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ON DERIVATION OF STRESS FUNCTIONS IN MICROPOLAR THEORY OF ELASTICITY

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The main result of the present work has been the proof of the possibility that for micropolar bodies the general and complete solution of equilibrium equations in terms of stress functions can be derived from the general primal form of principle of virtual work. The most important functionals of Lagrange's type have also been presented assuming that the micropolar body under consideration is linearly elastic. The vanishing of variations with respect to the strain fields of the functionals as a variational principle ensures, through stress functions, the fulfillment of equilibrium equations and stress boundary conditions.

1. **Introduction, Preliminaries**

1.1. Representation for equilibrated stress fields in terms of stress functions is one of the problems which has been solved in micropolar theory of elasticity. Stress function solution of simple structure for the equilibrium equations of micropolar theory was obtained by *M.Günther* in 1958 [4]. However he did not notice that the solution is complete only for such regions whose boundary consists of a single closed surface. If the region is bordered by more than one closed surface (multibordered region) which are assumed to be not intersecting then the solution is totally self-equilibrated on each surface therefore it can not be complete. By supplementing *Günther's* solution, but independently from each other, *H.Schaefer* and *D.Carlson* found formally different and complete solutions, however, they are equivalent [9,2].

1.2. Paper [11] by *M.Stippes* is devoted to the problem of how to find equilibrated and compatible stress fields in classical theory of elasticity. Since he seeks the solution with the aid of stress functions chosen suitably, he regards the derivation of the general solution for the equilibrium equations of classical elasticity from a variational principle as a first step.

The paper [11], however,

- — pays no attention to the analysis of surface integrals obtained by mathematical transformations; the author entirely leaves them out of consideration;
- — does not investigate the role of body forces;
- — includes no reference to the fact that for a region bordered by a single closed surface three stress functions are sufficient to the solution; perception of this requires the thorough investigation of the compatibility conditions as subsidiary conditions which was carried out by *I.Kozák* [5],[6].

Published in 1978 the book [1] written by *N.P.Abovski., N.P.Andreev* and *A.P.Deruga* provides a detailed representation of variational principles in classical elastostatics including those variational principles where the solutions of the equilibrium equations in terms of stress functions appear as *Euler* equations. In comparison

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with paper [4] there is a step ahead in the treatment of the boundary surface but all the terms needed to have a complete solution for multibordered regions are missing. The reason for this comes from the assumption that the particular solutions of equilibrium equations are assumed to be known therefore difference between homogeneous and particular solutions i.e. self-equilibrated stresses are given by the above mentioned *Euler* equations.

1.3. In micropolar theory of elasticity, to the best of the author's knowledge, no attempt has been made to derive complete solutions to equilibrium equations resulted from a variational principle or an equivalent form of principle of virtual work.

1.4. In view of the foregoing the present paper aims at

(a) deriving the complete solution of equilibrium equations in micropolar theory of elasticity from the general primal form of principle of virtual work striving, at the same time, for accuracy in handling volume integrals and body forces as well as the number of necessary stress functions;

(b) the presentation of the corresponding variational principles and their functionals.

The variational principle which has a functional obtained from the *Lagrange* functional by applying the method of the *Lagrange* multipliers is regarded as a primal one. As regards details in connection with primal systems we refer to *E. Tonti's* work [12].

2. **Complete Solution of Equilibrium Equations and Principle of Virtual Work**

2.1. The region occupied by the body and the surface of the body are denoted respectively by V and S. For the sake of simplicity we assume that the region V is simple-connected. The surface S may, however, consist of not only one but more

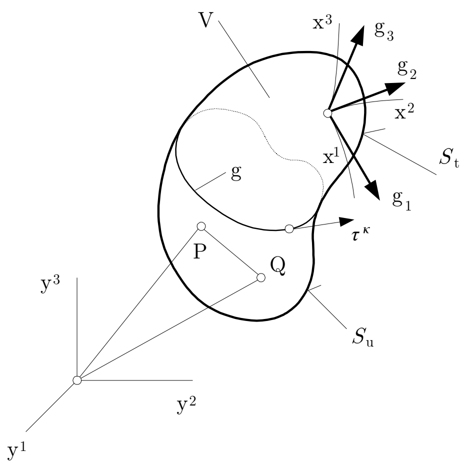

Fig. 1.

closed surfaces — multiply-bordered region — as well. The surface S is divided into parts S_u and S_t whose common bounding curve is denoted by g. The body represented in *Fig.1.* is bordered by a single closed surface.

If the body is bordered by N closed regular surfaces ($N \geq 2$) and each surface is divided into two parts $S_u^{(i)}$, $S_t^{(i)}$ separated from each other by a bounding curve $g^{(i)}$ ($i = 1, \ldots, N$) then S_u, S_t and g are the unions of the subsurfaces $S_u^{(i)}$ and $S_t^{(i)}$ and the bounding curves g_i, respectively.

Any of the surfaces $[S_u]$ $\{S_u^{(i)}\}$ or $[S_t]$ $\{S_t^{(i)}\}$ may be an empty set.

2.2. Indicial notations and three coordinate systems,

— the $(y^1 y^2 y^3)$ Cartesian
— the $(x^1 x^2 x^3)$ curvilinear and
— the $(\xi^1 \xi^2 \xi^3)$ curvilinear, defined on the surface S,

are applied throughout this paper.

Scalars and tensors, unless the opposite is stated, are denoted independently of the coordinate system by the same letter. Distinction is helped by the indication of the arguments y, x and ξ used to denote the totality of the corresponding coordinates.

Volume integrals — except the formulas (2.10) — and surface integrals are considered, respectively, in the coordinate systems $(x^1 x^2 x^3)$ and $(\xi^1 \xi^2 \xi^3)$, consequently, in the case of integrals, arguments are omitted.

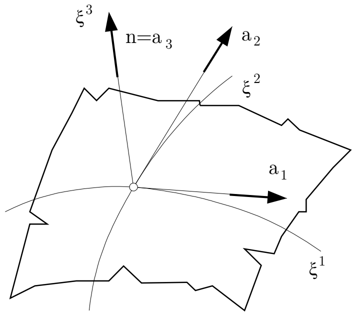

Fig. 2.

In accordance with the general rules of indicial notations summation over repeated indices is implied and subscripts preceded by a semicolon denote covariant differentiation with respect to the corresponding subscripts. Latin and Greek indices range over the integers 1, 2, 3 and 1, 2 respectively.

ϵ^{klm} and ϵ_{pqr} stand for the permutation tensors; δ_k^l is the *Kronecker* delta.

2.3. In Cartesian system $(y^1 y^2 y^3)$ $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the base vectors, besides covariant and contravariant components of tensors are coinciding.

2.4. In the system of material coordinates $(x^1 x^2 x^3)$ $\mathbf{g}_k$ and $\mathbf{g}^l$ are the covariant and contravariant base vectors. The corresponding metric tensors are denoted by g_{kl} and g^{pq}.

2.5. By assumption there exists one-to-one relationship $y^k = y^k(x^1, x^2, x^3)$ between the Cartesian coordinate y^k and the curvilinear coordinates x^1, x^2 and x^3 where

y^k is differentiable with respect to x^l as many times as required. Consequently

$$J_{y,x} = \left|\frac{\partial y^k}{\partial x^l}\right| \neq 0.$$

Contravariant and covariant vector fields B^l and C_b are transformed in accordance with the rules

$$C_b(x) = C_p(y)\frac{\partial y^p}{\partial x^b} \qquad B^k(x) = B^p(y)\frac{\partial x^k}{\partial y^p}\,. \tag{2.1}$$

2.6. Equations and calculations can be better understood by introducing a suitable surface oriented coordinate system. Let $x^k = x^k(\xi^1, \xi^2)$ be the equation of the surface S where ξ^1 and ξ^2 are the surface coordinates. Let ξ^3 be the perpendicular distance measured on the outward unit normal $\mathbf{n}$ to the surface S. On S $\xi^3 = 0$. [Base vectors] {Metric tensors} on S are denoted by [$\mathbf{a}^k$ and $\mathbf{a}_k$] {a_{kl} and a^{kl}}. In the coordinate system $(\xi^1\xi^2\xi^3)$

$$\mathbf{n} = \mathbf{a}_3 = \mathbf{a}^3\,, \qquad n^3 = 1 \qquad \text{and} \qquad n^\eta = 0.$$

The relationship $x^k = x^k(\xi^1, \xi^2, \xi^3)$ is assumed to be a one-to-one. Consequently the functional determinant is not vanishing:

$$J_{x,\xi} = \left|\frac{\partial x^k}{\partial \xi^l}\right| \neq 0.$$

Upon change of coordinates (x^1, x^2, x^3) and (ξ^1, ξ^2, ξ^3) a tensor $D^p_{.q}(x)$ of second order follows the transformation rules

$$D^k_{.l}(\xi) = D^p_{.q}(x)\frac{\partial \xi^k}{\partial x^p}\frac{\partial x^q}{\partial \xi^l}\,, \tag{2.2$_1$}$$

$$D^p_{.q}(x) = D^k_{.l}(\xi)\frac{\partial x^p}{\partial \xi^k}\frac{\partial \xi^p}{\partial x^k} \tag{2.2$_2$}$$

where

$$\frac{\partial x^k}{\partial \xi^l}\frac{\partial \xi^p}{\partial x^k} = \delta^p_l\,. \tag{2.2$_3$}$$

We shall assume that the vector and tensor fields involved in the investigations are sufficiently smooth.

2.7. Let u_k be the displacement field and φ^b be the rotation field (u_k and φ^b together are referred to as displacement fields or briefly, displacements). Furthermore let γ_{kl} be the asymmetric strain tensor and $\kappa_{a.}^{\ b}$ be the curvature twist tensor (together strain fields or briefly, strains).

By t^{kl} and $\mu^a_{.b}$ we denote, respectively, the asymmetric stress tensor and couple-stress tensor (together stress fields or briefly, stresses).

Displacements and strains will be assumed to be small.

Boundary conditions — inasmuch as there are some boundary conditions prescribed — have the following forms:

Displacement boundary conditions:

$$u_k = \hat{u}_k\,, \qquad \varphi^b = \hat{\varphi}^b \qquad \xi \in S_u \tag{2.3}$$

Stress boundary conditions:

$$n_k t^{kl} = \hat{t}^l\,, \qquad \mu^a_{.b} = \hat{\mu}_b \qquad \xi \in S_t \tag{2.4}$$

where $\hat{u}_k$ and $\hat{\varphi}^b$ are respectively prescribed displacement and rotation; $\hat{t}^l$ and $\hat{\mu}_b$ are prescribed tractions.

2.8. By generalizing the results of *I.Kozák* [7] valid for classical case paper [10] systematize [the general primal forms] {the primal form ordered to prescribed boundary conditions} of principle of virtual work, the corresponding assertions and, in addition to this, it gives the missing [general dual forms] {dual forms ordered to prescribed boundary conditions} and dual assertions together with their proofs.

The line of thought of the present section is based on a well known assertion related to the general primal form of principle of virtual work and on a proper choice of the corresponding subsidiary conditions.

2.9. Strains $\gamma_{kl}(x)$ and $\kappa_{a.}^{\ b}(x)$ are said to be [compatible] {kinematically admissible} if the differential equations

$$\gamma_{kl}(x) = u_{l;k} + \epsilon_{lks}\varphi^s \qquad x \in V \qquad (2.5)_1$$

$$\kappa_{a.}^{\ b}(x) = \varphi^{b}_{.;a} \qquad x \in V \qquad (2.5)_2$$

have a single-valued solution for the displacements $u_l(x)$ and $\varphi^b(x)$ $x \in V$ and the solution [does not satisfy other conditions] {satisfies the displacement boundary conditions (2.3)}.

By applying the above term sufficiently smooth — differentiable at least twice — displacements $u_l(x)$ and $\varphi^b(x)$ will also be referred to as compatible.

2.10. Let b^l and c_b be the body forces and body couples. By definition the stresses $t^{kl}(x)$ and $\mu^a_{.b}(x)$ $x \in V$ are said to be [equilibrated]{statically admissible} if they satisfy the equilibrium equations

$$t^{kl}_{..;k}(x) + b^l = 0 \qquad x \in V \qquad (2.6)_1$$

$$\mu^a_{.b;a}(x) + \epsilon_{bkl}t^{kl} + c_b = 0 \qquad x \in V \qquad (2.6)_2$$

and [do not meet other conditions]{the stress boundary conditions (2.4)}.

2.11. For a linearly elastic body the boundary conditions (2.3), (2.4) and field equations (2.5) (2.6) should be supplemented by the stress-strain relations. By assuming a centrosymmetric material the stress strain relations have the form

$$t^{kl} = A^{klpq}\gamma_{pq} \qquad x \in V \qquad (2.7)_1$$

$$\mu^{ab} = B^{abpq}\kappa_{pq} \qquad x \in V \qquad (2.7)_2$$

where A^{klpq} and B^{abpq} are the tensors of elastic coefficients.

2.12. Equation

$$\int_V (t^{kl}\gamma_{kl} + \mu^a_{.b}\kappa_a{}^b)dV = \int_V (b^l u_l + c_b\varphi^b)dV + \int_S (n_3 t^{3l} u_l + n_3\mu^3_{.b}\varphi^b)dA \qquad (2.8)$$

is the general primal form of principle of virtual work.

To the above equation the following direct assertion can be ordered:

Suppose that $\gamma_{kl}(x)$ and $\kappa_{a.}^{\ b}(x)$ as compatible strain fields are obtained from $(2.5)_{1,2}$. *If the equation* (2.8) *holds for any compatible displacement fields $u_k(x)$, $\varphi^b(x)$ then the stress fields $t^{kl}(x)$, $\mu^a_{.b}(x)$ are equilibrated.*

By substituting the kinematic equations $(2.5)_{1,2}$ as subsidiary conditions and performing partial integrations the assertion can easily be proved. Really, upon substitution of the integral

$$\int_V [t^{kl}(u_{l;k} + \epsilon_{lks}\varphi^s) + \mu^a_{.b}\varphi^b_{.;a}]dV = \int_S (n_3 t^{3l} u_l + n_3\mu^3_{.b}\varphi^b)dA + \int_V [t^{kl}_{..;k} u_l + (\mu^a_{.b;k} + \epsilon_{bkl}t^{kl})\varphi^b]dV$$

into (2.8) and subsequent rearrangement it follows the fulfillment of the equilibrium equations if we take into consideration that the coefficients u_k and φ^b in the resulting equation

$$\int_V (t^{kl}_{..;k} + b^l)u_l dV + \int_V (\mu^a_{.b;a} + \epsilon_{bkl}t^{kl} + c_b)\varphi^b dV = 0$$

are arbitrary in V.

2.13. It can be expected that the above assertion will remain valid when the subsidiary conditions (2.5) are replaced by such side conditions which have a different mathematical form but otherwise are equivalent to (2.5).

2.14. According to a fundamental result of potential theory [3] the body forces b^l and body couples c_b always admit the representation

$$b^l = -\Delta B^l = -g^{pq}B^l_{.;pq} \qquad x \in V \qquad (2.9)_1$$

$$c_b = -\Delta C^b = -g^{pq}C_{b;pq} \qquad x \in V \qquad (2.9)_2$$

where $B^l(x)$ and $C_b(x)$, provided that the integrals

$$B^l\big[y^r(Q)\big] = \frac{1}{4\pi}\int_V \frac{b^l\big[y^r(P)\big]}{|y^s(P) - y^s(Q)|}dV_P \qquad Q \in V \qquad (2.10)_1$$

$$C_b\big[y^r(Q)\big] = \frac{1}{4\pi}\int_V \frac{c_b\big[y^r(P)\big]}{|y^s(P) - y^s(Q)|}dV_P \qquad Q \in V \qquad (2.10)_2$$

have been determined first, are obtained from the transformation formulas (2.1). With reference to the above result we shall assume that the vector fields $B^l(x)$ and $C^b(x)$ are known.

2.15. After substituting formulas (2.9) and using then the *Gauss* integral theorem the volume integral

$$I_V^{BC} = \int_V (b^l u_l + c_b \varphi^b) dV \tag{2.11$_1$}$$

is changed to

$$I_V^{BC} = -\int_S (g^{pq} B^l_{.;pq} u_l + g^{pq} C_{b;pq} \varphi^b) dA = \int_S (n_3 a^{3l} B^s_{.;l} u_s + n_3 a^{3l} C_{b;l} \varphi^b) dA$$
$$- \int_V \left[g^{kl} B^s_{.;l} (u_{s;k} + \epsilon_{skp} \varphi^p) + g^{kl} (\epsilon_{lps} B^s + C_{p;l}) \varphi^p_{.;k} \right] dV. \tag{2.11$_2$}$$

Upon substitution of the integral $(2.11)_2$ into (2.8) and with regard to the kinematic equations $(2.5)_{1,2}$ we obtain the equation

$$\int_V \Big[(t^{kl} - g^{ks} B^l_{.;s}) \gamma_{kl} + \big(\mu^a_{.b} - g^{al}(\epsilon_{lbs} B^s + C_{b;l})\big) \kappa_{a.}^{\ b} \Big] dV =$$
$$\int_S \Big[n_3 (t^{3l} - B^l_{.;3}) u_l + n_3 \big(\mu^3_{.b} - (\epsilon_{3b\sigma} B^\sigma + C_{b;3})\big) \varphi^b \Big] dA \tag{2.12}$$

which is a transformation of the general primal form of principle of virtual work. It is noteworthy that in the above equation the kinematic variables appear

— either on the boundary S only as it is the case for u_l and φ^b

— or on the volume V as it is the case for γ_{kl} and $\kappa_{a.}^{\ b}$.

Paragraphs 2.16., 2.17. and 2.18 are devoted to the problem of how to find a proper form of the subsidiary conditions to equation (2.18).

2.16. With reference to that what has been said in paragraph 2.13. we have to raise the following two questions:

(a) Under what conditions are the strains

$$\gamma_{kl},\ \kappa_{a.}^{\ b} \qquad x \in V$$

compatible?

(b) What further conditions are needed if we want the displacements

$$u_k,\ \varphi^v \qquad x \in V$$

obtained from the above mentioned compatible strains to coincide with those being in the surface integral on the right hand side of (2.12).

2.17. Solution to problem (a) is presented here on the basis of paper [11]. To begin with, we have to introduce some new notations.

The index pairs that range over a subset of the nine possible values will be capitalized.

Let $\beta_{k.}^{\ l}$ and α_{ab} be arbitrary tensor fields on V. Furthermore let $r^l(x)$ and $w_b(x)$ $\ x \in V$ be two unknown vector fields.

By ${}_{K.}^{\ L}$ and ${}_{AB}$ we denote those subsets of the possible values of index pairs ${}_{k.}^{\ l}$ and ${}_{ab}$ for which the differential equations

$$r^L_{.;K} = \beta_{K.}^{\ L}(x) \qquad x \in V \tag{2.13$_1$}$$
$$w_{B;A} + \epsilon_{BAp} r^p = \alpha_{AB}(x) \qquad x \in V \tag{2.13$_2$}$$

have a solution for the vector fields $r^l(x)$ and $w_b(x)$.

It is obvious that the index pairs ${}_{K.}^{\;L}$ and ${}_{AB}$ may have only three-three different values.

Let $\{\,{}_{S.}^{\;T}\}\,[{}_{XY}]$ be the supplementary subsets of index pairs the union of which with $\{\,{}_{K.}^{\;L}\}\,[{}_{AB}]$ is the set of index pairs $\{{}_{k.}^{\;l}\}\,[{}_{ab}]$.

It is clear, that the index pairs ${}_{S.}^{\;T}$ and ${}_{XY}$ may have only six-six different values.

Tensors of incompatibility e^{mb} and $d^{m}_{\cdot l}$ are defined by the equations

$$e^{mb}(x) = \epsilon^{mpa}\kappa_{a.;p}^{\;\;b} \qquad x \in V \qquad (2.14)_1$$

$$\begin{aligned} d^{m}_{\cdot l}(x) &= \epsilon^{mpk}(\gamma_{kl;p} + \epsilon_{klb}\kappa_{p.}^{\;b}) \\ &= \epsilon^{mpk}\gamma_{kl;p} + \delta_l^m \kappa_{p.}^{\;p} - \kappa_{l.}^{\;m} \,. \qquad x \in V \qquad (2.14)_2 \end{aligned}$$

Returning to question (a) the independent necessary and sufficient conditions for the strains γ_{kl} and $\kappa_{a.}^{\;b}$ to be compatible [8] in a simple-connected region V are the fulfillment of differential equations of compatibility

$$e^{XY}(x) = \epsilon^{Xpk}\kappa_{k\,.;p}^{\;\;Y} = 0\,, \qquad x \in V \qquad (2.15)_1$$

$$d^{S}_{\cdot T}(x) = \epsilon^{Spq}(\gamma_{qT;p} + \epsilon_{qTb}\kappa_{p.}^{\;b}) = 0 \qquad x \in V \qquad (2.15)_2$$

and that of boundary conditions of compatibility

$$n_3 e^{3b}(\xi) = n_3\epsilon^{3\pi\chi}\kappa_{\chi\cdot;\pi}^{\;\;b} = 0\,, \qquad \xi \in S \qquad (2.16)_1$$

$$n_3 d^{3}_{\cdot l}(\xi) = n_3\epsilon^{3\pi\chi}(\gamma_{\chi l;\pi} + \epsilon_{\chi lb}\kappa_{\pi.}^{\;b}) = 0 \qquad \xi \in S \qquad (2.16)_2$$

We note, that $(2.15)_1$ and $(2.15)_2$ are equivalent to six-six scalar equations.

2.18. Referring again to [8] solution for problem (b) is provided by the following assertion:

Suppose that the strains $\gamma_{kl}(\xi)$ and $\kappa_{a.}^{\;b}(\xi)$ fulfil the kinematic boundary conditions

$$\kappa_{\eta.}^{\;b} - \varphi^{b}_{.;\eta} = 0\,, \qquad \xi \in S \qquad (2.17)_1$$

$$\gamma_{\chi l} - u_{l;\chi} - \epsilon_{l\chi b}\varphi^b = 0\,. \qquad \xi \in S \qquad (2.17)_2$$

Then

(1) the boundary conditions of compatibility $(2.16)_{1,2}$ are identically satisfied and
(2) the displacements $u_k(\xi)$, $\varphi^b(\xi)$ $\xi \in S$ can be determined in terms of strains $\gamma_{kl}(\xi)$, $\kappa_{a.}^{\;b}(\xi)$.

In paper [8] proof of part (1) of the assertion is not complete and that of part (2) is missing.

2.19. For the sake of completeness we shall overview the missing proofs. The main difficulty is inherent in the circumstance that the derivatives of displacements $u_k(\xi)$, $\varphi^b(\xi)$ $\xi \in S$ taken with respect to the normal will also appear in the formal transformations. As they cannot be calculated it is worth changing to symbolic notations mentioning that the normal derivatives are needed only apparently as it can be shown analytically too by a sophisticated analysis of the rules that apply to the calculations of derivatives taken on a surface. In the present case the symbolic writing leads to the desired results more quickly. Using symbolic notations and marking dot products to be carried out after the derivations from equations $(2.16)_{1,2}$ we obtain:

$$e^{3b}\mathbf{a}_b = \mathbf{a}_\nu \cdot \epsilon^{3\pi\nu}\frac{\partial}{\partial\xi^\pi}(\kappa_{\eta.}^{\ b}\mathbf{a}^\eta\mathbf{a_b}) = 0 \qquad \xi \in S \qquad (2.18)_1$$

$$d^3_{.l}\mathbf{a}^l = \mathbf{a}_\nu \cdot \epsilon^{3\pi\nu}\frac{\partial}{\partial\xi^\pi}(\gamma_{\chi l}\mathbf{a}^\chi\mathbf{a}^l) + \delta_l^3\kappa_{\pi.}^{\ \pi} - \delta_l^\pi\kappa_{\pi.}^{\ 3}\mathbf{a}^l = 0\,. \qquad \xi \in S \qquad (2.18)_2$$

As regards the left hand side of the second equation definition $(2.14)_2$ has also been taken into account. It follows from the fulfillment of condition $(2.17)_1$ that

$$\kappa_{\eta.}^{\ b}\mathbf{a}^\eta\mathbf{a}_b = \mathbf{a}^\eta\frac{\partial}{\partial\xi^\eta}(\varphi^b\mathbf{a}_b)\ . \qquad \xi \in S \qquad (2.19)$$

Substituting the above equation into $(2.18)_1$ and utilizing then the definition

$$\frac{\partial\mathbf{a}^\eta}{\partial\xi^\pi} = \Gamma^\eta_{\pi m}\mathbf{a}^m$$

for the Christoffel symbol $\Gamma^\eta_{\pi m}$ we arrive at

$$e^{3b}\mathbf{a}_b = \mathbf{a}_\nu \cdot \epsilon^{3\pi\nu}\frac{\partial}{\partial\xi^\pi}\left[\mathbf{a}^\eta\frac{\partial}{\partial\xi^\eta}(\varphi^b\mathbf{a}_b)\right] \qquad (2.13)$$

$$= \epsilon^{3\pi\eta}\frac{\partial^2}{\partial\xi^\pi\partial\xi^\eta}(\varphi^b\mathbf{a}_b) + \epsilon^{3\pi\nu}\Gamma^\eta_{\pi\nu}\frac{\partial}{\partial\xi^\eta}(\varphi^b\mathbf{a}_b) \equiv 0 \qquad (2.14)$$

which is nothing but the identical fulfillment of boundary condition of compatibility $(2.16)_1$.

Fulfillment of boundary condition of compatibility $(2.16)_2$ can be proved in an analogous way. For this reason a brief outline of the proof will only be presented herein. To begin with, substitute the following equation, which is equivalent to $(2.17)_2$, into $(2.18)_2$:

$$\gamma_{\chi l}\mathbf{a}^\chi\mathbf{a}^l = \mathbf{a}^\chi\frac{\partial}{\partial\xi^\chi}(u_l\mathbf{a}^l) + \epsilon_{l\chi b}\varphi^b\mathbf{a}^\chi\mathbf{a}^l \qquad \xi \in S \qquad (2.20)$$

Then, by making use of the equation

$$\mathbf{a}_\nu\epsilon^{3\pi\nu}\cdot\frac{\partial}{\partial\xi^\pi}\left[\epsilon_{l\chi b}\varphi^b\mathbf{a}^\chi\mathbf{a}^l\right] = \epsilon^{3\pi\chi}\epsilon_{l\chi b}\varphi^b_{.;\pi}\mathbf{a}_l =$$
$$= \left(-\delta_l^3\varphi^\pi_{.;\pi} + \delta_l^\pi\varphi^3_{.;\pi}\right)\mathbf{a}^l \quad \xi \in S$$

and repeating the steps of the formal transformations which follow equation (2.10) we can readily show the validity of $(2.18)_2$.

2.20. On the basis of *Fig.3.* it follows from equation $(2.17)_1$ that

$$\varphi^b(P)\mathbf{a}_b = \varphi^b(P_\circ)\mathbf{a}_b + \int_{P_\circ}^{P} \tau^\eta \kappa_{\eta.}^{\ b}\mathbf{a}_b\, ds \tag{2.21$_1$}$$

is the rotation at the point P of the surface S. From equation $(2.17)_2$ — referring again to *Fig.3.* — we obtain for the displacement at P:

$$\begin{aligned} u_l(P)\mathbf{a}^l &= u_l(P_\circ)\mathbf{a}^l + \int_{P_\circ}^{P} \tau^\chi(\gamma_{\chi l} + \epsilon_{l\chi b}\varphi^b)\mathbf{a}^l\, ds = \\ &= u_l(P_\circ)\mathbf{a}^l + \varphi^b(P_\circ)\mathbf{a}_b \times (\mathbf{r}_P - \mathbf{r}_{P_\circ}) \\ &+ \int_{P_\circ}^{P} \frac{d\xi^\chi}{ds}\big[\gamma_{\chi l} + \epsilon_{blk}\mathbf{a}^k \cdot (\mathbf{r} - \mathbf{r}_P)\kappa_{\chi.}^{\ b}\big]\mathbf{a}^l\, ds. \end{aligned} \tag{2.21$_2$}$$

2.21. All that has been said in paragraphs *2.19.* and *2.20.* proves the assertion given in paragraph *2.18.* Returning to the general primal form of principle of virtual work we should notice that equations $(2.15)_{1,2}$ and $(2.17)_{1,2}$ are the missing subsidiary

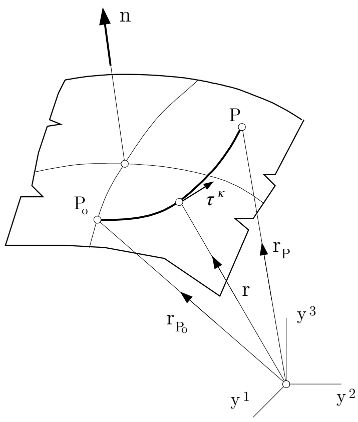

Fig. 3.

conditions. Since they cannot be substituted directly into the form (2.12) of principle of virtual work *Lagrange's* method of undetermined multipliers should be employed. Let

$H_{S.}^{\ T}$,	F_{XY}	$x \in V$
$\tilde{H}_{\eta.}^{\ l}$,	$\tilde{F}_{\eta l}$	$\xi \in S$

be the undetermined *Lagrange's* multipliers. Suppose that the side conditions $(2.15)_{1,2}$ and $(2.17)_{1,2}$ hold. Then both the volume integral I_1^V and the surface integral I_1^S are identically vanishing:

$$I_1^V = \int_V \big[\epsilon^{Spq}(\gamma_{qT;p} + \epsilon_{qTb}\kappa_{q.}^{\ b})H_S^{\ T} + \epsilon^{Xpk}\kappa_{k\ ;\pi}^{\ Y}F_{XY}\big]dV \equiv 0\,, \tag{2.22$_1$}$$

$$I_1^S = \int_S \big[(\gamma_{\chi l} - u_{l;\chi} - \epsilon_{l\chi b}\varphi^b)\epsilon^{3\chi\eta}\tilde{H}_{\eta.}^{\ l} + (\kappa_{\pi.}^{\ b} - \varphi^b_{.;\pi})\epsilon^{3\pi\eta}\tilde{F}_{\eta b}\big] dA \equiv 0 \qquad (2.22)_2$$

and moreover a form with no side conditions of principle of virtual work can be established by subtracting I_1^V and I_1^S from the left and right hand side of (2.12). To attain a more suitable form it is expedient to transform both I_1^V and I_1^S by performing partial integrations before the subtraction. When transforming I_1^V

– we replace $H_{S.}^{\ T}$ and F_{XY} by $H_{p.}^{\ l}$ and F_{yb} bearing in mind, however, that $H_{K.}^{\ L}$ and F_{AB} are obviously zero and moreover
– we shall assume, in conformity with paragraph *2.6.* — see $(2.12)_{1,2}$ —, that there exists a one-to-one relationship between *Lagrange* multipliers

$$H_{\eta.}^{\ l}(\xi), \qquad F_{\eta b}(\xi) \qquad\quad \xi \in S$$

and

$$H_{S.}^{\ T}(x), \qquad F_{XY}(x). \qquad\quad x \in V$$

Without detailing the transformation by partial integrations and suitable renaming dummy indices we obtain:

$$I_1^V = \int_V \big[\epsilon^{kyp}H_{p.;y}^{\ l}\gamma_{kl} + \epsilon^{kpy}(F_{yb;p} + \epsilon_{bpl}H_{y.}^{\ l})\kappa_{a.}^{\ b}\big] dV + \int_S (H_{\eta.}^{\ l}\epsilon^{3\chi\eta}\gamma_{\chi l} + F_{\eta b}\epsilon^{3\pi\eta}\kappa_{\pi.}^{\ b}) dA\,. \qquad (2.23)_1$$

Surface integral I_1^S can be transformed with the aid of *Stokes'* theorem employed here in such a form — see $(A.1)$ in Appendix — which is valid in a coordinate system defined on surface S. If, in addition to that $(A.2)_{1,2}$ is also taken into consideration we find that

$$I_1^S = \int_S (\tilde{H}_{\eta.}^{\ l}\epsilon^{3\chi\eta}\gamma_{\chi l} + \tilde{F}_{\eta b}\epsilon^{3\pi\eta}\kappa_{\pi.}^{\ b}) dA + \int_S \big[\epsilon^{3\chi\eta}\tilde{H}_{\eta.;\chi}^{\ l} u_l + \epsilon^{3\pi\eta}(\tilde{F}_{\eta b;\pi} + \epsilon_{b\pi l}\tilde{H}_{\eta.}^{\ l})\varphi^b\big] dA \qquad (2.23)_2$$

Subtraction of $(2.23)_1$ and $(2.23)_2$ from the left and right hand side of (2.12) and a subsequent rearrangement lead to the result

$$I_2^V + I_2^{S\chi\gamma} + I_2^{Su\varphi} = 0 \qquad (2.24)_1$$

where

$$I_2^V = \int_V (t^{kl} - \epsilon^{kyp}H_{p\ .;y}^{\ l} - g^{ks}B^l_{\ .;s})\gamma_{ks} dV + \int_V \big[\mu^a_{.b} - \epsilon^{apy}(F_{yb;p} + \epsilon_{bpl}H_{y.}^{\ l}) - g^{al}(\epsilon_{lbs}B^s + C_{b;l})\big]\kappa_{a.}^{\ b} dV \qquad (2.24)_2$$

$$I_2^{S\kappa\gamma} = \int_S n_3\big[(\tilde{H}_{\eta.}^{\ l} - H_{\eta.}^{\ l})\epsilon^{3\chi\eta}\gamma_{\chi l} + (\tilde{F}_{\eta b} - F_{\eta b})\epsilon^{3\pi\eta}\kappa_{\pi.}^{\ b}\big] dA \qquad (2.24)_3$$

and

$$\int_S n_3(t^{3l} - \epsilon^{3\chi\eta}\tilde{H}_{\eta\ .;\chi}^{\ l} - a^{3s}B^l_{.;s})u_l dA +$$
$$\int_S n_3\big[\mu^3_{.b} - \epsilon^{3\pi\eta}(\tilde{F}_{\eta b;\pi} + \epsilon_{b\pi l}\tilde{H}_{\eta .}^{\ l}) - a^{3l}(\epsilon_{lbs}B^s + C_{b;l})\big]\varphi^b dA\,. \qquad (2.24)_4$$

Since in (2.24) no conditions for

$$\gamma_{kl}(x), \qquad \kappa_{a.}^{\ b}(x) \qquad x \in V$$
$$\gamma_{\kappa l}(\xi), \qquad \kappa_{\pi .}^{\ b}(\xi) \qquad \xi \in S$$

and

$$u_l(\xi), \qquad \varphi^b(\xi) \qquad \xi \in S$$

are set down $\gamma_{kl}, \dots, \varphi^b$ are arbitrary. Consequently, from the disappearance of (2.24) it follows the fulfillment of the field equations

$$t^{kl} = \epsilon^{kyp}H_{p\ .;y}^{\ l} + g^{ks}B^l_{.;s} \qquad x \in V \qquad (2.25)_1$$
$$\mu^a_{.b} = \epsilon^{aby}(F_{yb;p} + \epsilon_{bpl}H_{y\ .}^{\ l}) + g^{al}(\epsilon_{lbs}B^s + C_{b;l}) \qquad x \in V \qquad (2.25)_2$$

and boundary conditions

$$\tilde{H}_{\eta .}^{\ l} - H_{\eta .}^{\ l} = 0\,, \qquad \tilde{F}_{\eta b} - F_{\eta b} = 0\,, \qquad \xi \in S \qquad (2.26)_1, 2$$

$$n_3 t^{3l} = n_3(\epsilon^{3\eta\pi}\tilde{H}_{\pi .;\eta}^{\ l} + a^{3s}B^{\ l}_{.;s})\,, \qquad \xi \in S \qquad (2.27)_1$$
$$n_3\mu^3_{.b} = n_3\big[\epsilon^{3\pi\eta}(\tilde{F}_{\eta b;\pi} + \epsilon_{b\pi l}\tilde{H}_{\eta .}^{\ l}) + a^{3l}(\epsilon_{lbs}B^s + C_{b;l})\big]\,. \qquad \xi \in S \qquad (2.27)_2$$

If we now substitute $(2.26)_{1,2}$ into $(2.27)_{1,2}$ and then compare $(2.27)_{1,2}$ with $(2.25)_{1,2}$ we get the result that the stresses t^{kl} and $\mu^a_{.b}$ can be calculated in the same way both in V and on S i.e. by using formulas $(2.25)_{1,2}$.

2.22. Equations $(2.25)_{1,2}$ provide equilibrated stresses as it can readily be shown by substituting them into the equilibrium equations $(2.6)_{1,2}$ and also by taking into account $(2.9)_{1,2}$. In addition to this they coincide with the representation found by *H.Schaeffer* [3]. For this reason multipliers $H_{y.}^{\ l}$ and F_{yb} will be referred to as stress functions.

2.23. It is worthy of special mention – with reference to paragraph 2.17. – that $H_{y.}^{\ l}$ and F_{yb} involve six-six scalar functions since $H_{K.}^{\ L} \equiv F_{AB} \equiv 0$. Inasmuch as $H_{y.}^{\ l}$ and F_{yb} are of nine-nine components fulfillment of the mentioned condition can always be ensured, by a proper choice of the vector components r^l and w_b, essentially, by the solution of the differential equations [2]

$$H_{K.}^{\ L} - r^L_{.;K} = 0\,, \qquad x \in V$$
$$F_{AB} - (w_{B:A} + \epsilon_{BAm}r^m) = 0\,. \qquad x \in V$$

[2]The stresses that are obtained from the stress functions

$$H_{y.}^{\ l} = r^l_{\ ;y}, \qquad F_{yb} = w_{b;y} + \epsilon_{bys}r^s$$

are identically zero [8].

These equations serve as a basis for the explanation why a proper choice of indices obey the rules presented in paragraph 2.17. [11].

3. Variational Principles of Lagrange's Type

3.1. In connection with the equation (2.24) obtained from the general primal form of principle of virtual work the question arises whether it is possible or not to establish such free variational problem where

- vanishing of variations with respect to the strain fields γ_{kl}, $\kappa_{a\,.}^{\;b}$ of the corresponding functional ensures the fulfillment of the field equations $(2.25)_{1,2}$ on the volume V of body and that of boundary conditions $(2.26)_{1,2}$ on part S_t of boundary
- furthermore vanishing of variations with respect to the displacements u_k, φ^b yields the fulfillment of the boundary conditions $(2.27)_{1,2}$ consequently the fulfillment of stress boundary conditions on S_t.

The sought functional can be derived from the functional of the total potential energy by applying the method of *Lagrange's* multipliers. The domain of the functional involves the strain fields

$$\gamma_{kl} \qquad \kappa_{a\,.}^{\;b} \qquad x \in V$$

the displacement fields

$$u_l, \qquad \varphi^b \qquad \xi \in S_t$$

and the stress functions

$$H_{S\,.}^{\;T}, \qquad F_{XY} \qquad x \in V$$
$$\tilde{H}_{\eta .}^{\;l}, \qquad \tilde{F}_{\eta l}\,. \qquad \xi \in S_t$$

In the latter case, as we have assumed so far, $H_{K\,.}^{\;L}$ and F_{AB} are regarded to be zero.

3.2. Equations of micropolar elasticity in terms of the above mentioned variables consist of the field equations

$$A^{klpq}\gamma_{pq} = \epsilon^{kpq}H_{q\;.;p}^{\;l} + g^{km}B^{l}_{\;.;m}\,, \qquad x \in V \qquad (3.1)_1$$

$$B_{.b\,.\,.}^{a\;pq}\kappa_{pq} = \epsilon^{aml}(F_{lb;m} + \epsilon_{bmq}H_{l\,.}^{\;q}) + g^{al}(\epsilon_{lbm}B^m + C_{b;l})\,, \qquad x \in V \qquad (3.1)_2$$

$$\epsilon^{Xpk}\kappa_{k\;.;p}^{\;Y} = 0\,, \qquad x \in V \qquad (3.2)_1$$

$$\epsilon^{Spq}(\gamma_{qT;p} + \epsilon_{qTb}\kappa_{p.}^{\;b}) = 0\,, \qquad x \in V \qquad (3.2)_2$$

the boundary conditions

$$\tilde{H}_{\eta .}^{\;l} - H_{\eta .}^{\;l} = 0\,, \qquad \xi \in S_t \qquad (3.3)_1$$

$$\tilde{F}_{\eta b} - F_{\eta b} = 0\,, \qquad \xi \in S_t \qquad (3.3)_2$$

$$\kappa_{\eta .}^{\;b} - \varphi^{b}_{\;.;\eta} = 0\,, \qquad \xi \in S_t \qquad (3.3)_1$$

$$\gamma_{\chi l} - u_{l;\chi} - \epsilon_{l\chi b}\varphi^b = 0\,, \qquad \xi \in S_t \qquad (3.4)_2$$

$$\kappa_{\eta\cdot}^{\ b} - \hat{\varphi}^b_{.;\eta} = 0\,, \qquad \xi \in S_u \qquad (3.5)_1$$

$$\gamma_{\chi l} - \hat{u}_{l;\chi} - \epsilon_{l\chi b}\hat{\varphi}^b = 0\,, \qquad \xi \in S_u \qquad (3.5)_2$$

$$\hat{t}^l - \epsilon^{3\chi\eta}\tilde{H}_{\eta\ .;\chi}^{\ l} - B^{\ l}_{\ .;3} = 0\,, \qquad \xi \in S_t \qquad (3.6)_1$$

$$\hat{\mu}_b - \epsilon^{3\pi\eta}(\tilde{F}_{\eta b;\pi} + \epsilon_{b\pi l}\tilde{H}_{\eta\ .}^{\ l}) - a^{3l}(\epsilon_{lb\sigma}B^\sigma + C_{b;l}) = 0 \qquad \xi \in S_t \qquad (3.6)_2$$

and the continuity condition

$$u_l = \hat{u}_l\,, \qquad \varphi^b = \hat{\varphi}^b\,. \qquad \xi \in g \qquad (3.7)_{1,2}$$

Really, simultaneous fulfillment of equations $(3.2)_{1,2}$, $(3.4)_{1,2}$, $(3.5)_{1,2}$ and $(3.7)_{1,2}$ ensures that the strain fields γ_{kl}, $\kappa_{a\cdot}^{\ b}$ are kinematically admissible. We note that — in accordance with paragraph *2.20.* and with regard to the continuity conditions $(3.7)_{1,2}$ – integration of equations $(3.4)_{1,2}$ yields the actual displacement fields $u_k(\xi)$, $\varphi^b(\xi)$ $\xi \in S_t$. If furthermore field equations $(3.1)_{1,2}$ are satisfied then the equilibrium on V is maintained while the simultaneous fulfillment of $(3.6)_{1,2}$ is equivalent to that of stress boundary conditions.

3.3. Now let

$$\Pi_2 = \Pi_2(\gamma_{kl}, \kappa_{a\cdot}^{\ b}, u_l, \varphi^b, H_{y\ .}^{\ l}, F_{yb}, \tilde{H}_{\eta\ .}^{\ l}, \tilde{F}_{\eta b}) = \Pi_2^{V1} + \Pi_2^{V2} + \Pi_2^{St} + \Pi_2^{Su} + \Pi_2^G + C_2^{Su} \qquad (3.8)_1$$

be the sought functional where

$$\Pi_2^{V1} = \frac{1}{2}\int_V \big(\gamma_{kl}A^{klpq}\gamma_{pq} + \kappa_{ab}B^{abpq}\kappa_{pq}\big)dV - \int_V \big[g^{km}B^{\ m}_{.;m}\gamma_{kl} + g^{al}(\epsilon_{lbm}B^m + C_{b;l})\kappa_{a\cdot}^{\ b}\big]dV\,, \qquad (3.8)_2$$

$$\Pi_2^{V2} = -\int_V \big[\epsilon^{Spq}(\gamma_{qT;p} + \epsilon_{qTb}\kappa_{p\cdot}^{\ b})H_{S\cdot}^{\ T} + \epsilon^{Xpk}\kappa_{k\ .;p}^{\ Y}F_{XY}\big]dV \qquad (3.8)_3$$

$$\Pi_2^{St} = -\int_{St}\big\{(\hat{t}^l - n_3a^{3l}B^{\ l}_{\ .;3})u_l + \big[\hat{\mu}_b - n_3a^{3l}(\epsilon_{lb\sigma}B^\sigma + C_{b;l})\big]\varphi^b\big\}dA + \int_{St}\big[(\gamma_{\chi l} - u_{l;\chi} - \epsilon_{l\chi b}\varphi^b)\epsilon^{3\chi\eta}\tilde{H}_{\eta\cdot}^{\ l} + (\kappa_{\pi\cdot}^{\ b} - \varphi^b_{.;\pi})\epsilon^{3\pi\eta}\tilde{F}_{\eta b}\big]dA\,, \qquad (3.8)_4$$

$$\Pi_2^{Su} = \int_{Su}\big[(\gamma_{\chi l} - \hat{u}_{l;\chi} - \epsilon_{l\chi b}\hat{\varphi}^b)\epsilon^{3\chi\eta}H_{\eta\cdot}^{\ l} + (\kappa_{\pi\cdot}^{\ b} - \hat{\varphi}^b_{.;\pi})\epsilon^{3\pi\eta}F_{\eta b}\big]dA\,, \qquad (3.8)_5$$

$$\Pi_2^G = \oint_g \big[\tau^\eta(u_l - \hat{u}_l)\tilde{H}_{\eta\cdot}^{\ l} + \tau^\eta(\varphi^b - \hat{\varphi}^b)\tilde{F}_{\eta b}\big]ds \qquad (3.8)_6$$

and

$$C_2^{Su} = \int_{Su} \big[n_3 a^{3m} B^l_{.;m} \hat{u}_l + n_3 a^{3l} (\epsilon_{lbm} B^m + C_{b;l}) \hat{\varphi}^b \big] dA = const. \qquad (3.8)_7$$

3.4. Vanishing of variation

$$\delta \Pi_2 = \delta_{\gamma,\kappa} \Pi_2 + \delta_{u,\varphi} \Pi_2 + \delta_{H,F} \Pi_2 = 0 \qquad (3.9)$$

as a variational principle ensures the fulfillment of the field equations (3.1) and (3.2), the boundary conditions (3.3), (3.4), (3.5) and (3.6) furthermore the continuity condition (3.7).

In the following we briefly outline the proof of the above assertion. Because of the independence of variations with respect to the distinct variables of functional (3.8) stationary condition (3.9) is equivalent to equations

$$\delta_{\gamma,\kappa} \Pi_2 = \delta_{\gamma,\kappa} \Pi_2^{V1} + \delta_{\gamma,\kappa} \Pi_2^{V2} + \delta_{\gamma,\kappa} \Pi_2^{St} + \delta_{\gamma,\kappa} \Pi_2^{Su} = 0 \,, \qquad (3.10)_1$$

$$\delta_{u,\varphi} \Pi_2 = \delta_{u,\varphi} \Pi_2^{St} + \delta_{u,\varphi} \Pi_2^{G} = 0 \qquad (3.10)_2$$

and

$$\delta_{H,F} \Pi_2 = \delta_{H,F} \Pi_2^{V2} + \delta_{H,F} \Pi_2^{St} + \delta_{H,F} \Pi_2^{Su} + \delta_{H,F} \Pi_2^{G} = 0 \,. \qquad (3.10)_3$$

Equation $(3.10)_1$ can be transformed by substituting $(2.22)_1$ and $(2.23)_1$ provided that in the latter equations strain fields are replaced by their variations. From the resulting equation, taking into account that the variations are arbitrary, we obtain the field equations $(3.1)_{1,2}$ and $(3.3)_{1,2}$.

Using transformation rules $(A.2)_{1,2}$ from equation $(3.10)_2$ we can readily derive the boundary conditions $(3.6)_{1,2}$.

Fulfillment of equation $(3.10)_3$ is equivalent to all the conditions i.e. to equations $(3.2)_{1,2}$, $(3.4)_{1,2}$, $(3.5)_{1,2}$ and $(3.7)_{1,2}$ kinematically admissible strain fields should meet.

3.5. If the strain fields are kinematically admissible and stress functions $\tilde{H}_{\eta.}^{\ l}$, $\tilde{F}_{\eta b}$ satisfying the conditions $(3.6)_{1,2}$ are known then functional (3.8) reduces to functional

$$\Pi_1 = \Pi(\gamma_{k,l}, \kappa_{a.}^{\ b}) = \Pi_1^V + \Pi_1^{St} + C_1^{Su} + C_1^G \qquad (3.11)_1$$

where

$$\Pi_1^V = \Pi_2^{V1} \,, \qquad (3.11)_2$$

$$\Pi_1^{St} = \int_{St} (n_3 \epsilon^{3\chi\eta} \gamma_{\chi l} \tilde{H}_{\eta.}^{\ l} + n_3 \epsilon^{3\chi\eta} \kappa_{\pi.}^{\ b} \tilde{F}_{\eta b}) dA \,, \qquad (3.11)_3$$

$$C_1^{Su} = C_2^{Su} = const \qquad (3.11)_4$$

and

$$C_1^G = -\oint_g \tau^\eta \tilde{H}_{\eta.}^{\ l} \hat{u}_l ds - \oint_g \tau^\eta \tilde{F}_{\eta b} \hat{\varphi}^b ds \,. \qquad (3.11)_5$$

During the transformations leading to (3.11) it had to be taken into account that due to their definition the kinematically admissible strain fields meet the preconditions

(3.2), (3.4), (3.5) and (3.7). In addition to this integral (A.3) should also be substituted by a suitable renaming of dummy indices and performing simultaneously some rearrangements.

3.6. Functional (3.11) is subjected to subsidiary conditions which should ensure that the strain fields are kinematically admissible. In contrast to the foregoing it is worthwhile to choose such subsidiary conditions on S_t which do not contain the displacement fields.

In accordance with all that has been said about the requirements the subsidiary conditions should meet when seeking the equations which follow from the stationarity of functional $\Pi_1(\gamma_{kl}, \kappa_{a.}^{\ b})$ one should supplement Π_1 by a sum of integrals

$$\Pi_S = \Pi_S^V + \Pi_S^{St} + \Pi_S^{Su} + \Pi_S^G \tag{3.12$_1$}$$

which vanishes if the subsidiary conditions are satisfied. Here

$$\Pi_S^V = I_1^V(H_{S.}^{\ T}, F_{XY}), \tag{3.12$_2$}$$

$$\Pi_S^{St} = \int_{St} \big[n_3 \epsilon^{3\pi\eta} \kappa_{\eta.;\pi}^{\ b} w_b + n_3 \epsilon^{3\pi\chi} (\gamma_{\chi l;\pi} + \epsilon_{\chi lb} \kappa_{\pi.}^{\ b}) r^l \big] dA \tag{3.12$_3$}$$

$$\Pi_S^{Su} = \Pi_2^{Su}(\overset{*}{H}{}_{\eta.}^{\ l}, \ \overset{*}{F}_{\eta b}) \tag{3.12$_4$}$$

and

$$\Pi_S^G = \oint_g \tau^\eta (\kappa_{\eta.}^{\ b} - \hat{\varphi}^b_{\ ;\eta}) \overset{*}{w}_b \, ds + \oint_g \tau^\chi \big[\gamma_{\chi l} - (\hat{u}_{l;k} + \epsilon_{l\chi b} \hat{\varphi}^b) \big] \overset{*}{r}{}^l \, ds \, . \tag{3.12$_5$}$$

Lagrange's multipliers in the above integrals are denoted by

$H_{S.}^{\ T}$,	F_{XY}	$x \in V$
w_b ,	$r^{\,l}$	$\xi \in S_t$
$\overset{*}{w}_b$,	$\overset{*}{r}{}^l$	$\xi \in g$
$\overset{*}{H}{}_{\eta.}^{\ T}$,	$\overset{*}{H}_{\eta b}$.	$x \in V$

Because of the same meaning the letters we used earlier are deliberately utilized again to designate unknown *Lagrange's* multipliers.

3.7. By varying the sum $\Pi_1 + \Pi_S$ with respect to the strain fields and utilizing appropriately the relations (A.1) and $(A.4)_{1,2}$ we obtain

$$\delta_{\kappa,\gamma} \Pi_1 + \delta_{\kappa,\gamma} \Pi_S = I_S^V + I_S^{St} + I_S^{Su} + I_S^G \tag{3.13$_1$}$$

where

$$I_S^V = \int_V \big[A^{klpq} \gamma_{pq} - (\epsilon^{kyp} H_{p.;y}^{\ l} + g^{ks} B^{\ l}_{.;s}) \big] \delta\gamma_{kl} \, dV + \\ \int_V \big[B^{a\,pq}_{.b\,..} \kappa_{pq} - \epsilon^{ami} (F_{ib;m} + \epsilon_{bmq} H_{i.}^{\ q}) - g^{al} (\epsilon_{lbm} B^m + C_{b;l}) \big] \delta\kappa_{a.}^{\ b} \, dV, \tag{3.13$_2$}$$

$$I_S^{St} = \int_{St} n_3 \epsilon^{3\pi\eta} \big[\tilde{F}_{\eta b} - F_{\eta b} - (w_{b;\eta} + \epsilon_{b\eta m} r^m) \big] \delta\kappa_{a.}^{\ b} \, dV + \\ \int_{St} n_3 \epsilon^{3\chi\eta} (\tilde{H}_{\eta.}^{\ l} - H_{\eta.}^{\ l} - r^{\ l}_{.;\eta}) \delta\gamma_{\chi l} \, dA, \tag{3.13$_3$}$$

$$I_S^{Su} = \int_{Su} \left[n_3 \epsilon^{3\pi\eta} (\overset{*}{F}_{\eta b} - F_{\eta b}) \delta\kappa_{\pi.}^{\ b} + n_3 \epsilon^{3\pi\eta} (\overset{*}{H}_{\eta.}^{\ l} - H_{\eta.}^{\ l}) \delta\gamma_{\chi l} \right] dA \qquad (3.13)_4$$

and

$$I_S^G = \oint_g \left[\tau^\eta (\overset{*}{w}_b - w_b) \delta\kappa_{\eta.}^{\ b} + \tau^\chi (\overset{*}{r}^l - r^l) \delta\gamma_{\chi l} \right] ds. \qquad (3.13)_5$$

3.8. Equation

$$\delta_{\kappa,\gamma}\Pi_1 + \delta_{\kappa,\gamma}\Pi_S = 0 \qquad (3.14)$$

as a variational principle ensures that the stress fields obtained from (2.7) are statically admissible. Really, if we compare the above equation and equation (3.13) we get the followings:

1. Vanishing of integral I_S^V yields the equilibrated representation $(3.1)_{1,2}$ for strain fields in V.
2. Vanishing of integral I_S^{Su} leads to the validity of the representation on S_u.
3. From the vanishing of I_S^{St} – taking into account, that the stress functions

$$\check{H}_{\eta.}^{\ l} = r^l_{.;\eta} \qquad \xi \in S_t \qquad (3.15)_1$$

$$\check{F}_{\eta b} = w_{b;\eta} + \epsilon_{b\eta m} r^m \qquad \xi \in S_t \qquad (3.15)_2$$

   result in identically zero stresses and utilizing furthermore that the stress functions $\tilde{F}_{\eta b}$ and $\tilde{H}_{\eta.}^{\ l}$ satisfy $(3.6)_{1,2}$ – it follows the fulfillment of the stress boundary conditions imposed on S_t.
4. Vanishing of integral I_S^G yields the fulfillment of continuity condition concerning *Lagrange's* multipliers.

3.9. It is worthy of special mention that to the fulfillment of stress boundary conditions there is no need to satisfy the boundary conditions

$$\tilde{H}_{\eta.}^{\ l} - H_{\eta.}^{\ l} = 0\,, \qquad \tilde{F}_{\eta b} - F_{\eta b} = 0\,. \qquad \xi \in S_t$$

Instead fulfillment of the weaker forms

$$\tilde{H}_{\eta.}^{\ l} - H_{\eta.}^{\ l} = \check{H}_{\eta.}^{\ l}\,, \qquad \tilde{F}_{\eta b} - F_{\eta b} = \check{F}_{\eta b} \qquad \xi \in S_t$$

is also sufficient. Although this result has been known [8], it appears here as a consequence of a variational principle.

4. **Concluding Remarks**

4.1. The main result of the present work has been the proof of the possibility, that for micropolar bodies the general and complete solution of equilibrium equations in terms of stress functions – valid therefore not only for a self-equilibrated case – can be derived from the general primal form of principle of virtual work. The proof is grounded on two circumstances. As well known, the general primal form of principle of virtual work ensures the fulfillment of equilibrium equations provided that the strain fields are kinematically admissible. Therefore an appropriate choice of the side conditions – and this is the second circumstance – leads to the desired final result as it has turned out. Since the side conditions involve six-six field equations any state of stress can be given in terms of six-six stress functions. Consequently three-three components of the corresponding stress function tensors $H_{y.}^{\ l}$ and F_{yb} can be set to zero. In addition to this special care should be taken to the calculations carried out on the boundary surface S. We note that the line of thought presented herein is of methodological significance and can be applied to other cases including the classical one. This work is now in progress.

4.2. A further question arises concerning the formulation of the corresponding variational principles. The most important functionals of *Lagrange's* type have also been presented assuming that the micropolar body under consideration is linearly elastic. The vanishing of variations with respect to the strain fields of the functionals as a variational principle ensures, through stress functions, the fulfillment of equilibrium equations and stress boundary conditions. Representation of stresses obtained in terms of stress functions from the stationarity condition coincide with the general and complete solution of equilibrium equations.

5. **Appendix**

5.1. Let $d_q(\xi)$ be sufficiently smooth vector field defined on S_t. According to *Stokes'* theorem

$$\int_{St} n_3 \epsilon^{3\chi\eta} d_{\eta;\chi}\, dA = \oint_g \tau^\eta d_\eta\, ds \tag{A.1}$$

If S_t is closed then the integral in the right hand side is vanishing.

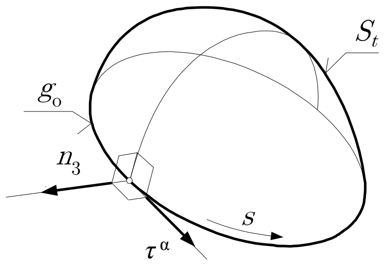

Fig. 4.

5.2. Integral appearing in equation (2.22) can be transformed with the aid of (A.1):

$$\int_{St} n_3 \epsilon^{3\chi\eta} u_{l;\chi} \tilde{H}_{\eta.}^{\ l}\, dA = \oint_g \tau^\eta u_l \tilde{H}_{\eta.}^{\ l}\, ds - \int_{St} n_3 \epsilon^{3\chi\eta} \tilde{H}_{\eta.;\chi}^{\ l} u_l\, dA\,, \qquad (A.2)_1$$

$$\int_{St} n_3 \epsilon^{3\pi\eta} \varphi^{\ b}_{.;\pi} \tilde{F}_{\eta b}\, dA = \oint_g \tau^\eta \varphi^b \tilde{F}_{\eta b}\, ds - \int_{St} n_3 \epsilon^{3\pi\eta} \tilde{F}_{\eta b;\pi} \varphi^b\, dA\,. \qquad (A.2)_2$$

If S_t is again closed i.e. $S_t \equiv S$ then the line integrals on the right hand side can be omitted.

5.3. Transformation of integral

$$I' = \int_{St} (\hat{t}^l u_l + \hat{\mu}_b \varphi^b)\, dA$$

appearing in $(3.8)_{1,2}$ requires the application of *Stokes'* theorem, the kinematic equations $(3.4)_{1,2}$ and the continuity condition $(3.7)_{1,2}$:

$$I' = \int_{St} \left[n^3 B^{\,l}_{.;3} u_l + n^3 (\epsilon_{3b\sigma} B^\sigma + C_{b;3}) \varphi^b \right] dA + \oint_g \tau^\eta \tilde{H}_{\eta.}^{\ l} \hat{u}_l \, ds + \oint_g \tau^\eta \tilde{F}_{\eta l} \hat{\varphi}^b \, ds + \int_{St} (n_3 \epsilon^{3\chi\eta} \gamma_{\chi l} \tilde{H}_{\eta.}^{\ l} + n_3 \epsilon^{3\pi\eta} \kappa_{\pi.}^{\ b} \tilde{F}_{\eta b})\, dA\,. \qquad \text{(A.3)}$$

5.4. Transformation of variation of integral $(3.12)_3$ can be carried out by the substitution of integrals

$$\int_{St} n_3 \epsilon^{3\pi\eta} \delta\kappa_{\eta;\pi}^{\ b} w_b \, dA = \oint_g \tau^\eta w_b \delta\kappa_{\eta.}^{\ b} \, ds + \int_{St} n_3 \epsilon^{3\pi\eta} w_{b;\eta} \delta\kappa_{\eta.}^{\ b} \, dA \qquad \text{(A.4)}_1$$

and

$$\int_{St} n_3 \epsilon^{3\pi\chi} \delta\gamma_{\chi l;\pi} r^l \, dA = \oint_g \tau^\chi r^l \delta\gamma_{\chi l} \, ds + \int_{St} n_3 \epsilon^{3\chi\eta} r^{\,l}_{.;\eta} \delta\gamma_{\chi l} \, dA \qquad \text{(A.4)}_2$$

obtained by partial integrations and a renaming of dummy indices.

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