillment of all conditions the strains should meet in order to be kinematically admissible. Section 5 is a short summary of the results. The last Section is an Appendix which contains some longer transformations.

2. Notations and preliminaries

2.1. For the sake of simplicity we shall assume that the volume region V occupied by the body under consideration is simple-connected. The boundary surface S is divided into two parts from which the boundary conditions on $S_{t\mu}$ are the same for the two problems considered. The common bounding curve is denoted by g. Figure 1 represents the region V and the parts $S_{t\varphi}$, $S_{t\mu}$ and $S_{u\mu}$, $S_{t\mu}$ for both problems. Indicial notations and two coordinate systems, the (x^1, x^2, x^3) curvilinear and the (ξ^1, ξ^2, ξ^3) curvilinear, defined on the surface S – see [7] for details – are employed throughout this paper. Scalars and tensors, unless otherwise stated, are denoted independently of the coordinate system by the same letter. Distinction is aided by the indication of the arguments x and ξ being used to denote the totality of the corresponding coordinates. Volume integrals and surface integrals are considered, respectively, in the coordinate systems $(x^1x^2x^3)$ and $(\xi^1\xi^2\xi^3)$. Consequently, in the case of integrals, arguments are omitted. In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a semicolon denote covariant differentiation with respect to the corresponding subscripts. Latin and Greek indices range over the integers 1, 2, 3 and 1, 2 respectively. ϵ^{klm} and ϵ_{pqr} stand for the permutation tensors; δ_k^l is the *Kronecker* delta. In the system of coordinates $(x^1x^2x^3)$ $\mathbf{g}_k$ and $\mathbf{g}^l$ are the covariant and contravariant base vectors. The corresponding metric tensors are denoted by g_{kl} and g^{pq}. The equation of the boundary surface is written as $x^k = x^k(\xi^1, \xi^2)$ where the coordinates ξ^1 and ξ^2 are the surface parameters. Let ξ^3 be the distance measured on the outward unit normal $\mathbf{n}$ to the surface. On S $\xi^3 = 0$. [Base vectors] {Metric tensors} on S are denoted by [$\mathbf{a}^k$ and $\mathbf{a}_k$] {a_{kl} and a^{kl}}. In the

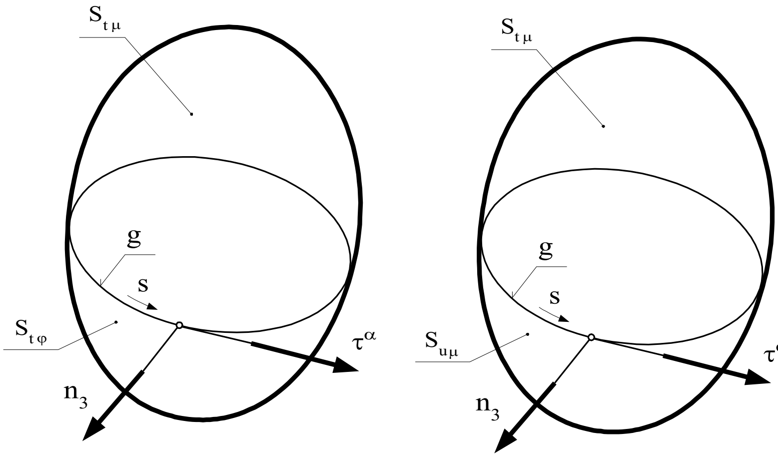

Figure 1.

coordinate system $(\xi^1\xi^2\xi^3)$

$$\mathbf{n} = \mathbf{a}_3 = \mathbf{a}^3\,, \qquad n^3 = 1 \qquad \text{and} \qquad n^\eta = 0\,. \tag{2.1}$$

If $\left|\xi^3\right| / (\min\{|R_1|, |R_2|\}) < 1$ in which R_1 and R_2 are the principal radii of curvature on S then the relationship $x^k = x^k(\xi^1, \xi^2, \xi^3)$ is always one-to-one.

2.2. The components of the asymmetric strain tensor and rotation tensor (together strains), and the force stress tensor and couple stress tensor (together stresses) are denoted by γ_{lk}, $\kappa_a{}^b$ and t^{kl}, $\mu^a{}_b$ respectively. In the primal system of micropolar elasticity the field equations can be given by the displacements u_l, φ^b as configuration variables, the strains γ_{lk}, $\kappa_a{}^b$ as intermediate variables of order one and the stresses t^{kl}, $\mu^a{}_b$ as intermediate variables of order two:

kinematic equations:

$$\gamma_{lk} = u_{l;k} + \varepsilon_{lks}\varphi^s, \qquad \kappa_a{}^b = \varphi^b{}_{;a} \qquad\qquad x \in V \tag{2.2}$$

Hook's law for centrosymmetric body [8]:

$$t^{kl} = \underset{-1}{a}{}^{klpq}\gamma_{pq}, \qquad \mu^{ab} = \underset{-1}{c}{}^{klpq}\kappa_{pq} \qquad\qquad x \in V \tag{2.3}$$

equilibrium equations:

$$t^{kl}{}_{;k} + b^l = 0, \qquad \mu^a{}_{b;a} + \varepsilon_{bpq}t^{pq} + c_b = 0 \qquad x \in V \tag{2.4}$$

where $\underset{-1}{a}{}^{klpq}$ and $\underset{-1}{c}{}^{klpq}$ are the tensors of elastic coefficients while b^l and c_b are body forces and couples. These equations are associated with the boundary conditions

$$n_3 t^{3l} = \hat{t}^l, \qquad \varphi^a = \hat{\varphi}^a \qquad\qquad \xi \in S_{t\varphi} \tag{2.5a}$$

$$n_3 t^{3l} = \hat{t}^l, \qquad n_3\mu^3{}_b = \hat{\mu}_b \qquad\qquad \xi \in S_{t\mu} \tag{2.5b}$$

for the first problem and with the boundary conditions

$$u_l = \hat{u}_l\,, \qquad n_3 \mu^3{}_b = \hat{\mu}_b \qquad \xi \in S_{u\mu} \tag{2.6a}$$

$$n_3 t^{3l} = \hat{t}^l\,, \qquad n_3 \mu^3{}_b = \hat{\mu}_b \qquad \xi \in S_{t\mu} \tag{2.6b}$$

for the second one.

2.3. The stresses t^{kl} and $\mu^a{}_b$ are said to be equilibrated {statically admissible} if they satisfy the equilibrium equations (2.4) {and the traction boundary conditions $(2.5\text{a})_1$, (2.5b) or $(2.6\text{a})_2$, (2.6b) }. Let

$$\overset{\circ}{t}{}^{kl} = p^l{}_{;m} g^{mk} \qquad \text{and} \qquad \overset{\circ}{\mu}{}^a{}_b = g^{am}(\varepsilon_{mby} p^y + q_{b;m}) \qquad x \in V \tag{2.7a}$$

where

$$g^{mn} p^l{}_{;mn} = -b^l \qquad \text{and} \qquad g^{mn} q_{b;mn} = -c_b\,. \qquad x \in V \tag{2.7b}$$

Further let $\mathcal{F}_y{}^l$ and $\mathcal{H}_{yb}$ $\;x \in V\;$ be stress function tensors. Then the stresses

$$t^{kl} = \varepsilon^{kmy} \mathcal{F}_y{}^l{}_{;m} + \overset{\circ}{t}{}^{kl}, \qquad \mu^a{}_b = \varepsilon^{apy}\left(\mathcal{H}_{yb;p} + \varepsilon_{bps}\mathcal{F}_y{}^s\right) + \overset{\circ}{\mu}{}^a{}_b \qquad x \in V \tag{2.8}$$

are equilibrated and $\overset{\circ}{\mu}{}^a{}_b$, $\overset{\circ}{t}{}^{kl}$ are particular solutions to the equilibrium equations (2.4) [9,10]. We shall assume that the particular solutions are known.

Remark 2.1.: Let $\alpha_{yb}(x)$ and $\beta^y{}_l(x)$ be differentiable otherwise arbitrary tensors on V. Further let ${}_{AB}$ and ${}_K^L$ be the subsets of the index pairs ${}_{yb}$ and ${}_y{}^l$ for which the differential equations

$$r^L{}_{;K} = \beta^L{}_K(x) \qquad \text{and} \qquad w_{B;A} + \varepsilon_{BAs} r^s = \alpha_{AB}(x) \qquad x \in V \tag{2.9}$$

always have a solution for the vector fields r^l and w_b. It can be shown that the stress function triplets $\mathcal{H}_{AB}$ and $\mathcal{F}_K^L$ can be set to zero [1].

Remark 2.2.: The proof of Remark 2.1 [1] is based on the observation that there belong identically zero stresses to the stress functions

$$\mathcal{F}_y{}^l = r^l{}_{;y}(x) \qquad \text{and} \qquad \mathcal{H}_{yb} = w_{b;y}(x) + \varepsilon_{bys} r^s(x)\,. \qquad x \in V \tag{2.10}$$

Remark 2.3.: Let ${}_{XY}$ and ${}_S^T$ be the complementary subsets of the index pairs ${}_{AB}$ and ${}_K^L$. It follows from Remark 2.1 that any stress condition can be given in terms of the stress functions $\mathcal{F}_S{}^T$ and $\mathcal{H}_{AB}$, i.e., by means of six-six stress functions.

Remark 2.4.: For this reason we shall assume that the stress functions and their variations have only six independent components each which are identified by the index pairs ${}_{XY}$ and ${}_S^T$.

2.4. The strains γ_{lk}, $\kappa_a{}^b$ are said to be compatible {kinematically admissible} if the kinematic equations (2.2) have single valued solutions for the displacements u_l, φ^b {and the solutions meet the boundary condition $(2.5\text{a})_2$ or $(2.6\text{a})_2$}.

The incompatibility tensors $\mathcal{Y}^{xy}$ and $\mathcal{D}^{s}{}_{t}$ are defined by the equations

$$\mathcal{Y}^{xy}(x) = \varepsilon^{xpa}\kappa_{a}{}^{y}{}_{;p} \quad \text{and} \quad \mathcal{D}^{s}{}_{t}(x) = \varepsilon^{spk}\left(\gamma_{kt;p} + \varepsilon_{ktb}\kappa_{p}{}^{b}\right) . \qquad x \in V \qquad (2.11)$$

It can be seen [1] that the strains γ_{lk}, $\kappa_{a}{}^{b}$ are compatible on a simple connected domain V if the six-six compatibility field equations

$$\mathcal{Y}^{XY}(x) = 0 \qquad \text{and} \qquad \mathcal{D}^{S}{}_{T}(x) = 0 \qquad x \in V \qquad (2.12a)$$

and the compatibility boundary conditions

$$n_3\mathcal{Y}^{3b}(x) = 0 \qquad \text{and} \qquad n_3\mathcal{D}^{3}{}_{t}(x) = 0 \qquad \xi \in S \qquad (2.12b)$$

are satisfied [1].

2.5. In the dual system of micropolar elasticity the stress functions $\mathcal{F}_{S}{}^{T}$, $\mathcal{H}_{XY}$ are the configuration variables ($\mathcal{F}_{K}^{L}$ and $\mathcal{H}_{AB}$ are set to zero), the stresses t^{kl}, $\mu^{a}{}_{b}$ are the intermediate variables of order one and the strains γ_{lk}, $\kappa_{a}{}^{b}$ are the intermediate variables of order two. The field equations consists of

the dual kinematic equations

$$t^{kl} - \overset{\circ}{t}{}^{kl} = \varepsilon^{kmy}\mathcal{F}_{y}{}^{l}{}_{;m}, \qquad \mu^{a}{}_{b} - \overset{\circ}{\mu}{}^{a}{}_{b} = \varepsilon^{apy}\left(\mathcal{H}_{yb;p} + \varepsilon_{bps}\mathcal{F}_{y}{}^{s}\right) \qquad x \in V \qquad (2.13)$$

Hook's law

$$\gamma_{kl} = a_{klpq}t^{pq}, \qquad \kappa_{ab} = c_{abpq}\mu^{pq} \qquad x \in V \qquad (2.14)$$

(a_{klpq} and c_{abpq} are the inverses of $\underset{-1}{a}{}^{klpq}$ and $\underset{-1}{c}{}^{klpq}$)

and the dual equilibrium equations

$$\mathcal{Y}^{XY}(x) = \varepsilon^{Xpa}\kappa_{a}{}^{Y}{}_{;p} = 0\,, \qquad \mathcal{D}^{S}{}_{T}(x) = \varepsilon^{Spk}\left(\gamma_{kT;p} + \varepsilon_{kTb}\kappa_{p}{}^{b}\right) = 0\,. \qquad x \in V \qquad (2.15)$$

In view of (2.13) it follows from equations (2.5a,b) and (2.6a,b) that the field equations (2.13), (2.14) and (2.15) should be associated with the traction boundary conditions

$$n_3\overset{\circ}{t}{}^{3l} + n_3\varepsilon^{3\mu\eta}\mathcal{F}_{\eta}{}^{l}{}_{;\mu} = \hat{t}^{l} \qquad \xi \in S \qquad (2.16a)$$

$$n_3\overset{\circ}{\mu}{}^{3}{}_{b} + n_3\varepsilon^{3\pi\eta}\left(\mathcal{H}_{\eta b;\pi} + \varepsilon_{b\pi s}\mathcal{F}_{\eta}{}^{s}\right) = \hat{\mu}_{b} \qquad \xi \in S_{t\mu} \qquad (2.16b)$$

for the first problem and with the traction boundary conditions

$$n_3\overset{\circ}{t}{}^{3l} + n_3\varepsilon^{3\mu\eta}\mathcal{F}_{\eta}{}^{l}{}_{;\mu} = \hat{t}^{l} \qquad \xi \in S_{t\mu} \qquad (2.17a)$$

$$n_3\overset{\circ}{\mu}{}^{3}{}_{b} + n_3\varepsilon^{3\pi\eta}\left(\mathcal{H}_{\eta b;\pi} + \varepsilon_{b\pi s}\mathcal{F}_{\eta}{}^{s}\right) = \hat{\mu}_{b} \qquad \xi \in S \qquad (2.17b)$$

for the second one.

It is obvious that the compatibility boundary conditions (2.12b) should also be fulfilled.

We remind again the reader of the fact that no boundary conditions can be imposed directly on the displacements u_l and φ^b since these variables do not belong to the set of dual variables.

The strains γ_{kl}, κ_{ab} are said to be statically admissible if they are given in terms of statically admissible stresses t^{pq} and μ^{pq} by means of Hook's law (2.14).

3. Kinematic admissibility of strains for the first problem

3.1. The total complementary energy functional for the first problem assumes the form

$$K_I = \frac{1}{2}\int_V \left(t^{kl}\gamma_{kl} + \mu^a{}_b\kappa_a{}^b\right)\,\mathrm{d}V - \int_{S_{t\varphi}} n_3\mu^3{}_b\hat{\varphi}^b\,\mathrm{d}A \tag{3.1}$$

where both the stresses t^{kl}, $\mu^a{}_b$ and the strains γ_{kl}, $\kappa_a{}^b$ are statically admissible.

According to the principle of minimum complementary energy, the first variation of the functional K_I should vanish:

$$\delta K_I = I_1^V + I_1^{S_{t\varphi}} = \int_V \left(\gamma_{kl}\delta t^{kl} + \kappa_a{}^b\delta\mu^a{}_b\right)\,\mathrm{d}V - \int_{S_{t\varphi}} n_3\delta\mu^3{}_b\hat{\varphi}^b\,\mathrm{d}A = 0\,. \tag{3.2}$$

Because the stresses and strains are statically admissible, variations δt^{kl} and $\delta\mu^a{}_b$ of the force and couple stresses can not be arbitrary but should meet the side conditions

$$\delta t^{kl}{}_{;k} = 0\,, \qquad \delta\mu^a{}_{b;a} + \varepsilon_{bkl}\delta t^{kl} = 0 \qquad x \in V \tag{3.3a}$$

and

$$n_3\delta t^{3l} = 0\,, \qquad \xi \in S \tag{3.3b}$$

$$n_3\delta\mu^3{}_b = 0\,. \qquad \xi \in S_{t\mu} \tag{3.3c}$$

In the sequel we shall assume that the variations of the particular solutions $\mathring{t}^{kl}$ and $\mathring{\mu}^a{}_b$ are equal to zero.

It can be proved by direct substitutions that the side conditions (3.3a) are identically fulfilled if the variations of stresses are given in terms of the variations of stress functions as follows:

$$\delta t^{kl} = \varepsilon^{kmy}\delta\mathcal{F}_y{}^l{}_{;m}\,, \qquad x \in V \tag{3.4a}$$

$$\delta\mu^a{}_b = \varepsilon^{apy}\left(\delta\mathcal{H}_{yb;p} + \varepsilon_{bps}\delta\mathcal{F}_y{}^s\right)\,. \qquad x \in V \tag{3.4b}$$

Let r^l and w_b be arbitrary differentiable vector fields on S and $S_{t\mu}$, respectively. Further let the variations of the stress functions on $S_{t\mu}$ and $S_{t\varphi}$ be given in terms of the variations of r^l and w_b as follows:

$$\delta\mathcal{F}_\eta{}^l = \delta r^l{}_{;\eta}\,, \qquad \xi \in S \tag{3.5a}$$

$$\delta\mathcal{H}_{\eta b} = \delta w_{b;\eta} + \varepsilon_{b\pi l}r^l\,. \qquad \xi \in S_{t\mu} \tag{3.5b}$$

If this is the case, side conditions (3.3b) and (3.3c) are also identically satisfied.

3.2. If we utilize that

$$\delta K_I = I_2^V + I_1^{S_{t\varphi}} + I_1^S = I_2^V + I_2^{S_{t\varphi}} + I_3^{S_{t\varphi}} + I_1^{S_{t\mu}} + I_2^{S_{t\mu}} + I_2^S + I_1^G$$

(the details of the manipulations leading to this form are given in Subsection 6.2) and substitute equations (6.5), (6.6) and (6.7) into the above equation then the stationary condition (3.2) yields

$$\begin{aligned}\delta K_I = &\int_V (\mathcal{Y}^{XY}\delta\mathcal{H}_{XY} + \mathcal{D}^S{}_T\delta\mathcal{F}_S{}^T)\,\mathrm{d}V - \int_{S_{t\varphi}} \varepsilon^{3\pi\eta}(\kappa_\pi{}^b - \hat{\varphi}^b{}_{;\pi})\delta\mathcal{H}_{xy}\,\mathrm{d}A \\ &- \int_{S_{t\varphi}} \varepsilon^{3\eta\chi}(\gamma_{\chi l;\eta} - \varepsilon_{\chi lb}\hat{\varphi}^b{}_{;\eta})\delta r^l\,\mathrm{d}A - \int_{S_{t\mu}} \left[n_3\mathcal{Y}^{3b}\delta w_b + n_3\mathcal{D}^3{}_l\delta r^l\right]\mathrm{d}A \\ &+ \int_g \tau^\eta(\kappa_\eta{}^b - \hat{\varphi}^b{}_{;\eta})\delta w_b\,\mathrm{d}s = 0\,. \end{aligned} \tag{3.6}$$

With regard to the arbitrariness of the variations $\delta\mathcal{H}_{XY},\ldots,\delta w_b$ this equation is equivalent to the compatibility field equations (2.15), the strain boundary conditions

$$\kappa_\pi{}^b - \hat{\varphi}^b{}_{;\pi} = 0\,, \qquad \xi \in S_{t\varphi} \tag{3.7}$$

$$\varepsilon^{3\eta\chi}(\gamma_{\chi l;\eta} - \varepsilon_{\chi lb}\hat{\varphi}^b{}_{;\eta}) = 0\,, \qquad \xi \in S_{t\varphi} \tag{3.8}$$

the compatibility boundary conditions (2.12b) on $S_{t\mu}$ and the continuity condition $d\hat{\varphi}^b/ds - \tau^\eta\kappa_\eta{}^b = 0$ on g.

REMARK 3.1.: It can be shown with ease that the fulfillment of the strain boundary conditions (3.7) and (3.8) ensures that of the compatibility boundary conditions (2.12b) on $S_{t\varphi}$.

4. Kinematic admissibility of strains for the second problem

4.1. The total complementary energy functional and the corresponding stationary condition are of the form

$$K_{II} = \frac{1}{2}\int_V \left(t^{kl}\gamma_{kl} + \mu^a{}_b\kappa_a{}^b\right)\,\mathrm{d}V - \int_{S_{u\mu}} n_3 t^{3l}\hat{u}_l\,\mathrm{d}A \tag{4.1}$$

and

$$\delta K_{II} = I_1^V + I_1^{S_{u\mu}} = \int_V \left(\gamma_{kl}\delta t^{kl} + \kappa_a{}^b\delta\mu^a{}_b\right)\,\mathrm{d}V - \int_{S_{u\mu}} n_3\delta t^{3l}\hat{u}_l\,\mathrm{d}A = 0\,, \tag{4.2}$$

respectively. In this case the variations δt^{kl} and $\delta\mu^a{}_b$ of the force and couple stresses should meet the side conditions (3.3a) and

$$n_3\delta t^{3l} = 0\,, \qquad n_3\delta\mu^3{}_b = 0\,, \qquad \xi \in S_{t\mu} \tag{4.3a}$$

$$n_3\delta\mu^3{}_b = 0\,. \qquad \xi \in S_{u\mu} \tag{4.3b}$$

Side conditions (3.3a) are again identically fulfilled if the variations of stresses are given in terms of the variations of stress functions in the same form as for the first problem – see equations (3.4a,b) for details.

Let r^l and w_b be arbitrary differentiable vector fields on $S_{t\mu}$, respectively. If the variations of the stress functions on $S_{t\mu}$ are given in terms of the variations of r^l and w_b :

$$\delta\mathcal{F}_\eta{}^l = \delta r^l{}_{;\eta} \qquad \delta\mathcal{H}_{\eta b} = \delta w_{b;\eta} + \varepsilon_{b\pi l} r^l \quad \xi \in S_{t\mu} \tag{4.4}$$

then side conditions (4.3a) are fulfilled. If the variations of the couple stresses are given in terms of the variations of stress functions, then substitution of representation (3.4b) into the side condition (4.3b) yields

$$n_3 \delta\mu^3{}_b = \varepsilon^{3\pi\eta} \left(\delta\mathcal{H}_{\eta b;\pi} + \varepsilon_{b\pi s}\delta\mathcal{F}_\eta{}^s\right) = 0 \quad \xi \in S_{u\mu} \tag{4.5}$$

from which it follows with regard to (6.2,c) that the variations $\delta\mathcal{F}_2{}^2$ and $\delta\mathcal{F}_\beta{}^3$ can not be arbitrary but should meet the conditions

$$\delta\mathcal{F}_2{}^2 = -\delta\mathcal{F}_1{}^1 - \varepsilon^{3\pi\eta}(\delta\mathcal{H}_{\eta 3||\pi} + b^\sigma_\pi \delta\mathcal{H}_{\eta\sigma} - b_{\eta\pi}\delta\mathcal{H}_{33}) , \qquad \xi \in S_{tu} \tag{4.6a}$$

$$\delta\mathcal{F}_\rho{}^3 = \varepsilon^{3\pi\eta}\delta\mathcal{H}_{\eta\rho;\pi} = \varepsilon^{3\pi\eta}\left(\delta\mathcal{H}_{\eta\rho||\pi} - b_{\eta\pi}\delta\mathcal{H}_{3\rho} - b_{\rho\pi}\delta\mathcal{H}_{\eta 3}\right) . \qquad \xi \in S_{tu} \tag{4.6b}$$

Derivation of the equations that follow from the stationary condition (4.2) requires a lengthy formal transformation which is based on the use of equations (3.4a,b) and (4.4), (4.6a,b) since their fulfillment ensures that of the side conditions on V and S. In addition integrations by parts should be performed by utilizing the Gauss and Stokes theorems. As regards the transformation the details are given in Subsections 6.3 to 6.4. Here we confine ourselves only to gathering the results. Comparison of the stationary condition (4.2) with equations (6.8), (6.9), (6.13) and (6.15) yields

$$\begin{aligned}\delta K_{II} &= I_1^V + I_1^{S_{u\mu}} = I_2^V + I_2^{S_{u\mu}} + I_3^{S_{u\mu}} + I_3^{S_{t\mu}} + I_2^G = \\ &= I_2^V + I_4^{S_{u\mu}} + I_3^{S_{t\mu}} + I_2^G + I_3^G = I_2^V + I_4^{S_{u\mu}} + I_4^{S_{t\mu}} + I_2^G + I_3^G + I_3^G = 0 .\end{aligned} \tag{4.7}$$

The final form of stationary condition (4.7) is obtained by substituting (6.4) for I_2^V, (6.14a) for $I_4^{S_{u\mu}}$, (6.16a) for $I_4^{S_{t\mu}}$ and (6.17) for $I_2^G + I_3^G + I_3^G$. After making some rearrangements we have

$$\begin{aligned}\delta K_{II} = &\int_V (\mathcal{Y}^{XY}\delta\mathcal{H}_{XY} + \mathcal{D}^S{}_T\delta\mathcal{F}_S{}^T)\,\mathrm{d}V + \int_{S_{u\mu}} \varepsilon^{3\pi\eta}(\tilde{\varphi}^\rho{}_{||\pi} - b^\rho_\pi\tilde{\varphi}^3 - \kappa_\pi{}^\rho)\delta\mathcal{H}_{\eta\rho}\,\mathrm{d}A + \\ &+ \int_{S_{u\mu}} \left[\varepsilon^{312}\left(\hat{u}_{1;1} - \gamma_{11}\right)\delta\mathcal{F}_2{}^1 + \varepsilon^{321}\left(\hat{u}_{2;2} - \gamma_{22}\right)\delta\mathcal{F}_1{}^1\right]\,\mathrm{d}A + \\ &+ \int_{S_{u\mu}} \varepsilon^{3\pi\eta}(\tilde{\varphi}^3{}_{||\pi} + b_{\rho\pi}\tilde{\varphi}^\rho - \kappa_\pi{}^3)\delta\mathcal{H}_{\eta 3}\,\mathrm{d}A - \\ &- \int_{S_{u\mu}} \varepsilon^{312}\left[\hat{u}_{2;1} + \hat{u}_{1;2} - (\gamma_{12} + \gamma_{21})\right]\delta\mathcal{F}_1{}^1\mathrm{d}A - \int_{S_{t\mu}} \left[n_3\mathcal{Y}^{3b}\delta w_b + n_3\mathcal{D}^3{}_l\delta r^l\right]\,\mathrm{d}A \\ &- \oint_g (\frac{d\tilde{\varphi}^b}{ds} - \tau^\eta\kappa_\eta{}^b)\delta w_b\,\mathrm{d}s - \oint_g \left[\frac{d\hat{u}_\lambda}{ds} - \tau^\eta\left(\gamma_{\eta\lambda} + \varepsilon_{\eta\lambda 3}\tilde{\varphi}^3\right)\right]\delta r^\lambda\,\mathrm{d}s = 0 .\end{aligned} \tag{4.8}$$

Since the variations $\delta\mathcal{H}_{XY},\dots,\delta r^{\lambda}$ in the stationary condition $\delta K_{II}=0$ are arbitrary we obtain

– the compatibility field equations

(2.12a);

– the strain boundary conditions

$$\tilde{\varphi}^{\rho}{}_{||\pi} - b^{\rho}_{\pi}\tilde{\varphi}^{3} - \kappa_{\pi}{}^{\rho} = 0\,, \qquad \xi\in S_{u\mu} \tag{4.9a}$$

$$\tilde{\varphi}^{3}{}_{||\pi} + b_{\rho\pi}\tilde{\varphi}^{\rho} - \kappa_{\pi}{}^{3} = 0\,, \qquad \xi\in S_{u\mu} \tag{4.9b}$$

and

$$\hat{u}_{1;1} - \gamma_{11} = 0\,, \qquad \hat{u}_{2;2} - \gamma_{22} = 0\,, \qquad \xi\in S_{u\mu} \tag{4.10a}$$

$$\hat{u}_{2;1} + \hat{u}_{1;2} - (\gamma_{12}+\gamma_{21}) = 0\,; \qquad \xi\in S_{u\mu} \tag{4.10b}$$

– the compatibility boundary conditions

(2.12b)

on $S_{t\mu}$

and

the continuity conditions

$$\frac{d\tilde{\varphi}^{b}}{ds} - \tau^{\eta}\kappa_{\eta}{}^{b} = 0\,, \qquad \frac{d\tilde{u}_{\lambda}}{ds} - \tau^{\eta}\left(\gamma_{\eta\lambda} + \varepsilon_{\eta\lambda 3}\tilde{\varphi}^{3}\right)\,. \qquad \xi\in g \tag{4.11}$$

Remark 4.1.: It can be shown by performing lengthy hand made calculations, which require some attention, that the fulfillment of the strain boundary conditions (4.9a,b) and (4.10a,b) ensures that of the compatibility boundary conditions (2.12b) on $S_{u\mu}$.

5. Concluding remarks

5.1. We have clarified what boundary conditions the strains of the micropolar theory should meet in order to be kinematically admissible if

– microrotations and force stresses

or

– displacements and couple stresses

are imposed on a part of the boundary surface. The corresponding boundary conditions – like those found by Kozák-Szeidl in 1981 [2], - are referred to as strain boundary conditions. We draw the reader's attention to the fact that the fulfillment of the strain boundary conditions ensures that of the compatibility boundary conditions for both problems – see Remarks 3.1 and 4.1.

5.2. It is a further issue what form the strain boundary conditions have if for instance displacements are given in the tangent plane to the surface and force stress is prescribed perpendicularly to it etc. Investigations to find an appropriate reply to the latter problem are in progress.

6. Appendix

6.1. Let b^σ_π and $b_{\eta\pi}$ be the mixed and covariant components of the tensor of curvature on S. Further let $\tilde{\varphi}^b$ and $\mathcal{H}_{\eta b}$ be differentiable vector and tensor fields defined on V and S. The covariant derivatives taken on the surface with respect to the surface coordinates and the surface covariant derivatives are denoted by $\tilde{\varphi}^\rho{}_{|\pi}$, $\mathcal{H}_{\eta b|\pi}$ and $\tilde{\varphi}^\rho{}_{||\pi}$, $\mathcal{H}_{\eta b||\eta}$, respectively. The following relations hold

$$\tilde{\varphi}^\rho{}_{;\pi} = \tilde{\varphi}^\rho{}_{|\pi} = \tilde{\varphi}^\rho{}_{||\pi} - b^\rho_\pi \tilde{\varphi}^3 \,, \tag{6.1}$$

$$\mathcal{H}_{\eta\beta;\pi} = \mathcal{H}_{\eta\beta|\pi} = \mathcal{H}_{\eta\beta||\pi} - b_{\eta\pi}\mathcal{H}_{3\beta} - b_{\beta\pi}\mathcal{H}_{\eta 3} \,, \tag{6.2}$$

$$\mathcal{H}_{\eta 3;\pi} = \mathcal{H}_{\eta 3|\pi} = \mathcal{H}_{\eta 3||\pi} + b^\sigma_\pi \delta\mathcal{H}_{\eta\sigma} - b_{\eta\pi}\delta\mathcal{H}_{33} \,. \tag{6.3}$$

6.2. Transformation of the volume integral I^V_1 – see equation (3.2) – requires

- substitution of (3.4a,b) for the variations δt^{kl} and $\delta\mu^a{}_b$
- performance of integrations by parts making use of the Gauss theorem

and

- an appropriate rearrangement after utilizing the definitions given by (2.11) for $\mathcal{Y}^{xy}$ and $\mathcal{D}^s{}_t$.

In addition one should utilize the assumption in Remark 2.4. After performing the steps listed above we have

$$I^V_1 = I^V_2 + I^S_1 = \\ = \int_V (\mathcal{Y}^{XY}\delta\mathcal{H}_{XY} + \mathcal{D}^S{}_T \delta\mathcal{F}_S{}^T)\,\mathrm{d}V - \int_S (\varepsilon^{3\chi\eta}\gamma_{\chi l}\delta\mathcal{F}_\eta{}^l + \varepsilon^{3\pi\eta}\kappa_\pi{}^b \delta\mathcal{H}_{\eta b})\,\mathrm{d}A \,. \tag{6.4}$$

This transformation is valid both for the first problem and for the second one.

Integral $I^{S_{t\varphi}}_1$ of equation (3.2) can be manipulated into a more suitable form by applying the Stokes theorem:

$$I^{S_{t\varphi}}_1 = \oint_g \tau^\eta \hat{\varphi}^b \delta\mathcal{H}_{\eta b}\,\mathrm{d}s + \int_{S_{t\varphi}} \varepsilon^{3\pi\eta}\hat{\varphi}^b{}_{;\eta}\delta\mathcal{H}_{\eta b}\mathrm{d}A - \int_{S_{t\varphi}} \varepsilon^{3\pi\eta}\varepsilon_{b\pi s}\delta\mathcal{F}_\eta{}^s \hat{\varphi}^b\,\mathrm{d}A \,.$$

Substituting (3.5a,b) into the sum $I^{S_{t\varphi}}_1 + I^S_1$ and taking the relation $S = S_{t\varphi} \cup S_{t\mu}$ into account we have

$$I^{S_{t\varphi}}_1 + I^S_1 = I^{S_{t\varphi}}_2 + I^{S_{t\varphi}}_3 + I^{S_{t\mu}}_1 + I^{S_{t\mu}}_2 + I^S_2 + I^G_1 \tag{6.5}$$

where

$$\begin{aligned} I^{S_{t\varphi}}_2 &= -\int_{S_{t\varphi}} \varepsilon^{3\pi\eta}\left(\kappa_\pi{}^b - \hat{\varphi}^b{}_{;\eta}\right)\delta\mathcal{H}_{\eta b}\,\mathrm{d}A \,, & I^{S_{t\varphi}}_3 &= -\int_{S_{t\varphi}} \varepsilon^{3\pi\eta}\varepsilon_{b\pi s}\delta r^s{}_{;\eta}\hat{\varphi}^b\,\mathrm{d}A \,, \\ I^{S_{t\mu}}_1 &= -\int_{S_{t\mu}} \varepsilon^{3\pi\eta}\kappa_\pi{}^b \delta w_{b;\eta}\,\mathrm{d}A \,, & I^{S_{t\mu}}_2 &= -\int_{S_{t\mu}} \varepsilon^{3\pi\eta}\varepsilon_{b\pi l}\kappa_\pi{}^b \delta r^l \mathrm{d}A \,, \\ I^S_2 &= -\int_S \varepsilon^{3\chi\eta}\gamma_{\chi l}\delta r^l{}_{;\eta}\,\mathrm{d}A \,, & I^G_1 &= \oint_g \tau^\eta \hat{\varphi}^b \left(\delta w_{b;\eta} + \varepsilon_{b\pi l}\delta r^l\right)\,\mathrm{d}s \,. \end{aligned} \tag{6.6}$$

If we now integrate the right sides of $I_2^{S_{t\varphi}}$, $I_1^{S_{t\mu}}$ and I_3^S by parts – this transformation is based upon the Stokes theorem – then perform integrations by parts along the curve g, we obtain

$$
\begin{aligned}
I_3^{S_{t\varphi}} + I_1^{S_{t\mu}} + I_2^{S_{t\mu}} + I_2^S + I_1^G = &- \int_{S_{t\varphi}} \varepsilon^{3\eta\chi} \left(\gamma_{\chi l;\eta} + \varepsilon_{\chi lb}\hat{\varphi}^b{}_{;\eta}\right) \delta r^l \, \mathrm{d}A \\
&- \int_{S_{t\mu}} \left[n_3 \mathcal{Y}^{3b} \delta w_b + n_3 \mathcal{D}^3{}_l \delta r^l\right] \mathrm{d}A + \int_g \tau^\eta (\kappa_\eta{}^b - \hat{\varphi}^b{}_{;\eta}) \delta w_b \, \mathrm{d}s \,. \qquad (6.7)
\end{aligned}
$$

6.3. Substituting equation (3.3a) into the integral $I_1^{S_{u\mu}}$ and making use of the Stokes theorem we have:

$$
I_1^{S_{u\mu}} = I_2^{S_{u\mu}} + I_2^G = \int_{S_{u\mu}} \varepsilon^{3\chi\eta} \hat{u}_{l;\chi} \delta \mathcal{F}_\eta{}^l \, \mathrm{d}A + \oint_g \tau^\eta \hat{u}_l \delta \mathcal{F}_\eta{}^l \, \mathrm{d}A \,. \qquad (6.8)
$$

Recalling the relation $S = S_{u\mu} \cup S_{t\mu}$ the integral taken over S in (6.4) can be resolved into two parts

$$
I_1^S = I_3^{S_{t\mu}} + I_3^{S_{u\mu}} = \int_{S_{t\mu}} \ldots \mathrm{d}A + \int_{S_{u\mu}} \ldots \mathrm{d}A \,. \qquad (6.9)
$$

In view of (6.9) it is now our aim to transform the sum

$$
\begin{aligned}
I_2^{S_{u\mu}} + I_3^{S_{u\mu}} = &- \int_{S_{u\mu}} \varepsilon^{3\pi\eta} \kappa_\pi{}^3 \delta \mathcal{H}_{\eta 3} \, \mathrm{d}A - \int_{S_{u\mu}} \varepsilon^{3\pi\eta} \kappa_\pi{}^\beta \delta \mathcal{H}_{\eta\beta} \, \mathrm{d}A \\
&+ \int_{S_{u\mu}} \left(\hat{u}_{3;\chi} - \gamma_{\chi 3}\right) \varepsilon^{3\chi\rho} \delta \mathcal{F}_\rho{}^3 \, \mathrm{d}A + \int_{S_{u\mu}} \left(\hat{u}_{\lambda;\chi} - \gamma_{\chi\lambda}\right) \varepsilon^{3\chi\rho} \delta \mathcal{F}_\rho{}^\lambda \, \mathrm{d}A \qquad (6.10)
\end{aligned}
$$

into a more appropriate form. This aim can be achieved

- if we introduce the notations

$$
\tilde{\varphi}^\rho = -\varepsilon^{3\chi\rho} \left(\hat{u}_{3;\chi} - \gamma_{\chi 3}\right) \qquad \xi \in S_{u\mu} \qquad (6.11)
$$

  and

$$
\tilde{\varphi}^3 = -\varepsilon^{132} \left(\hat{u}_{2;1} - \gamma_{12}\right) \,, \qquad \xi \in S_{u\mu} \qquad (6.12)
$$

- if we substitute the representation (as a matter of fact side condition) (4.6b) for $\delta\mathcal{F}_\rho{}^3$,
- if we write (4.6b) for $\delta\mathcal{F}_2{}^2$ in the last integral of equation (6.10),
- if we substitute the right sides of the equations

$$
\begin{aligned}
- \int_{S_{u\mu}} \tilde{\varphi}^\rho \varepsilon^{3\pi\eta} \delta \mathcal{H}_{\eta\rho||\pi} \, \mathrm{d}A &= \int_{S_{u\mu}} \varepsilon^{3\pi\eta} \tilde{\varphi}^\rho{}_{||\pi} \delta \mathcal{H}_{\eta\rho} \, \mathrm{d}A + \oint_g \tau^\eta \tilde{\varphi}^\rho \delta \mathcal{H}_{\eta\rho} \, \mathrm{d}s \\
- \int_{S_{u\mu}} \tilde{\varphi}^3 \varepsilon^{3\pi\eta} \delta \mathcal{H}_{\eta 3||\pi} \, \mathrm{d}A &= \int_{S_{u\mu}} \varepsilon^{3\pi\eta} \tilde{\varphi}^3{}_{||\pi} \delta \mathcal{H}_{\eta 3} \, \mathrm{d}A + \oint_g \tau^\eta \tilde{\varphi}^3 \delta \mathcal{H}_{\eta 3} \, \mathrm{d}s
\end{aligned}
$$

  each obtained by making use of the Stokes theorem and

– if we make an appropriate rearrangement.

After performing the aforementioned steps we get

$$I_2^{S_{u\mu}} + I_3^{S_{u\mu}} = I_4^{S_{u\mu}} + I_3^G \tag{6.13}$$

where

$$\begin{aligned} I_4^{S_{u\mu}} = & \int_{S_{u\mu}} \left[\varepsilon^{312} \left(\hat{u}_{1;1} - \gamma_{11}\right) \delta\mathcal{F}_2{}^1 + \varepsilon^{312} \left(\hat{u}_{2;2} - \gamma_{22}\right) \delta\mathcal{F}_1{}^2\right] \mathrm{d}A \\ & - \int_{S_{u\mu}} \varepsilon^{312} \left[\hat{u}_{2;1} + \hat{u}_{1;2} - (\gamma_{12} + \gamma_{21})\right] \delta\mathcal{F}_1{}^1 \, \mathrm{d}A \\ & + \int_{S_{u\mu}} \varepsilon^{3\pi\eta} \left[\tilde{\varphi}^\rho{}_{||\pi} - b^\rho_\pi \tilde{\varphi}^3 - \kappa_\pi{}^\beta\right] \delta\mathcal{H}_{\eta\rho} \, \mathrm{d}A \\ & - \int_{S_{u\mu}} \varepsilon^{3\pi\eta} \left[\tilde{\varphi}^3{}_{||\pi} - b_{\rho\pi} \tilde{\varphi}^\rho - \kappa_\pi{}^3\right] \delta\mathcal{H}_{\eta 3} \, \mathrm{d}A \end{aligned} \tag{6.14a}$$

and

$$I_3^G = \oint_g \tau^\eta \tilde{\varphi}^\rho \delta\mathcal{H}_{\eta\rho} \, \mathrm{d}s + \oint_g \tau^\eta \tilde{\varphi}^3 \delta\mathcal{H}_{\eta 3} \, \mathrm{d}s \,. \tag{6.14b}$$

6.4. The integral

$$I_3^{S_{t\mu}} = -\int_{S_{t\mu}} (\varepsilon^{3\chi\eta} \gamma_{\chi l} \delta\mathcal{F}_\eta{}^l + \varepsilon^{3\pi\eta} \kappa_\pi{}^b \delta\mathcal{H}_{\eta b}) \, \mathrm{d}A$$

can be manipulated into the form

$$I_3^{S_{t\mu}} = I_4^{S_{t\mu}} + I_4^G \tag{6.15}$$

where

$$I_4^{S_{t\mu}} = -\int_{S_{t\mu}} \left[n_3 \mathcal{Y}^{3b} \delta w_b + n_3 \mathcal{D}^3{}_l \delta r^l\right] \mathrm{d}A \tag{6.16a}$$

and

$$I_4^G = \oint_g \tau^\eta \gamma_{\eta l} \delta r^l_b \, \mathrm{d}s + \oint_g \tau^\eta \kappa_\eta{}^b \delta w_b \, \mathrm{d}s \tag{6.16b}$$

if we substitute $(4.4)_{1,2}$ for $\delta\mathcal{F}_\eta{}^l$ and $\delta\mathcal{H}_{\eta b}$ and then perform integrations by parts – the latter step is based on the Stokes theorem.

The sum of the line integrals $I_3^G + I_4^G$ can be cast into the final form

$$I_2^G + I_3^G + I_4^G = -\oint_g \left(\frac{d\tilde{\varphi}^b}{ds} - \tau^\eta \kappa_\eta{}^b\right) \delta w_b \, \mathrm{d}s - \oint_g \left[\frac{d\tilde{u}_b}{ds} - \tau^\eta \left(\gamma_{\eta b} + \varepsilon_{\eta b 3} \tilde{\varphi}^3\right)\right] \delta r^\lambda \, \mathrm{d}s \tag{6.17}$$

if we substitute $(4.4)_{1,2}$ for $\delta\mathcal{F}_\eta{}^l$ and $\delta\mathcal{H}_{\eta b}$ and then perform integrations by parts in order to remove the covariant derivatives from the integrand obtained after the substitution.

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