

## ON COUPLED VIBRATIONS OF BEAMS WITH LATERAL LOADS

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**Abstract.** The objective of this paper is to analyze the free vibration and mode shapes of straight beams where the coupling between the bending and torsion is induced by steady state lateral loads. The governing differential equations and boundary conditions for the coupled vibrations of Euler-Bernoulli-Vlasov beams are derived by using the virtual work principle which includes the second order terms of finite beam rotations. Closed form solution is found for the coupled frequencies and mode shapes of a symmetric beam with simply supported ends under uniform bending. A finite element model with seven degrees of freedoms per node is also presented. To illustrate the accuracy of this formulation, numerical solutions are presented and compared with available solutions.

*Mathematical Subject Classification:* 74K10, 70J25, 74S05

*Keywords:* coupled free vibrations, buckling, torsional warping

### 1. INTRODUCTION

The determination of natural frequencies and modes is a significant problem in the dynamic analysis of thin-walled beams and it is of great importance in designing of beam structures subject to dynamic loadings. For a beam having cross sectional symmetry in two perpendicular directions the solution of independent bending and torsional free vibration frequencies and mode shapes is well known (Bishop and Johnson, [1]). If the shear center  $\mathbf{S}$  and the centroid  $\mathbf{C}$  are not coincident, coupling between bending and torsion should be considered during free vibration.

A number of studies dealing with coupled vibrations of thin-walled beams were developed and only a few is mentioned here. Dokumaci [2] derived the exact analytical expression for the solution of the bending-torsion equations and his results were later extended by Bishop et. al. [3] to include torsional warping which is important for thin-walled section beams. Afterwards, Tanaka and Bercin [4] and then Arpacı and Bozdağ [5] extended the approach to triply coupled vibrations of thin-walled beams and Prokić [6] analysed the fivefold coupled vibrations of Timoshenko beams. Banerjee and Williams took into account the effect of axial load [7]. In Ref. [8] Kollár presented the analysis of the natural frequency of composite beams. A dynamic transfer matrix method has been presented by Li et al. [9]. Recently Chen and Hsiao [10] investigated the coupled vibration induced by the boundary conditions.

In the most general case the coupling between different vibration modes is induced not only by the eccentricity of geometry but by the steady state lateral loads and internal stress resultants. The investigation presented in this paper is motivated by the fact that for dynamic structural analyses that are sensitive to the modal vibration properties small errors in the natural frequencies and mode shapes may produce sizable errors in the modal time history and associated structural response (e.g., earthquake response). A literature survey on the subject has revealed that studies of this kind of couplings are limited.

The important points presented in this study are summarized as follows:

The potential energy of nonsymmetric beams subjected to initial loads and stress resultants based on semitangential rotations and moments is firstly derived.

Next, the finite element model is defined by introducing seven nodal parameters.

Equations of motion and the closed form solution are derived from the potential energy principle for simply supported beam under uniform bending load.

Finally, numerical solution is presented for a cantilever under eccentric steady state tip load.

Accordingly the main objective of the present paper is to develop an accurate numerical procedure to account for static loads in linear dynamic analysis of spatial beam structures with arbitrary cross-sections.

## 2. FORMULATION

**2.1. Equations of the problem.** In this work, the basic assumptions are as follows: the beam member is straight and prismatic, the cross-section is rigid in its plane but is subjected to torsional warping, rotations are large but strains are small, the material is homogeneous, isotropic and linearly elastic. Figure 1 shows a straight, prismatic beam member with an arbitrary cross-section.

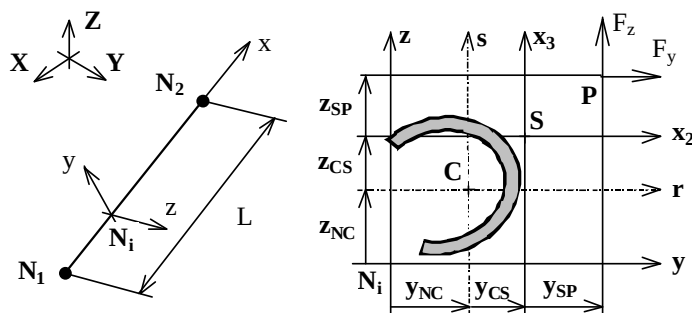


Figure 1. A beam element with local coordinate systems and eccentricities

The local axis  $x$  of the right-handed orthogonal system is parallel to the axis of the beam and passes through the end nodes  $N_1$  and  $N_2$ . The co-ordinate axes  $y$  and

$z$  are parallel to the principal axes, marked as  $r$  and  $s$ , respectively. The positions of the centroid  $\mathbf{C}$  and shear center  $\mathbf{S}$  in the plane of the cross section are given by the co-ordinates  $y_{NC}$ ,  $z_{NC}$  and  $y_{CS}$ ,  $z_{CS}$ . The external loads are applied along points  $\mathbf{P}$  located  $y_{SP}$  and  $z_{SP}$  from the shear center  $\mathbf{S}$ .

Based on large rotation theory, the displacement vector consisting of two parts due to the translational and rotational deformations is obtained as (Kim et al. [11], [12]):

$$\mathbf{u} = \mathbf{U} + \mathbf{U}^* \quad (1)$$

where  $\mathbf{U}$  and  $\mathbf{U}^*$  are the displacements corresponding to the linear and second order terms of displacement parameters due to large rotation effects. In the explicit form these components can be written as:

$$\mathbf{U} = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} u + \vartheta\varphi \\ v \\ w \end{bmatrix} + \begin{bmatrix} \beta(s - z_{CS}) - \gamma(r - y_{CS}) \\ -\alpha(s - z_{CS}) \\ \alpha(r - y_{CS}) \end{bmatrix}, \quad (2)$$

$$\mathbf{U}^* = \begin{bmatrix} U_x^* \\ U_y^* \\ U_z^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha\beta(r - y_{CS}) + \alpha\gamma(s - z_{CS}) \\ -(\alpha^2 + \gamma^2)(r - y_{CS}) + \beta\gamma(s - z_{CS}) \\ \beta\gamma(r - y_{CS}) - (\alpha^2 + \beta^2)(s - z_{CS}) \end{bmatrix}. \quad (3)$$

Displacement parameters are defined at the shear center  $\mathbf{S}$  as shown in Figure 2. Accordingly,  $u$ ,  $v$ , and  $w$  are the rigid body translations in the directions  $x$ ,  $y$ , and  $z$  and  $\alpha$ ,  $\beta$  and  $\gamma$  denote rigid body rotations about the shear center axes parallel to  $x$ ,  $y$  and  $z$ , respectively. The small out-of-plane torsional warping displacement is defined by the  $\vartheta(x)$  warping parameter and the warping function  $\varphi(r, s)$  is normalized with respect to the shear center. In the following the warping function  $\varphi$  and the shear center location are the same as in the case of free torsion. For thin-walled sections  $\varphi = -\omega$ , the sector area coordinate introduced in Vlasov's model. When the shear deformation effects are not considered, the Euler-Bernoulli and the Vlasov internal kinematical constraints are adopted as:

$$\beta = -w', \quad \gamma = v', \quad \vartheta = \alpha' \quad (4)$$

in which the prime denotes differentiation with respect to variable  $x$ .

The stress resultants shown in Figure 2 are defined as follows:

$$\begin{aligned} N &= \int_A \sigma_x dA, \quad V_r = \int_A \tau_{xr} dA, \quad V_s = \int_A \tau_{xs} dA, \\ M_t &= \int_A (r\tau_{xs} - s\tau_{xr}) dA, \quad M_r = \int_A s\sigma_x dA, \quad M_s = - \int_A r\sigma_x dA, \quad B = \int_A \varphi\sigma_x dA, \\ M_1 &= M_t - V_sy_{CS} + V_rz_{CS}, \quad M_2 = M_r - z_{CS}N, \quad M_3 = M_s + y_{CS}N, \\ M_W &= \int_A ((r - y_{CS})^2 + (s - z_{CS})^2) \sigma_x dA = Ni_p^2 + M_r\beta_r - M_s\beta_s + B\beta_\omega \end{aligned} \quad (5)$$

where  $N$  is the axial force,  $V_r$  and  $V_s$  are the shear forces acting at the shear center,  $M_1$ ,  $M_2$  and  $M_3$  are the total twisting and bending moments with respect to shear

center, respectively, and  $B$  is the bimoment. The stress resultant  $M_W$  is known as the Wagner effect. Also, the sectional properties are defined as

$$\begin{aligned}
 I_r &= \int_A s^2 dA, \quad I_s = \int_A r^2 dA, \quad I_\omega = \int_A \varphi^2 dA, \\
 I_p &= I_s + I_r + A(y_{CS}^2 + z_{CS}^2), \quad i_p^2 = \frac{I_{pS}}{A}, \quad J = I_r + I_s - \int_A \left( s \frac{\partial \phi}{\partial r} - r \frac{\partial \phi}{\partial s} \right) dA, \\
 \beta_r &= \frac{1}{I_r} \int_A s(r^2 + s^2) dA - 2z_{CS}, \quad \beta_s = \frac{1}{I_s} \int_A r(r^2 + s^2) dA - 2y_{CS}, \\
 \beta_\omega &= \frac{1}{I_\omega} \int_A \varphi(r^2 + s^2) dA,
 \end{aligned} \tag{6}$$

where  $A$  denotes the cross sectional area.

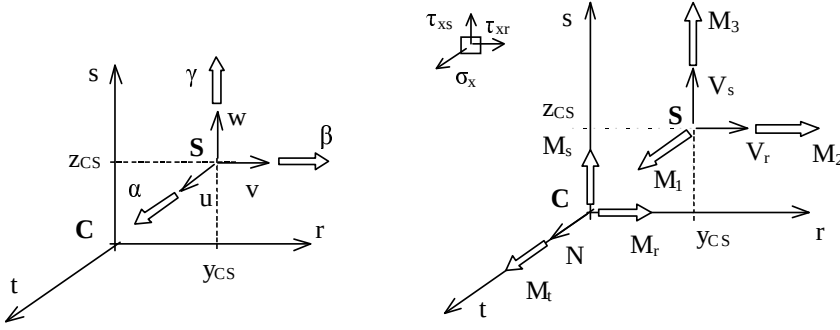


Figure 2. Definition of displacement parameters and stress resultants

The final form of the virtual work principle for the beam structure subjected to initial stresses may be expressed as

$$\delta \Pi = \delta (\Pi_L + \Pi_{Gi} - \Pi_{Ge}) - \delta W = 0 \tag{7}$$

where  $\Pi_L$ ,  $\Pi_{Gi}$ ,  $\Pi_{Ge}$  are the linear elastic strain energy, the energy change due to initial stress resultants and the potential energy due to eccentric initial external loads, respectively, and  $W$  is the work of external load increments on incremental displacements.

The first two terms of total potential (6) can be written as:

$$\Pi_L = \frac{1}{2} \int_0^L [EA(\bar{u}')^2 + EI_r(w'')^2 + EI_s(v'')^2 + EI_\omega(\alpha'')^2 + GJ(\alpha')^2] dx, \tag{8}$$

where  $E$  and  $G$  are the Young's and shear moduli, respectively, and

$$\begin{aligned}
 \Pi_{Gi} &= \frac{1}{2} \int_0^L [N((v')^2 + (w')^2) + M_W(\alpha')^2 + \\
 &\quad + M_1(v''w' - v'w'') + M_2(v''\alpha - v'\alpha') + M_3(w''\alpha - w'\alpha') + \\
 &\quad + (V_r w' - V_s v')\alpha - 2(V_r v' + V_s w')(w' - v''y_{CS} - w''z_{CS})] dx. \tag{9}
 \end{aligned}$$

The new displacement parameter, i.e. the overall average  $\bar{u}$  of the axial displacement  $U_x$  is defined as

$$\bar{u} = \frac{1}{A} \int_A U_x dA = u + y_{CS}v' + z_{CS}w'. \quad (10)$$

It should be mentioned that energy functional (7) was consistently obtained corresponding to semitangential internal moments because equation (9) due to initial bending and torsion moments was derived on the basis of including the second order terms of semitangential rotations in equation (3). For a detailed derivation of  $\Pi_L$  and  $\Pi_{Gi}$  the reader is referred to - among others - Kim et al. [12], [13] and Vörös [14].

The third term of equation (7) is the incremental work of initial loads. Considering conservative initial external forces  $F_x$ ,  $F_y$  and  $F_z$  each acting at the material point  $\mathbf{P}(y_{SP}, z_{SP})$  of the  $i$ -th nodal section - see Figure 1 for the details - the incremental work of these forces is

$$\begin{aligned} \Pi_{Ge} &= [F_x U_x^* + F_y U_y^* + F_z U_z^*]_{Pi} = \\ &= \frac{1}{2} [F_x (y_{SP}\beta + z_{SP}\gamma)\alpha + F_y (z_{SP}\beta\gamma - y_{SP}(\gamma^2 + \alpha^2)) + \\ &\quad + F_z (y_{SP}\beta\gamma - z_{SP}(\beta^2 + \alpha^2))]_i \end{aligned} \quad (11)$$

For time dependent dynamic problems, according to the d'Alembert's principle the volume load increment  $\mathbf{q}$  is the inertia force and the appropriate virtual work, neglecting the second order terms, can be written in the following form

$$\delta W = \int_V \mathbf{q} \delta \mathbf{u} dV = - \int_V \rho (\ddot{\mathbf{U}} + \ddot{\mathbf{U}}^*) \delta (\mathbf{U} + \mathbf{U}^*) dV \approx - \int_V \rho \ddot{\mathbf{U}} \delta \mathbf{U} dV = \delta \Pi_M$$

where  $\rho$  is the mass density per unit volume and dot denotes differentiation with respect to time variable  $t$ . Substituting the linear displacements from equation (2) and taking the definitions for the section properties from equation (6) into account the following expression is obtained:

$$\begin{aligned} \delta \Pi_M &= - \int_0^L \rho [A \ddot{u} \delta \bar{u} + A (\ddot{v} + \ddot{\alpha} z_{CS}) \delta v + A (\ddot{w} - \ddot{\alpha} y_{CS}) \delta w + \\ &\quad + A (\ddot{v} z_{CS} - \ddot{w} y_{CS} + \ddot{\alpha} i_p^2) \delta \alpha + I_s \ddot{v}' \delta v' + I_r \ddot{w}' \delta w' + I_\omega \ddot{\alpha}' \delta \alpha'] dx. \end{aligned} \quad (12)$$

**2.2. Finite element discretisation.** The derivation of finite element matrices is based on the assumed displacement field. The nodal vector of seven local displacement parameters is defined as

$$\Delta_i = [\bar{u} \quad v \quad w \quad \alpha \quad \beta \quad \gamma \quad \vartheta]^T_i, \quad \mathbf{U}_E = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}. \quad (13)$$

A linear interpolation is adopted for the axial displacement and a cubic Hermitian function for the lateral deflections and twist:

$$\begin{aligned}\bar{u}(\xi) &= \bar{u}_1(1 - \xi) + \bar{u}_2\xi, \\ v(\xi) &= v_1F_1 + \gamma_1LF_2 + v_2F_3 + \gamma_2LF_4, \\ w(\xi) &= w_1F_1 - \beta_1LF_2 + w_2F_3 - \beta_2LF_4, \\ \alpha(\xi) &= \alpha_1F_1 + \vartheta_1F_2 + \alpha_2N_3 + \vartheta_2LN_4\end{aligned}\tag{14}$$

where

$$F_1 = 1 - 3\xi^2 + 2\xi^3, \quad F_2 = \xi - 2\xi^2 + \xi^3, \quad F_3 = 3\xi^2 - 2\xi^3, \quad F_4 = \xi^3 - \xi^2, \quad \xi = \frac{x}{L}.$$

Substituting the shape functions into equations (8), (9), (11) and (12) and integrating along the element length  $L$ , the elementary matrices obtained can be defined as:

$$\begin{aligned}\delta\mathbf{\Pi}_L &= \delta\mathbf{U}_E^T \mathbf{k}_L \mathbf{U}_E, & \delta\mathbf{\Pi}_{Gi} &= \delta\mathbf{U}_E^T \mathbf{k}_{Gi} \mathbf{U}_E, \\ \delta\mathbf{\Pi}_{Ge} &= \delta\mathbf{U}_E^T \mathbf{k}_{Ge} \mathbf{U}_E, & \delta\mathbf{\Pi}_M &= \delta\mathbf{U}_E^T \mathbf{m} \mathbf{U}_E.\end{aligned}\tag{15}$$

The  $\mathbf{k}_L$  linear stiffness is the same as that published by Attard [15] and the exactly integrated 14x14 element geometric stiffness and consistent mass matrices  $\mathbf{k}_{Gi}$ ,  $\mathbf{k}_{Ge}$  and  $\mathbf{m}$  were presented by Vörös [14], [16]. In this study it is assumed that the  $M_r$  and  $M_s$  initial bending moments are linearly varying along a beam element length, while the other internal force components are uniform. In this case the uniform shear forces can be approximated as  $V_s = (M_{r2} - M_{r1})/L$  and  $V_r = -(M_{s2} - M_{s1})/L$ , where 1 and 2 in the indices refer to the  $\mathbf{M}_1$  and  $\mathbf{M}_2$  nodal moments.

This procedure was implemented in a conventional finite element program called VEM7 to obtain numerical results given in the following sections.

### 3. FREE VIBRATION OF A SIMPLY SUPPORTED BEAM

To study the effect of the initial bending on dynamic behaviour, consider a straight beam under a uniform bending moment. Both end loads are quasitangential moments and each moment is equivalent to a couple with a small rigid lever as shown in Figure 3. A quasitangential bending moment can be regarded as the simultaneous action of two eccentric axial forces. Therefore, making use of equation (11) at first with  $F_x = F$ ,  $z_{SP} = a/2$  than  $F_x = -F$ ,  $z_{SP} = -a/2$  at the  $x = L$  end and the same but with opposite forces at the  $x = 0$  end of the beam, the virtual work of quasitangential moment under small spatial rotations can be evaluated as follows:

$$\Pi_{Ge} = -\frac{1}{2} [Fa\gamma\alpha]_{x=L} - \frac{1}{2} [-Fa\gamma\alpha]_{x=0} = -\frac{1}{2} M [\gamma\alpha]_0^L, \tag{16}$$

where  $M = Fa$ .

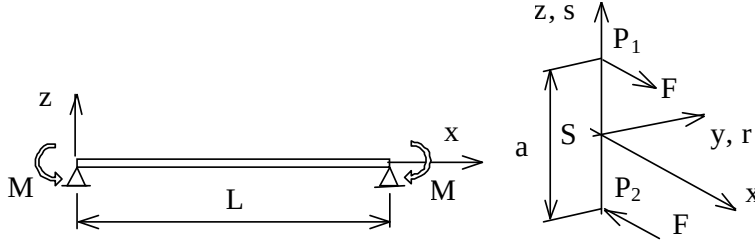


Figure 3. Simply supported beam with quasitangential end moments

To derive a closed form solution for the simplest case, a doubly symmetric section is taken and the only non zero initial stress resultant is the uniform bending moment. Now applying the  $\beta_r = 0$  and  $M_2 = M_r = M$  conditions in equation (9) and adopting the internal constraints (5) in equation (16), the actual form of the virtual work principle (7) is obtained as:

$$\begin{aligned} \delta\Pi = & \delta \int_0^L \frac{1}{2} (EA(\bar{u}')^2 + EI_r w'' + EI_s (v'')^2 + EI_\omega (\alpha'')^2 + GJ(\alpha')^2) dx + \\ & + \int_0^L \rho [A(\ddot{u}\delta\bar{u} + \ddot{v}\delta v + \ddot{w}\delta w) + I_p \ddot{\alpha}\delta\alpha + I_s \ddot{v}'\delta v' + I_r \ddot{w}'\delta w' + I_\omega \ddot{\alpha}'\delta\alpha'] dx + \\ & + \delta \int_0^L \frac{1}{2} M (v''\alpha - v'\alpha') dx + -\frac{1}{2} M \delta [v'\alpha]_0^L = 0. \quad (17) \end{aligned}$$

It can be seen that due to the initial bending moment  $M_r = M$  the lateral displacement  $v(x, t)$  and the torsional rotation  $\alpha(x, t)$  are coupled. Neglecting the third mixed derivative terms (effects of rotary and warping inertia) in the above principle and focusing on the coupled parameters only from the variation of equation (17) with respect to  $v$  and  $\alpha$  the equations of motion and boundary conditions are derived as follows:

$$\begin{aligned} EI_s v'''' + M\alpha'' + \rho A \ddot{v} &= 0 \\ EI_\omega \alpha'''' - GJ\alpha'' + Mv'' + \rho I_p \ddot{\alpha} &= 0. \end{aligned} \quad (18a)$$

$$\begin{aligned} [(EI_s v'''' - M\alpha'')\delta v]_0^L &= 0, \quad [(EI_s v'')\delta v']_0^L = 0, \\ [(-EI_\omega \alpha'''' + GJ\alpha' - Mv'')\delta\alpha]_0^L &= 0, \quad [(EI_\omega \alpha'')\delta\alpha']_0^L = 0. \end{aligned} \quad (18b)$$

In the case of simply supports at each end (fork like supports which prevent torsional rotation and allow free warping) the boundary conditions are:

$$x = 0, \quad x = L: \quad v = 0, \quad \alpha = 0, \quad v'' = 0, \quad \alpha'' = 0. \quad (18c)$$

For free harmonic vibrations the boundary conditions are satisfied by taking the trial solutions as

$$v(x, t) = v_0 \sin\left(i\frac{\pi}{L}x\right) \sin\omega t, \quad \alpha(x, t) = \alpha_0 \sin\left(i\frac{\pi}{L}x\right) \sin\omega t, \quad (19)$$

where  $v_0, \alpha_0$  are the modal amplitudes and  $\omega$  is the circular frequency. Substituting equation (19) into equation (18a) results in the following linear homogeneous equations:

$$\begin{bmatrix} \left(EI_s \left(i\frac{\pi}{L}\right)^4 - \rho A \omega^2\right) & -M \left(i\frac{\pi}{L}\right)^2 \\ -M \left(i\frac{\pi}{L}\right)^2 & \left(EI_\omega \left(i\frac{\pi}{L}\right)^4 + GJ \left(i\frac{\pi}{L}\right)^2 - \rho I_p \omega^2\right) \end{bmatrix} \begin{bmatrix} v_0 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (20)$$

Obviously, if there is no moment load on the beam, that is  $M = 0$ , then the uncoupled lateral bending and torsional frequencies are

$$\omega_{bi}^2 = \left(i\frac{\pi}{L}\right)^4 \frac{EI_s}{\rho A}, \quad \omega_{ti}^2 = \left(i\frac{\pi}{L}\right)^2 \frac{GJ}{\rho I_p} \left[1 + \left(i\frac{\pi}{L}\right)^2 \frac{EI_\omega}{GJ}\right], \quad i = 1, 2, \dots \quad (21)$$

Taking the solution of equation (20) in the static case for which  $\omega = 0$  the following moment eigenvalues are obtained

$$M_i^2 = \left(i\frac{\pi}{L}\right)^2 EG I_s J \left(1 + \left(i\frac{\pi}{L}\right)^2 \frac{EI_\omega}{GJ}\right), \quad i = 1, 2, \dots \quad (22)$$

and the smallest moment obtained for  $i = 1$  is the critical bending load:

$$M_{cr} = M_1 = \pm \frac{\pi}{L} \sqrt{EG I_s J} \sqrt{1 + \frac{\pi^2 EI_\omega}{L^2 GJ}} = \pm \frac{L^2}{\pi^2} \rho \sqrt{AI_p} \omega_{b1} \omega_{t1}. \quad (23)$$

Introducing the steady state moment load factor as

$$\mu_i = M/M_i \quad (24)$$

and substituting equations (21) and (22) into equation (20) yields

$$\begin{bmatrix} A(\omega_{bi}^2 - \omega^2) & -\sqrt{AI_p} \mu_i \omega_{bi} \omega_{ti} \\ -\sqrt{AI_p} \mu_i \omega_{bi} \omega_{ti} & I_p (\omega_{ti}^2 - \omega^2) \end{bmatrix} \begin{bmatrix} v_0 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (25)$$

The non-trivial solution is obtained by setting the determinant of the above system equal to zero. For every  $i$  two natural frequencies can be calculated:

$$\omega_{i,1,2}^2 = \frac{\omega_{ti}^2 + \omega_{bi}^2}{2} \pm \sqrt{\left(\frac{\omega_{ti}^2 - \omega_{bi}^2}{2}\right)^2 + \mu_i^2 \omega_{ti}^2 \omega_{bi}^2}. \quad (26)$$

It can be seen from this result, that only the bending and torsion modes with the same  $i$  index are coupled. Now if  $i = 1$  and  $\omega_{b1} < \omega_{t1}$  then the first two coupled frequencies are

$$\omega_{1,1}^2 = \frac{\omega_{t1}^2 + \omega_{b1}^2}{2} - \sqrt{\left(\frac{\omega_{t1}^2 - \omega_{b1}^2}{2}\right)^2 + \mu_1^2 \omega_{t1}^2 \omega_{b1}^2}, \quad (27)$$

$$\omega_{1,2}^2 = \frac{\omega_{t1}^2 + \omega_{b1}^2}{2} + \sqrt{\left(\frac{\omega_{t1}^2 - \omega_{b1}^2}{2}\right)^2 + \mu_1^2 \omega_{t1}^2 \omega_{b1}^2}. \quad (28)$$



The corresponding diagrams of the bending moment and frequencies are shown in Figure 4. From equations (27,b) the extreme values of the plots are:

$$\begin{aligned} M=0, \quad \mu_1=0, \quad \omega_{1,1}^2 &= \omega_{b1}^2, \quad \omega_{1,2}^2 = \omega_{t1}^2, \\ M = \pm M_{cr}, \quad \mu_1 = \pm 1, \quad \omega_{1,1}^2 &= 0, \quad \omega_{1,2}^2 = \omega_{t1}^2 + \omega_{b1}^2. \end{aligned} \quad (29)$$

Once the natural frequencies are found, the modal amplitudes can be calculated in the usual way. The ratio of first modal amplitudes, or the modal bending-torsion mixing factor, from the first row of equation (25) is:

$$\frac{\alpha_0 i_p}{v_0} = \frac{\omega_{b1}^2 - \omega_{1,1}^2}{\mu_1 \omega_{b1} \omega_{t1}}. \quad (30)$$

The mixing factor-moment plot can also be seen in Figure 4. In a limit case, when  $M = M_{cr}$  and  $\mu_1 = 1$ , the value of mode mixing factor is

$$\frac{\alpha_0 i_p}{v_0} = \frac{\omega_{b1}}{\omega_{t1}}. \quad (31)$$

Briefly, the bending-torsion mixing factor of the lateral bending-torsion buckling mode shape is proportional to the ratio of uncoupled natural frequencies.

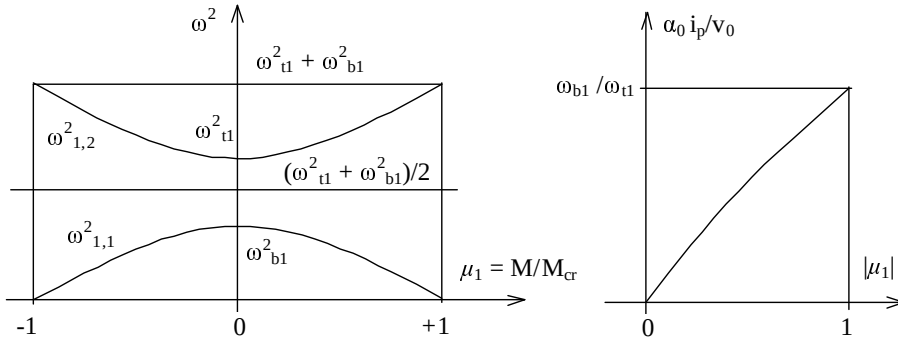


Figure 4. Change of first pair of coupled frequencies and mixing factor of the first mode

| M (106 Nmm) | $\mu_1$ | FEM7           |                |                      | Eq.(27a)       | Eq.(27b)       | Eq.(29)              |
|-------------|---------|----------------|----------------|----------------------|----------------|----------------|----------------------|
|             |         | $\omega_{1,1}$ | $\omega_{1,2}$ | $\alpha_0 i_p / v_0$ | $\omega_{1,1}$ | $\omega_{1,2}$ | $\alpha_0 i_p / v_0$ |
| 0           | 0       | 68.93          | 161.4          | 0                    | 69.00          | 161.4          | 0                    |
| 10          | 0.284   | 65.51          | 162.8          | 0.145                | 65.58          | 162.8          | 0.146                |
| 20          | 0.569   | 54.91          | 166.7          | 0.274                | 54.93          | 166.7          | 0.275                |
| 35          | 0.995   | 6.39           | 175.4          | 0.425                | 6.34           | 175.4          | 0.426                |

Table 1. Comparison of coupled frequencies (1/sec) and mode mixing factors

To examine the validity and accuracy of the proposed VEM7 model, numerical and the closed form solutions are compared. Material and cross sectional properties used in this example are listed in Figure 5, and the length of the simply supported beam is  $L = 4$  m. The first uncoupled lateral bending and torsional frequencies, using

equation (21) are  $\omega_{b1} = 69.00 \text{ sec}^{-1}$ ,  $\omega_{t1} = 161.4 \text{ sec}^{-1}$  and the buckling moment from equation (23)  $M_1 = M_{cr} = \pm 35.18 \times 10^6 \text{ Nmm}$ . Table 1 shows that the FEM7 solutions using 10 elements are practically identical with the closed form solutions.

#### 4. CANTILEVER WITH LATERAL LOAD

In the following a straight cantilever beam of length  $L$  with a monosymmetric and uniform cross section is considered. The beam is shown in Figure 5. The distance between the centroid and shear center is denoted by  $z_{CS}$  and the initial lateral load  $F$  is applied at the cantilever right end with an eccentricity  $z_{SP}$ .

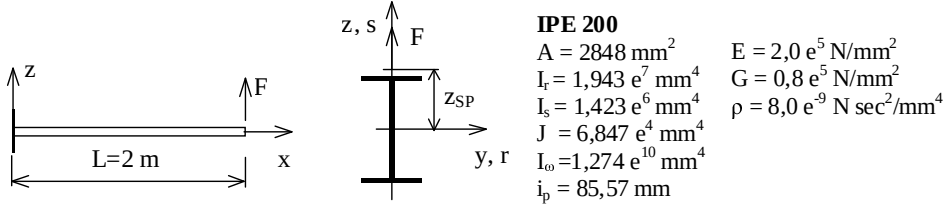


Figure 5. Cantilever beam with a tip load

Because the lateral bending in  $y$  direction and the torsional vibrations are coupled, only the  $v(x, t)$  and  $\alpha(x, t)$  increments are considered here. Applying the non zero initial stress resultants  $M_2 = M_r$  and  $V_s$  in equation (9) the final form of the virtual work principle (7) is obtained as:

$$\begin{aligned} \delta \Pi = & \delta \int_0^L \frac{1}{2} (EI_s (v'')^2 + EI_\omega (\alpha'')^2 + GJ (\alpha')^2) dx + \\ & + \delta \int_0^L \frac{1}{2} [M_r (\beta_r \alpha^2 + v'' \alpha - v' \alpha') - V_s v' \alpha] dx + \\ & + \int_0^L \rho [A (\ddot{v} + z_{CS} \ddot{\alpha}) \delta v + (I_p \ddot{\alpha} + A z_{CS} \ddot{v}) \delta \alpha + I_s \ddot{v}' \delta v' + I_\omega \ddot{\alpha}' \delta \alpha'] dx + \\ & + \frac{1}{2} \delta [F z_{SP} \alpha^2]_{x=L} = 0. \quad (32) \end{aligned}$$

Now, in due course, at first integrating by part the variation of equation (32) then replacing the internal equilibrium equation  $dM_r/dx = V_s$  the equations of motion can be derived in the following form:

$$\begin{aligned} EI_s v'''' + (M_r \alpha)'' + \rho A (\ddot{v} + z_{CS} \ddot{\alpha}) - \rho I_s \ddot{v}'' &= 0, \\ EI_\omega \alpha'''' - GJ \alpha'' - \beta_r (M_r \alpha')' + M_r v'' + \rho I_p \ddot{\alpha} + \rho A z_{CS} \ddot{v} - \rho I_\omega \ddot{\alpha}'' &= 0. \end{aligned} \quad (33a)$$

The corresponding boundary conditions with  $M_r(L) = 0$  are

$$\begin{aligned} x = 0 : v &= 0, \quad \alpha = 0, \quad v' = 0, \quad \alpha' = 0, \\ x = L : v'' &= 0, \quad -EI_s v''' + \rho I_s \ddot{v}' = 0, \\ \alpha'' &= 0, \quad -EI_\omega \alpha''' + GJ \alpha' + \rho I_\omega \ddot{\alpha}' + F z_{SP} \alpha = 0. \end{aligned} \quad (33b)$$

It should be noted, that equation (33a) is coupled for the following two reasons: asymmetry of the section ( $z_{CS}$ ,  $\beta_r$ ) and the internal stress resultant ( $M_r$ ).

| F ( $10^3$ kN) | $\omega_{1.1}$ (1/sec) | $\omega_{1.2}$ (1/sec) | $\alpha_0 i_p / v_0$ |
|----------------|------------------------|------------------------|----------------------|
| 0              | 98.21                  | 150.1                  | 0                    |
| $\pm 20$       | 89.39                  | 148.0                  | 0.250                |
| $\pm 30$       | 76.07                  | 145.8                  | 0.373                |
| $\pm 46$       | 7.71                   | 142.0                  | 0.546                |

Table 2. Frequencies and mode mixing factor,  $z_{SP} = 0$  mm,  $F_{cr} = \pm 46.13$  kN

| F ( $10^3$ kN) | $\omega_{1.1}$ (1/sec) | $\omega_{1.2}$ (1/sec) | $\alpha_0 i_p / v_0$ |
|----------------|------------------------|------------------------|----------------------|
| -22            | 19.70                  | 107.4                  | 1.311                |
| 0              | 98.21                  | 150.1                  | 0                    |
| +30            | 87.33                  | 197.7                  | 0.141                |
| +65            | 9.30                   | 204.9                  | 0.215                |

Table 3. Frequencies and mode mixing factor,  $z_{SP} = 100$  mm,  $F_{cr} = -22.38$  kN / + 65.26 kN

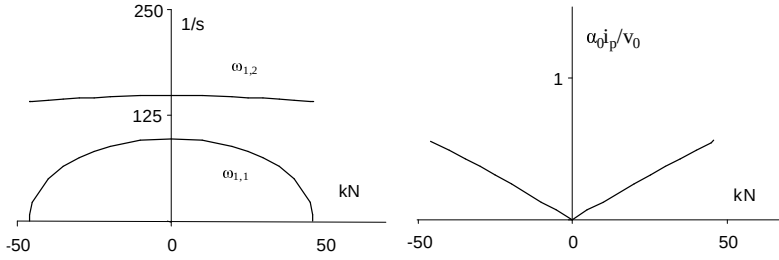


Figure 6. Change of coupled frequencies and mode mixing factor,  $z_{SP} = 0$  mm

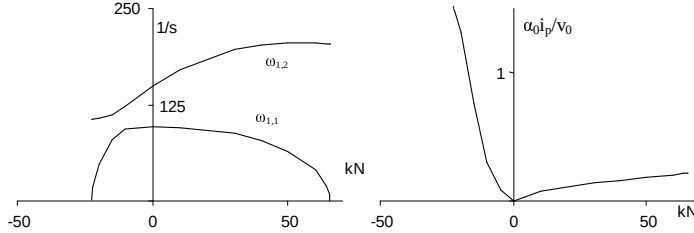


Figure 7. Change of coupled frequencies and mode mixing factor,  $z_{SP} = 100$  mm

In general - even if the section is symmetric and the rotary and warping acceleration terms are neglected - it is not possible to derive a closed form solution for the coupled vibration of the cantilever, so numerical solutions by FEM7 model are presented here.

The first two coupled bending-torsion natural frequencies and the mode mixing factor - defined in equation (30) - of the first mode shape are given in Tables 2 and 3 for various eccentricities. As can be seen in Figures 6 and 7 the load eccentricity has a significant influence on the natural frequencies and particularly on the mode shapes even if the section is symmetric. For example, if a vertical tip load of magnitude  $F = -10 \text{ kN} \approx F_{cr}/2$  pointing downwards acts on the top flange of IPE200 section ( $z_{SP} = 100 \text{ mm}$ ) the decrease of the first two frequencies approximately are 4% and 20% respectively, and 30% of the lateral displacement of the load point of action comes from the torsional component of the first mode shape.

## 5. CONCLUSION

The paper presented a numerical method analysing the effect of steady state bending on the coupled bending torsional vibration and mode shapes of Bernoulli-Vlasov beams. It was found that the initial bending moment has a significant effect on the mode shapes. This prove the need of second order dynamics in structural analysis. The FEM7 finite element approach can be advantageously extended to more complex problems of spatial beam structures.

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