

VIBRATION ANALYSIS OF A PIEZOELECTRIC CYLINDER PANEL

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Abstract. A new iterative model is presented for the dynamic analysis of a piezoelectric cylinder panel. The differential equations which govern the free vibrations of a piezoelectric panel as well as the boundary conditions are derived from Hamilton's principle by taking the bending, shear and normal deformation into account. The author developed a new iterative process to successively refine the stress and strain fields and the electrical field in the panel. The model includes the effects of transverse shear and rotary inertia. The iterative model can be applied to predict the modal frequencies and the stress field in the piezoelectric medium under the conditions of cylindrical bending and periodic motion.

Mathematical Subject Classification:

Keywords: piezoelectric cylinder panel, stress and strain fields, electrical field, iterative model

1. INTRODUCTION

In recent years piezo-electric materials and their use in structural dynamic have become a growing area of research. These materials induce electric potential /charge when they are subjected to mechanical deformations by virtue of direct piezoelectric effect and are deformed due to the externally applied voltage by virtue of converse piezoelectric effect. The load-bearing structures fitted with piezoelectric layers, called intelligent structures, are subjected to the coupled action of electric and elastic fields.

The theory of piezoelectricity was first formulated by Voigt in 1894. The basic theory of linear piezoelectricity has been outlined by several authors [1, 2, 3]. Early computational studies of piezoelectric solids include those of Eer Nisse [4] and Holland [5] for electroelastic vibration analysis using Ritz method. The work of Tiersten [6] provided the theoretical foundations and numerous examples of the dynamic behavior of piezoelectric plates. Finite element modelling of the electroelastic behaviour of a piezoelectric sensor/actuator structural system was presented by Tzou and Tseng [7] and Tzou et al. [8]. Robbins and Reddy [9] developed the finite element models of a piezoelectrically actuated beam using four different displacement-based one-dimensional beam theories, derivable from the generalized laminated plate theory of

Reddy [10] to investigate the behaviour of a piezoelectrically actuated beam. Very few exact solutions of the three-dimensional field equations are available for the coupled response of piezoelectric elements to electromechanical loading. Ray et al. [10, 12] presented exact solutions for the static analysis of a simply-supported piezoelectric plate and a layered intelligent plate under cylindrical bending. A three-dimensional analysis of semi-infinite crack embedded in a transversely isotropic piezoelectric material was performed by Sosa and Pak [13]. Heylinger and Brooks [14] have presented exact solutions for the free vibration behavior of piezoelectric laminates in cylinder bending. The equations of motion, the charge equation, and the boundary and interface conditions are satisfied exactly. A discrete-layer laminate model is presented for the analysis of laminated piezoelectric composite shells in [15]. The variational formulation and corresponding finite element model are developed using the equations of motion and the charge equation. Páczelt and Kovács [16] have studied a semicircular planar prismatic beam subjected to bending moments resulting from piezoelectric actuator devices bonded to the parallel surfaces of the beam. The bending and torsional deformations are investigated with the Finite Element Method and the closed-form solution of the linear theory of 3D curved beams, respectively.

The predictor-corrector procedures appear to have high potential for the accurate prediction of vibration frequencies, stresses and deformations in multilayered composite plates and shells. Zapfe and Lesieutre [17] developed an iterative process to refine successively the shape of the stress/strain distribution for the dynamic analysis of laminated beams. The iterative model is used to predict the modal frequencies and damping of simply supported beams with integral viscoelastic layers. Noor and Burton [18] presented a predictor-corrector approach for the analysis of composite plates. The authors used a plate model based on first-order shear deformation theory, coupled with integration of the equilibrium equations, to refine the estimate of the local stress field through the thickness of the laminate. The refined stress field was also used to generate improved estimates of the shear correction factors in the first-order shear deformation model, leading to improved estimates of the plate displacements and natural frequencies. Lee and Cao [19] have presented a predictor-corrector approach for the numerical analysis of general thick laminated plates which ensures that the transverse shear stress distributions from both constitutive and equilibrium considerations are sufficiently close to each other before the computations of the displacements and stresses are considered sufficiently accurate. In the predictor phase, a linear or cubic zig-zag model is adopted and the layerwise polynomial approximation of the transverse shear stresses through the thickness is determined from the equilibrium equations of elasticity.

The present research extends the iterative model developed by Zapfe and Lesieutre to the dynamic analysis of a cylinder panel of piezoelectric medium. The current model is developed for the specific case of a simply supported cylinder panel under the conditions of cylindrical bending and periodic motion.

2. EQUATIONS OF MOTION

The geometry of interest and the notations are shown in Figure 1. We consider an infinitely long (perpendicular to the plane of paper) cylinder panel of piezoelectric medium. The free vibration of the cylinder panel is confined to the $r-\varphi$ plane. The z axis is out-of-plane, with the r and φ coordinates corresponding to the respective radial and circumferential directions of the panel. Owing to the cylinder panel is assumed to be infinitely long in the z - direction, the piezoelectric medium is under cylindrical bending (i.e. plane strain state) in the $r-\varphi$ plane. As indicated in the Figure, the panel ends are simply supported as described by the end conditions. The panel is considered to be thin and of radius of curvature R of the middle surface, and thickness h . The arc length s is measured along the middle surface of the panel in the circumferential direction, which is defined by s (Fig.1), where $s = R\varphi$. The length of the panel in the circumferential direction is denoted by L . In this case $L = R\vartheta$ where ϑ is the opening angle of the curved panel. The displacements of the panel in the φ - and r - directions are denoted by u and w respectively. Under the conditions of cylindrical bending (i.e. plane strain state), the displacements field takes the form $\mathbf{t} = w(r, \varphi, t)\mathbf{e}_r + u(r, \varphi, t)\mathbf{e}_\varphi$, and the electrostatic potential has the form $\Phi = \Phi(r, \varphi, t)$. Hence all variables are independent of the z coordinate, and terms containing a gradient in the z -direction vanish. For the analysis of the response of piezoelectric structural components we need to deal with two different field equations that are coupled to each other. One is the balance of the momentum equation and the other is the balance of the charge equation. These two field equations are coupled to each other through the constitutive equations of the material.

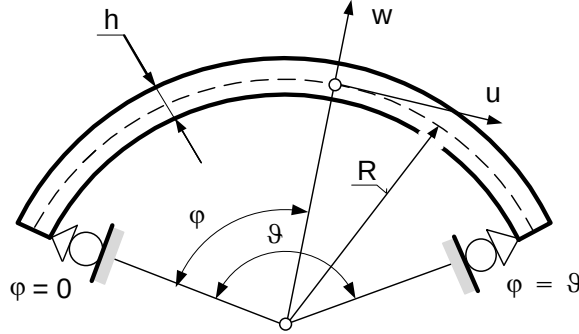


Figure 1. Parameters for a cylinder panel.

We start with a linear, anisotropic, and electro-elastic solid for which the coupled constitutive relation can be written as

$$\sigma_r = C_{11}\varepsilon_r + C_{12}\varepsilon_\varphi - e_{33}E_r, \quad (1)$$

$$\sigma_\varphi = C_{12}\varepsilon_r + C_{22}\varepsilon_\varphi - e_{31}E_r, \quad (2)$$

$$\tau_{r\varphi} = C_{33}\gamma_{r\varphi} - e_{15}E_\varphi, \quad (3)$$

$$D_r = e_{31}\varepsilon_\varphi + e_{33}\varepsilon_r + \varepsilon_{33}E_r, \quad (4)$$

$$D_\varphi = e_{15}\gamma_{r\varphi} + \varepsilon_{11}E_\varphi, \quad (5)$$

where σ_r, σ_φ and $\tau_{r\varphi}$ are the stresses, $\varepsilon_r, \varepsilon_\varphi$ and $\gamma_{r\varphi}$ are the strains, D_r and D_φ denote the electric displacements, E_r and E_φ are the electric fields, C_{11}, C_{12}, C_{22} and C_{33} are the elastic stiffnesses, e_{31}, e_{33} and e_{15} are the electric permittivity components while ε_{11} and ε_{33} stand for the dielectric constants of piezoelectric in the polar coordinate system (r, φ) .

In the present model the displacement field can be given in the following form

$$\begin{aligned} \mathbf{t}(r, \varphi, t) &= w(r, \varphi, t)\mathbf{e}_r + u(r, \varphi, t)\mathbf{e}_\varphi = \\ &= \left[v_0(\varphi, t) - \frac{r-R}{R} \left(\frac{\partial w_0}{\partial \varphi} - v_0(\varphi, t) \right) + f(r)v_1(\varphi, t) \right] \mathbf{e}_\varphi + \\ &\quad + [w_0(\varphi, t) + g(r)w_1(\varphi, t)] \mathbf{e}_r \end{aligned} \quad (6)$$

where $f(R) = 0$ and $g(R) = 0$ which means that (v_0, w_0) denote the displacement at a point (R, φ) on the centre-line along the circumferential and radial directions, respectively.

The terms $f(r)v_1(\varphi, t)$ and $g(r)w_1(\varphi, t)$ can be thought to be the corrections needed to account for the transverse shear and normal deformation effects, respectively. The functions $f(r)$ and $g(r)$ represent the shape of the corrections through the thickness of the cylinder panel, while $v_1(\varphi, t)$ and $w_1(\varphi, t)$ determine their distributions along the circumferential direction. The solution of a given problem requires the determination of the unknown functions $v_0(\varphi, t), v_1(\varphi, t), w_0(\varphi, t), w_1(\varphi, t), f(r)$ and $g(r)$. By using the standard expressions

$$\mathbf{t} = w\mathbf{e}_r + u\mathbf{e}_\varphi, \quad \varepsilon_\varphi = \frac{1}{r} \frac{\partial u}{\partial \varphi} + \frac{w}{r}, \quad \gamma_{r\varphi} = \frac{1}{r} \frac{\partial w}{\partial \varphi} + \frac{\partial u}{\partial r} - \frac{u}{r}, \quad \varepsilon_r = \frac{\partial w}{\partial r}$$

and using equation (6) we can determine the components of the strain tensor

$$\varepsilon_\varphi = \frac{1}{r} \left[\frac{\partial v_0}{\partial \varphi} - \frac{r-R}{R} \left(\frac{\partial^2 w_0}{\partial \varphi^2} - \frac{\partial v_0}{\partial \varphi} \right) + f(r) \frac{\partial v_1}{\partial \varphi} + w_0 + g(r)w_1(\varphi, t) \right], \quad (7)$$

$$\gamma_{r\varphi} = \left[\frac{df}{dr} - \frac{f(r)}{r} \right] v_1(\varphi, t) + \frac{g(r)}{r} \frac{\partial w_1}{\partial \varphi}, \quad (8)$$

$$\varepsilon_r = \frac{dg}{dr} w_1(\varphi, t), \quad (9)$$

where $f(r)$ and $g(r)$ are single-valued functions at each point through the thickness.

For a piezoelectric body we have two balance equations to satisfy simultaneously. The balance of momentum and balance of electric charge require the fulfilment of equations

$$\operatorname{div} \mathbf{T} = \rho \ddot{\mathbf{t}}, \quad (10)$$

$$\operatorname{div} \mathbf{D} = 0 \quad (11)$$

where ρ is the density, $\mathbf{D} = D_r(r, \varphi, t)\mathbf{e}_r + D_\varphi(r, \varphi, t)\mathbf{e}_\varphi$ is the electrical displacement, $\mathbf{T}$ is the stress tensor and the partial derivative with respect to time is denoted by a dot over the letter.

Assume that the cylinder panel considered is in plane strain. Then the equations of motion (10) and the balance of electric charge (11) can be manipulated into the following forms

$$\frac{\partial}{\partial r} [r^2 \tau_{r\varphi}] + r \frac{\partial \sigma_\varphi}{\partial \varphi} = r^2 \rho \frac{\partial^2 u}{\partial t^2}, \quad (12)$$

$$r \frac{\partial \sigma_r}{\partial r} + \sigma_r + \frac{\partial \tau_{r\varphi}}{\partial \varphi} - \sigma_\varphi = r \rho \frac{\partial^2 w}{\partial t^2}, \quad (13)$$

$$\frac{\partial D_r}{\partial r} + \frac{D_r}{r} + \frac{1}{r} \frac{\partial D_\varphi}{\partial \varphi} = 0. \quad (14)$$

The components of the electric field are related to the electrostatic potential $\Phi(r, \varphi, t)$ via the relations

$$E_r = -\frac{\partial \Phi}{\partial r}, \quad E_\varphi = -\frac{1}{r} \frac{\partial \Phi}{\partial \varphi} \quad (15)$$

If the panel is in plain strain $E_z = 0$.

Here the electric potential can be given the following form

$$\Phi(r, \varphi, t) = p(r)q(\varphi, t). \quad (16)$$

In addition to these equations the boundary conditions on the upper and bottom surface of the panel must be specified. For the study of free vibration these two surfaces are assumed to be traction free. Consequently the following boundary conditions should be prescribed

$$\tau_{r\varphi}(r = R + \frac{h}{2}, \varphi, t) = \tau_{r\varphi}(r = R - \frac{h}{2}, \varphi, t) = 0, \quad (17)$$

$$\sigma_r(r = R + \frac{h}{2}, \varphi, t) = \sigma_r(r = R - \frac{h}{2}, \varphi, t) = 0 \quad (18)$$

In addition to the mechanical boundary conditions, the electric surface conditions should also be satisfied. This is accomplished in this study by specifying a homogeneous electrostatic potential:

$$\Phi(r = R + \frac{h}{2}, \varphi, t) = \Phi(r = R - \frac{h}{2}, \varphi, t) = 0 \quad (19)$$

Substitution of equation (16) into equation (19) yields the following boundary conditions on the upper and bottom surface of the panel

$$p(r = R + \frac{h}{2}) = p(r = R - \frac{h}{2}) = 0. \quad (20)$$

The Lagrangian for the cylinder panel is given by [6]:

$$L = \frac{1}{2} \int_{\varphi=0}^{\vartheta} \int_{R-h/2}^{R+h/2} \left[-\sigma_r \varepsilon_r - \sigma_\varphi \varepsilon_\varphi - \tau_{r\varphi} \gamma_{r\varphi} + D_\varphi E_\varphi + D_r E_r + \rho (\dot{\mathbf{t}})^2 \right] r dr d\varphi. \quad (21)$$

Substituting equations (7-9) and (15-16) into equations (1-5) and introducing the results into the Hamilton's principle $\delta \int_{t_1}^{t_2} L dt = 0$ one obtains

$$\begin{aligned}
& A_{1,1} \frac{\partial^4 w_0}{\partial \varphi^4} + A_{1,2} \frac{\partial^2 w_0}{\partial \varphi^2} + A_{1,3} \frac{\partial^2 w_1}{\partial \varphi^2} + A_{1,4} \frac{\partial^3 v_0}{\partial \varphi^3} + A_{1,5} \frac{\partial^3 v_1}{\partial \varphi^3} + \\
& + A_{1,6} \frac{\partial^2 q}{\partial \varphi^2} + A_{1,7} \frac{\partial v_0}{\partial \varphi} + A_{1,8} \frac{\partial v_1}{\partial \varphi} + A_{1,9} w_1 + A_{1,10} w_0 + A_{1,11} q = \\
& = D_{1,1} \frac{\partial^4 w_0}{\partial \varphi^2 \partial t^2} + D_{1,2} \frac{\partial^3 v_0}{\partial \varphi \partial t^2} + D_{1,3} \frac{\partial^3 v_1}{\partial \varphi \partial t^2} + D_{1,4} \frac{\partial^2 w_0}{\partial t^2} + D_{1,5} \frac{\partial^2 w_1}{\partial t^2} , \quad (22a)
\end{aligned}$$

$$\begin{aligned}
& A_{2,1} \frac{\partial^2 w_0}{\partial \varphi^2} + A_{2,2} \frac{\partial^2 w_1}{\partial \varphi^2} + A_{2,3} \frac{\partial^2 q}{\partial \varphi^2} + A_{2,4} \frac{\partial v_1}{\partial \varphi} + A_{2,5} \frac{\partial v_0}{\partial \varphi} + \\
& + A_{2,6} w_1 + A_{2,7} w_0 + A_{2,8} q = D_{2,1} \frac{\partial^2 w_0}{\partial t^2} + D_{2,2} \frac{\partial^2 w_1}{\partial t^2} , \quad (22b)
\end{aligned}$$

$$\begin{aligned}
& A_{3,1} \frac{\partial^3 w_0}{\partial \varphi^3} + A_{3,2} \frac{\partial^2 v_1}{\partial \varphi^2} + A_{3,3} \frac{\partial^2 v_0}{\partial \varphi^2} + A_{3,4} \frac{\partial w_0}{\partial \varphi} + A_{3,5} \frac{\partial w_1}{\partial \varphi} + \\
& + A_{3,6} \frac{\partial q}{\partial \varphi} + A_{3,7} v_1 = D_{3,1} \frac{\partial^2 v_1}{\partial t^2} + D_{3,2} \frac{\partial^2 v_0}{\partial t^2} + D_{3,3} \frac{\partial^3 w_0}{\partial \varphi \partial t^2} , \quad (22c)
\end{aligned}$$

$$\begin{aligned}
& A_{4,1} \frac{\partial^3 w_0}{\partial \varphi^3} + A_{4,2} \frac{\partial^2 v_0}{\partial \varphi^2} + A_{4,3} \frac{\partial^2 v_1}{\partial \varphi^2} + A_{4,4} \frac{\partial w_0}{\partial \varphi} + \\
& + A_{4,5} \frac{\partial w_1}{\partial \varphi} + A_{4,6} \frac{\partial q}{\partial \varphi} = D_{4,1} \frac{\partial^3 w_0}{\partial \varphi \partial t^2} + D_{4,2} \frac{\partial^2 v_0}{\partial t^2} + D_{4,3} \frac{\partial^2 v_1}{\partial t^2} , \quad (22d)
\end{aligned}$$

$$A_{5,1} \frac{\partial^2 w_0}{\partial \varphi^2} + A_{5,2} \frac{\partial^2 w_1}{\partial \varphi^2} + A_{5,3} \frac{\partial^2 q}{\partial \varphi^2} + A_{5,4} w_0 + A_{5,5} w_1 + A_{5,6} q = 0 , \quad (22e)$$

the equations of motion for the four unknown functions, $w_0(\varphi, t)$, $w_1(\varphi, t)$, $v_0(\varphi, t)$, $v_1(\varphi, t)$ and $q(\varphi, t)$. In equations (22a-22e) A_{ij} and D_{ij} are given in the Appendix.

The section stiffness K_{1-27} and the mass coefficients M_{1-8} are provided by the following equations

$$K_{[1,\dots,10]} = \int_{R-h/2}^{R+h/2} C_{22} \left[\frac{1}{r}, r, \frac{1}{r} f^2, \frac{1}{r} g^2, 1, \frac{1}{r} f, \frac{1}{r} g, f, g, \frac{1}{r} f g \right] dr , \quad (23a)$$

$$K_{[11]} = \int_{R-h/2}^{R+h/2} C_{11} r \left(\frac{dg}{dr} \right)^2 dr, \quad (23b)$$

$$K_{[12,\dots,15]} = \int_{R-h/2}^{R+h/2} C_{12} \left[\frac{dg}{dr}, r \frac{dg}{dr}, f \frac{dg}{dr}, g \frac{dg}{dr} \right] dr, \quad (23c)$$

$$K_{[16,\dots,18]} = \int_{R-h/2}^{R+h/2} C_{33} \left[r \left(\frac{df}{dr} - \frac{f}{r} \right)^2, \left(\frac{df}{dr} - \frac{f}{r} \right) g, \frac{1}{r} g^2 \right] dr, \quad (23d)$$

$$K_{[23]} = \int_{R-h/2}^{R+h/2} e_{33} r \left(\frac{dg}{dr} \right) \left(\frac{dp}{dr} \right) dr, \quad (23e)$$

$$K_{[24,25]} = \int_{R-h/2}^{R+h/2} e_{15} \left[p \left(\frac{df}{dr} - \frac{f}{r} \right), \frac{1}{r} gp \right] dr, \quad (23f)$$

$$K_{[26]} = \int_{R-h/2}^{R+h/2} \varepsilon_{11} \frac{1}{r} p^2 dr, \quad (23g)$$

$$K_{[27]} = \int_{R-h/2}^{R+h/2} \varepsilon_{33} r \left(\frac{dp}{dr} \right)^2 dr, \quad (23h)$$

$$M_{[1,\dots,8]} = \int_{R-h/2}^{R+h/2} \rho [r, r^2, r^3, rf, r^2f, rf^2, rg, rg^2] dr, \quad (23i)$$

The kinematic and natural boundary conditions specified at $\varphi = 0$ and $\varphi = \vartheta$, are given by

$$v_0 = 0, \quad F_{11} \frac{\partial^2 w_0}{\partial \varphi^2} + F_{12} \frac{\partial v_0}{\partial \varphi} + F_{13} \frac{\partial v_1}{\partial \varphi} + F_{14} w_0 + F_{15} w_1 + F_{16} q = 0 \quad (24a)$$

$$w_0 = 0, \quad F_{21} \frac{\partial^3 w_0}{\partial \varphi^3} + F_{22} \frac{\partial^2 v_0}{\partial \varphi^2} + F_{23} \frac{\partial^2 v_1}{\partial \varphi^2} + F_{24} \frac{\partial^3 w_0}{\partial \varphi^2 \partial t} + F_{25} \frac{\partial^2 v_0}{\partial t^2} + F_{26} \frac{\partial^2 v_1}{\partial t^2} + F_{27} \frac{\partial w_0}{\partial \varphi} + F_{28} \frac{\partial w_1}{\partial \varphi} + F_{29} \frac{\partial q}{\partial \varphi} = 0 \quad (24b)$$

$$v_1 = 0, \quad F_{31} \frac{\partial^2 w_0}{\partial \varphi^2} + F_{32} \frac{\partial v_0}{\partial \varphi} + F_{33} \frac{\partial v_1}{\partial \varphi} + F_{34} w_0 + F_{35} w_1 + F_{36} q = 0 \quad (24c)$$

$$\frac{\partial w_0}{\partial \varphi} = 0, \quad F_{41} \frac{\partial^2 w_0}{\partial \varphi^2} + F_{42} \frac{\partial v_0}{\partial \varphi} + F_{43} \frac{\partial v_1}{\partial \varphi} + F_{44} w_0 + F_{45} w_1 + F_{46} q = 0 \quad (24d)$$

$$w_1 = 0, \quad F_{51} \frac{\partial w_1}{\partial \varphi} + F_{52} v_1 + F_{53} \frac{\partial q}{\partial \varphi} = 0 \quad (24e)$$

$$q = 0, \quad F_{61} \frac{\partial w_1}{\partial \varphi} + F_{62} v_1 + F_{63} \frac{\partial q}{\partial \varphi} = 0 \quad (24f)$$

where F_{ij} are constants. For the special case of a simply supported curved panel, the first, third and fourth natural boundary conditions are combined with the kinematic condition, $w_0 = w_1 = q = 0$. In this case the edges of panel are assumed to be grounded.

3. SOLUTION FOR A SIMPLY SUPPORTED PANEL

The displacement functions and the potential function satisfying the boundary conditions can be given by

$$w_0(\varphi, t) = W_0 \sin(k_n \varphi) e^{i\omega_n t} \quad (25a)$$

$$w_1(\varphi, t) = W_1 \sin(k_n \varphi) e^{i\omega_n t} \quad (25b)$$

$$v_0(\varphi, t) = V_0 \cos(k_n \varphi) e^{i\omega_n t} \quad (25c)$$

$$v_1(\varphi, t) = V_1 \cos(k_n \varphi) e^{i\omega_n t} \quad (25d)$$

$$q(\varphi, t) = Q \sin(k_n \varphi) e^{i\omega_n t} \quad (25e)$$

where $k_n = (n\pi)/\vartheta$. Substitution of equations (25) into equations (22-23) results in a set of five simultaneous, homogeneous algebraic equations with symmetric coefficients. In matrix form, these equations are

$$[-\omega_n^2 [M] + [Y]] \{U\} = 0, \quad \{U\} = \{W_0, W_1, V_0, V_1, Q\} \quad (26)$$

where M_{ij} and Y_{ij} are given in the Appendix. While the assumed form of the correction functions $f^{(k)}(r)$, $g^{(k)}(r)$ and $p^{(k)}(r)$ will change from one iteration to the next, in each iteration step they can be treated as known functions. In this phase an estimation of the eigenvalues $W_0^{(k)}$, $W_1^{(k)}$, $V_0^{(k)}$, $V_1^{(k)}$ and $Q^{(k)}$ and frequency of vibration $\omega_n^{(k)}$ are obtained by solving equation (26). This approximation is then used in the corrector phase in order to improve the predictions for the correction functions $f^{(k+1)}(r)$, $g^{(k+1)}(r)$ and $p^{(k+1)}(r)$.

4. IMPROVED ESTIMATE FOR CORRECTION FUNCTIONS

An improved estimate for the correction functions $f^{(k+1)}(r)$, $g^{(k+1)}(r)$ and $p^{(k+1)}(r)$ is derived from the three-dimensional equilibrium equations and the three-dimensional constitutive relations. The transverse shear and normal stress are obtained by integrating the 3D equilibrium equations (12-13) in the thickness direction as follows:

$$\tau_{r\varphi}^{(k+1)} = \frac{1}{r^2} \int_{R-h/2}^r r^2 \rho \frac{\partial^2 u^{(k)}}{\partial t^2} - r \frac{\partial \sigma_\varphi^{(k)}}{\partial \varphi} dr + C_1, \quad (27)$$

$$\sigma_r^{(k+1)} = \frac{1}{r} \int_{R-h/2}^r r \rho \frac{\partial^2 w^{(k)}}{\partial t^2} + \sigma_\varphi^{(k)} - \frac{\partial \tau_{r\varphi}^{(k+1)}}{\partial \varphi} dr + C_2, \quad (28)$$

where $\sigma_\varphi^{(k)} = C_{12}\varepsilon_r^{(k)} + C_{22}\varepsilon_\varphi^{(k)} - e_{31}E_r^{(k)}$ is obtained from equation (2). The constants C_1 and C_2 can be determined from the stress boundary conditions imposed on the outer surfaces of the panel. Owing to equations (17) and (18) we have $C_1 = C_2 = 0$.

The transverse electric displacement D_r is calculated by integrating the 3D charge equation of electrostatic (14) in the thickness direction

$$D_r^{(k+1)} = \frac{1}{r} \int_{R-h/2}^r -\frac{\partial D_\varphi^{(k)}}{\partial \varphi} dr + C_3, \quad (29)$$

where $D_\varphi^{(k)} = e_{15}\varepsilon_{r\varphi}^{(k)} + \varepsilon_{11}E_\varphi^{(k)}$ is obtained from equation (5). The integration constant C_3 is obtained by setting $D_r^{(k+1)}$ at $r = R$ to be equal to the transverse electric displacement of the previous iteration $D_r^{(k)}$.

Substitution of equations (7-9), (15-16) and (25) in equations (1), (3) and (4) yield the following set of differential equations in terms of the new unknown correction functions $f^{(k+1)}(r)$, $g^{(k+1)}(r)$ and $p^{(k+1)}(r)$.

$$C_{11}W_1^{(k)} \frac{dg^{(k+1)}}{dr} - \frac{C_{12}}{r} k_n V_1^{(k)} f^{(k+1)}(r) + \frac{C_{12}}{r} W_1^{(k)} g^{(k+1)}(r) - e_{33}Q^{(k)} \frac{dp^{(k+1)}}{dr} =$$

$$= \sigma_r^{(k+1)} - \frac{C_{12}}{r}(-V_0^{(k)}k_n + \frac{r-R}{R}(k_n^2W_0^{(k)} - k_nV_0^{(k)}) + W_0^{(k)}) \quad (30a)$$

$$C_{33}V_1^{(k)}\frac{df^{(k+1)}}{dr} - \frac{C_{33}}{r}k_nV_1^{(k)}f^{(k+1)}(r) + \frac{C_{33}}{r}W_1^{(k)}k_ng^{(k+1)}(r) + \frac{e_{15}}{r}k_nQ^{(k)}p^{(k+1)} = \tau_r^{(k+1)} \quad (30b)$$

$$e_{33}W_1^{(k)}\frac{dg^{(k+1)}}{dr} - e_{33}Q^{(k)}\frac{dp^{(k+1)}}{dr} - \frac{e_{31}}{r}k_nV_1^{(k)}f^{(k+1)}(r) + \frac{e_{31}}{r}W_1^{(k)}g^{(k+1)}(r) = \\ = D_r^{(k+1)} - \frac{e_{31}}{r}(-V_0^{(k)}k_n + \frac{r-R}{R}(k_n^2W_0^{(k)} - k_nV_0^{(k)}) + W_0^{(k)}) \quad (30c)$$

where $\tau_r^{(k+1)}$, $\sigma_r^{(k+1)}$ and $D_r^{(k+1)}$ are obtained from equations (27-29). As implied above, in addition to these differential equations, the new correction functions must satisfy the following boundary conditions:

$$f^{(k+1)}(R) = g^{(k+1)}(R) = p^{(k+1)}(R - h/2) = p^{(k+1)}(R + h/2) = 0. \quad (31)$$

The distribution of the new correction functions in the thickness direction is obtained by integrating the equations (30) subject to the boundary conditions of equations (31). The integrals in equations (30) are evaluated numerically using a trapezoidal method. This new estimates of $f^{(k+1)}(r)$, $g^{(k+1)}(r)$ and $p^{(k+1)}(r)$ are used as the correction functions for the next iteration. As with any smeared laminate model,

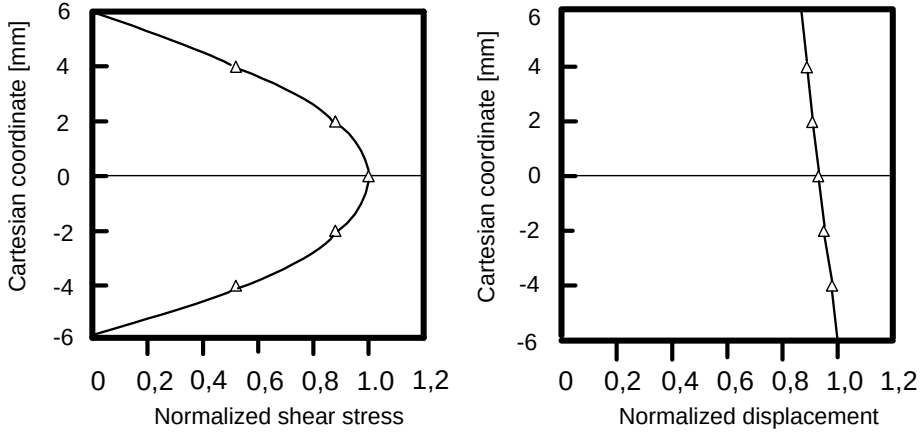


Figure 2. Through-thickness mode shapes for piezoelectric cylinder panel.

there are two distinct ways to calculate the shear stress distribution: from the material constitutive relations; or by elemental stress equilibrium. The ultimate goal of the iterative analysis is the determination of the functions, $f(r)$ and $g(r)$, that causes the two stress distributions to be equal. This defines the convergence point for

the iterative functions $f(r)$ and $g(r)$, the point at which the stresses and strains are self-consistent.

5. RESULTS AND DISCUSSION

We consider a layer of piezoelectric material. The input data used here were $h = 12.0$ mm, $R = 0.8$ m, $L = 0.8$ m, $\vartheta = 1.0$ rad, $C_{11} = 6.45 \cdot 10^{10}$ N/m², $C_{22} = 8.13 \cdot 10^{10}$ N/m², $C_{12} = 0.0432 \cdot 10^{10}$ N/m², $C_{33} = 2.56 \cdot 10^{10}$ N/m², $\rho = 2.7 \cdot 10^3$ kg/m³, $e_{33} = 15.8$ C/m², $e_{31} = -5.2$ C/m², $e_{15} = 12.72$ C/m², $\varepsilon_{11} = 1.51 \cdot 10^{-8}$ F/m, $\varepsilon_{33} = 1.3 \cdot 10^{-8}$ F/m. In the example the thickness-to-radius ratio was 0.015 i.e. the panel is considered to be thin and of radius of curvature R of the middle surface, and thickness h .

Table 1 Iteration improvement of frequency factor for the panel

Iteration number	frequency factor ω_n^2
1	$6.297877 \cdot 10^4$
2	$6.399191 \cdot 10^4$
3	$6.386247 \cdot 10^4$
4	$6.386247 \cdot 10^4$

For the minimum vibration frequencies ($n = 1$) representative through-thickness distributions of the elastic and electric field variables are shown.

These have been normalized by dividing through by the maximum value for each field variable. The through-thickness distributions for axial displacement u , transverse shear stress $\tau_{r\varphi}$, transverse normal stress σ_r , and transverse electric field E_r are shown in Fig. 2-3, respectively, associated with minimum vibration frequencies. The plot for the transverse displacement w is not shown because for the thin panel the distribution is essentially constant through the thickness.

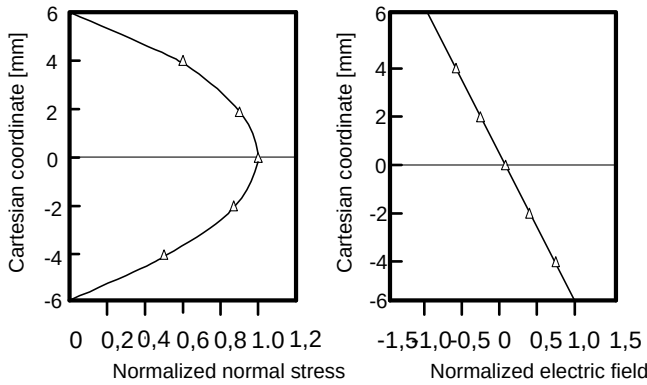


Figure 3. Through-thickness mode shapes for the piezoelectric cylinder panel.

Since $f(r)$ and $g(r)$ are determined from the previous iteration, the question arises as to what to use for the first iteration. Almost any reasonable displacement assumption is acceptable, however experience has shown that a quadratic shear stress distribution is a very good starting point. Table 1. shows the improvement in the prediction of the frequency factor ω_n^2 for the cylinder panel example using a quadratic shear stress as a starting point for the first iteration.

6. CONCLUSIONS

A new iterative model has been presented for a thin piezoelectric cylindrical panel that can accurately determine the dynamic stress distribution in the panel. This represents an advantage over previous smeared laminate models, in which accurate estimates of the stress field were only possible if the assumed displacement field was a reasonable approximation of the actual displacement field. Accurate determination of the stress and displacement fields is particularly important for "stress critical" calculations such as damping and delamination.

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APPENDIX A. CONSTANTS

Equations (22) to (23) in the main text are given in terms of A_{ij} and D_{ij} which are presented below:

$$\begin{aligned}
 A_{1,1} &= K_1 + K_2 - 2K_5/R, & A_{1,2} &= 2K_1 - 2K_5/R \\
 A_{1,3} &= K_7 + K_{12} - K_9/R - K_{13}/R \\
 A_{1,4} &= -K_2 + K_5/R, & A_{1,5} &= K_6 - K_8/R \\
 A_{1,6} &= K_{19} - K_{20}/R, & A_{1,7} &= K_5/R \\
 A_{1,8} &= K_6, & A_{1,9} &= K_{12} + K_7, & A_{1,10} &= K_1, & A_{1,11} &= K_{19} \\
 A_{2,1} &= -K_7 - K_{12} + K_9/R + K_{13}/R, & A_{2,2} &= K_{18}, & A_{2,3} &= K_{27}, \\
 A_{2,4} &= K_{17} - K_{10} - K_{14}, & A_{2,5} &= -K_9/R - K_{13}/R \\
 A_{2,6} &= -K_4 - K_{11} - K_{15} \\
 A_{2,7} &= -K_7 - K_{12}, & A_{2,8} &= -K_{22} - K_{23} \\
 A_{3,1} &= K_6 - K_8/R, & A_{3,2} &= K_3, & A_{3,3} &= K_8/R \\
 A_{3,4} &= K_6, & A_{3,5} &= K_{10} + K_{14} - K_{16} \\
 A_{3,6} &= K_{21} - K_{26}, & A_{3,7} &= -K_{16} \\
 A_{4,1} &= K_5/R - K_2, & A_{4,2} &= K_2, & A_{4,3} &= K_8/R, & A_{4,4} &= K_5/R \\
 A_{4,5} &= K_9/R + K_{13}/R, & A_{4,6} &= K_{20}/R \\
 A_{5,1} &= K_{20}/R - K_{19}, & A_{5,2} &= K_{27}, & A_{5,3} &= -K_{30}, & A_{5,4} &= -K_{19} \\
 A_{5,5} &= -K_{22} - K_{23}, & A_{5,6} &= K_{31} \\
 D_{1,1} &= M_1 - 2M_2/R + M_3/R^2 \\
 D_{1,2} &= M_2/R - M_3/R^2, & D_{1,3} &= M_4 - M_5/R \\
 D_{1,4} &= -M_1, & D_{1,5} &= -M_7, & D_{2,1} &= M_7, & D_{2,2} &= M_8 \\
 D_{3,1} &= M_6, & D_{3,2} &= M_5/R, & D_{3,3} &= M_4 - M_5/R \\
 D_{4,1} &= M_2/R - M_3/R^2, & D_{4,2} &= M_3/R^2, & D_{4,3} &= M_5/R.
 \end{aligned}$$

Equation (26) in the main text contain the quantities Y_{ij} and M_{ij} which are presented below:

$$\begin{aligned}
Y_{11} &= k_n^4 (K_1 + K_2 - 2K_5/R) - 2k_n^2 (K_1 - K_5/R) + K_1 \\
Y_{12} = Y_{21} &= K_7 + K_{12} - k_n^2 (K_7 + K_{12} - K_9/R - K_{13}/R), \\
Y_{13} = Y_{31} &= k_n^3 (K_5/R - K_2) - k_n K_5/R \\
Y_{14} = Y_{41} &= k_n^3 (K_6 - K_8/R) - k_n K_6 \\
Y_{15} = Y_{51} &= K_{19} - k_n^2 (K_{19} - K_{20}/R) \\
Y_{22} &= k_n^2 K_{18} + K_4 + K_{11} + K_{15}, \quad Y_{23} = Y_{32} = k_n (-K_9/R - K_{13}/R) \\
Y_{24} = Y_{42} &= k_n (K_{17} - K_{10} - K_{14}), \quad Y_{25} = Y_{52} = k_n^2 K_{27} + K_{22} + K_{23} \\
Y_{33} &= k_n^2 K_2, \quad Y_{34} = Y_{43} = k_n^2 K_8/R, \quad Y_{35} = Y_{53} = -k_n K_{20}/R \\
Y_{44} &= k_n^2 K_3 + K_{16}, \quad Y_{45} = Y_{54} = k_n (K_{26} - K_{21}), \quad Y_{55} = -K_{31} - k_n^2 K_{30} \\
M_{11} &= M_1 + k_n^2 (M_1 + M_3/R^2 - 2M_2/R), \quad M_{12} = M_{21} = M_7 \\
M_{13} = M_{31} &= k_n (M_2/R - M_3/R^2), \quad M_{14} = M_{41} = k_n (M_4 - M_5/R) \\
M_{33} &= M_3/R^2, \quad M_{34} = M_{43} = M_5/R, \quad M_{44} = M_6, \quad M_{22} = M_8 \\
M_{23} = M_{32} = M_{24} = M_{42} &= 0.
\end{aligned}$$