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BAR ELEMENT WITH VARIATION OF CROSS-SECTION FOR GEOMETRIC NON-LINEAR ANALYSIS

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Abstract. This paper deals with a new bar element with varying cross-sectional area which can be used for geometric non-linear analysis. Shape functions of the bar element include transfer functions and transfer constants, which respect variation of cross-sectional area. Main FE equations are assembled using non-incremental non-linearized method. The von Mises two bar structure with varying cross-sectional area was analyzed. The results obtained with our new element were compared with ANSYS bar element results.

Mathematical Subject Classification: 74S05, 74B20
Keywords: bar element, variation of cross-sectional area, geometric non-linear analysis, finite element method

1. Introduction

Even though the solution of geometric non-linear problems is possible, a great deal of time and effort is spent on improving effectiveness and accuracy of non-linear analyses. Commonly used FEM programs use incremental methods, where the Green-Lagrange strain tensor is linearized in total as well as in updated formulation [1]. Furthermore constitutive law is often linearized - relationship between increment of stress tensor and increment of strain tensor.

A new non-incremental Lagrange formulation without linearization has recently been published in [2]. Non-incremental equations are simpler and contain full non-linear stiffness matrices.

In our paper, we use these non-incremental equations to derive a full non-linear stiffness matrix and a full non-linear tangent matrix for a bar element with variation of the cross-sectional area. Variation of the cross-sectional area is defined as polynomial. New shape functions are used - shape functions which reflect variation of cross-sectional area exactly [3].

2. The basic equations set up in a local co-ordinate system

The Green-Lagrange strain tensor of finite deformation in Lagrange formulation can be written as

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) = e_{ij} + \eta_{ij} \ , \tag{1}$$

where e_{ij} is the linear part of the Green-Lagrange strain tensor and η_{ij} is its non-linear part. u_i represents the i-th component of displacement and $u_{i,j}$ is the gradient of displacement u_i.

The constitutive law can be written as

$$S_{ij} = C_{ijkl} E_{kl} \ , \tag{2}$$

where C_{ijkl} is the tensor of elastic constants and S_{ij} is II. Piola-Kirchhoff stress tensor.

The principle of virtual work can be written as

$$\delta W^{\mathrm{int}} = \delta W^{\mathrm{ext}} \ , \tag{3}$$

where δW^{int} and δW^{ext} are the internal and external virtual works. The internal virtual work assumes the form

$$\delta W^{\mathrm{int}} = \int_{V^0} S_{ij} \, \delta E_{ij} \, \mathrm{d}V \, . \tag{4}$$

For the external virtual work we can write

$$\delta W^{\mathrm{ext}} = \int_{A^0} F_i \, \delta u_i \, \mathrm{d}A + \hat{F}_k \, \delta q_k \tag{5}$$

where F_i is the i-th surface load and δu_i is the appropriate virtual displacement, $\hat{F}_k$ is discrete load at a node and δq_k is the virtual displacement at the same node. The integration is done through the initial volume V^0 and the initial area A^0.

Applying equations (4) and (5) to (3) and considering the displacement as

$$u_i = \phi_{ik} \, q_k \ , \tag{6}$$

where ϕ_{ik} are shape functions and q_k is nodal displacement, we obtain the classical FEM equation

$$\mathbf{K}(\mathbf{q}) \, \mathbf{q} = \mathbf{F} \, . \tag{7}$$

Matrix $\mathbf{K}(\mathbf{q})$ is a full non-linear stiffness matrix, $\mathbf{q}$ is the vector of local displacements and $\mathbf{F}$ is the vector of external local loads.

The full non-linear stiffness matrix has a linear and a non-linear part

$$\mathbf{K}(\mathbf{q}) = \mathbf{K}^L + \mathbf{K}^{NL}(\mathbf{q}) = \mathbf{K}^L + \mathbf{K}^{NL1}(\mathbf{q}) + \mathbf{K}^{NL2}(\mathbf{q}) + \mathbf{K}^{NL3}(\mathbf{q}) \, . \tag{8}$$

The nm-th members of the single matrices can be written in the forms

$$K_{nm}^{L} = \frac{1}{4}\int_{V^0} C_{ijkl}(\phi_{km,l} + \phi_{lm,k})(\phi_{in,j} + \phi_{jn,i})\,\mathrm{d}V\ , \tag{9a}$$

$$K_{nm}^{NL1} = \frac{1}{4}\int_{V^0} C_{ijkl}\phi_{pm,k}\phi_{pr,l}(\phi_{in,j} + \phi_{jn,i})q_r\,\mathrm{d}V\ , \tag{9b}$$

$$K_{nm}^{NL2} = \frac{1}{2}\int_{V^0} C_{ijkl}\phi_{pr,i}\phi_{pn,j}(\phi_{km,l} + \phi_{lm,k})q_r\,\mathrm{d}V\ , \tag{9c}$$

$$K_{nm}^{NL3} = \frac{1}{2}\int_{V^0} C_{ijkl}\phi_{pm,k}\phi_{pv,l}\phi_{rq,i}\phi_{rn,j}q_v q_q\,\mathrm{d}V\ , \tag{9d}$$

where $\phi_{pm,k}$ is the first derivative of shape function ϕ_{pm} with respect to the k-th coordinate. Other derivatives in the previous equations have a similar meaning.

3. **Stiffness matrices of bar element with variation of cross-section**

3.1. **Introductory remarks.** The matrices, which were derived above, are valid for all types of elements whose displacements are described by equation (6). That means that we can use these equations also for a bar element with variation of cross-sectional area.

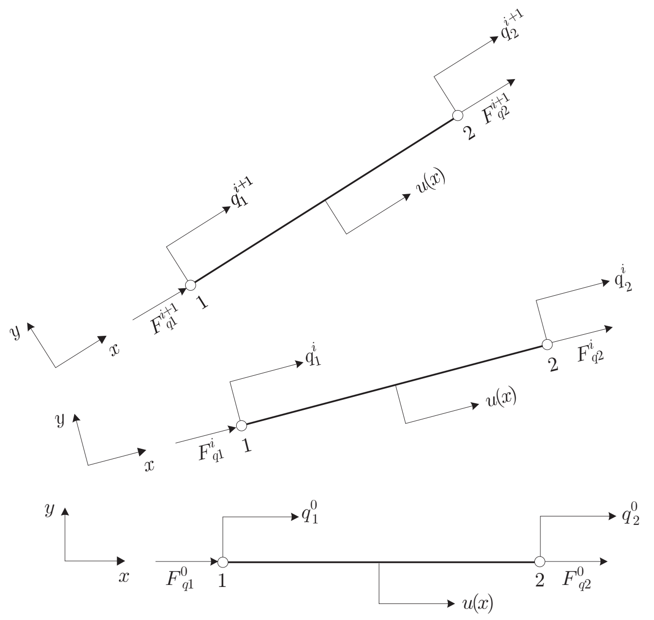

Figure 1. Local nodal displacements and forces in single iterations

In the classical FE codes (e.g. ANSYS), linear interpolation is used for shape functions. But such functions do not respect variation of the element's cross-sectional area and in a very coarse mesh they are responsible for an increase in inaccuracy.

This behavior of bar elements with variation of cross-section with classical linear shape function is also included in linear theory [4].

Figure 1 shows a bar element in a local co-ordinate system with local forces and local displacements in single iterations. The vectors of local displacements and forces have the forms

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T \tag{10}$$

$$\mathbf{F} = \begin{bmatrix} F_{q1} & F_{q2} \end{bmatrix}^T \tag{11}$$

The variation of cross-sectional area $A^0(x)$ is defined as

$$A^0(x) = A_1^0 \eta_A(x) = A_1^0 \Big(1 + \sum_{k=1}^{p} \eta_{Ak} x^k\Big) \tag{12}$$

where A_1^0 is cross sectional area at node 1 and polynomial $\eta_A(x)$ describes variation of the cross-sectional area (η_{Ak} are the coefficients of the polynomial $\eta_A(x)$).

3.2. **Shape functions for bar element with variation of cross-sectional area.** New shape functions for a bar element with variation of cross-sectional area are derived from the direct stiffness method and the whole procedure is published in [3]. The new shape functions contain the transfer functions and transfer constants, which characterize the solution of the linear differential equation with non-constant parameters [5] and depend on polynomial $\eta_A(x)$. For displacement in location x in the local co-ordinate system, we can write

$$u(x) = u_1 - \frac{d'_{N2}(x)}{d'_{N2}} u_1 + \frac{d'_{N2}(x)}{d'_{N2}} u_2 \,, \tag{13}$$

where u_1 and u_2 are displacements in node 1 and 2, respectively, $d'_{N2}(x)$ is transfer function and d'_{N2} is transfer constant (transfer constant is transfer function for $x = L$ and L is length of element).

From equation (13) we can write for shape functions

$$\phi_{11} = 1 - \frac{d'_{N2}(x)}{d'_{N2}} \qquad \phi_{12} = \frac{d'_{N2}(x)}{d'_{N2}} \tag{14}$$

and their derivatives

$$\phi_{11,1} = -\frac{d''_{N2}(x)}{d'_{N2}} \qquad \phi_{12,1} = \frac{d''_{N2}(x)}{d'_{N2}} \tag{15}$$

3.3. **Full non-linear stiffness matrix.** Members of single matrices of a full non-linear stiffness matrix for a bar element with variation of cross-sectional area also have the forms (9a), (9b), (9c) and (9d). Shape functions and their derivations are defined by (14) and (15). For bar elements with linear elastic deformation the tensor of elastic constants C_{ijkl} is characterized by the Young modulus of elasticity E and $\mathrm{d}V$ can be written as $A^0\mathrm{d}x$, where A^0 is undeformed cross-sectional area, which is defined by equation (12). Considering all these equations, we can write for members of $\mathbf{K}^L$

$$K^L_{mn} = A^0_1 E \int_{L^0} \eta_A(x)\phi_{1m,1}\phi_{1n,1}\mathrm{d}x \,. \tag{16}$$

After the integration [3] for the whole matrix $\mathbf{K}^L$ we can write

$$\mathbf{K}^L = \frac{A^0_1 E}{d'_{N2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \tag{17}$$

For member of $\mathbf{K}^{NL1}$ we obtained

$$K^{NL1}_{nm} = \frac{1}{4} A^0_1 E \int_{L^0} \eta_A(x)\phi_{1m,1}\left(\phi_{11,1}q_1 + \phi_{12,1}q_2\right) 2\phi_{1n,1}\mathrm{d}x \tag{18}$$

and for the whole matrix $\mathbf{K}^{NL1}$

$$\mathbf{K}^{NL1}(\mathbf{q}) = \frac{1}{2}\frac{A^0_1 E}{(d'_{N2})^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (q_2 - q_1)\, \bar{d}'_{N2} \,. \tag{19}$$

Similarly we can derive $\mathbf{K}^{NL2}$ and $\mathbf{K}^{NL3}$

$$\mathbf{K}^{NL2}(\mathbf{q}) = \frac{A^0_1 E}{(d'_{N2})^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (q_2 - q_1)\, \bar{d}'_{N2} \tag{20}$$

$$\mathbf{K}^{NL3}(\mathbf{q}) = \frac{1}{2}\frac{A^0_1 E}{(d'_{N2})^4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (q_2 - q_1)^2\, \bar{\bar{d}}'_{N2} \tag{21}$$

The final full non-linear stiffness matrix (8) can be written using (17), (19), (20) and (21) as

$$\mathbf{K}(\mathbf{q}) = (k^L + k^{NL}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} , \tag{22}$$

where

$$k^L = \frac{A^0_1 E}{d'_{N2}} \,, \tag{23}$$

$$k^{NL} = k^L \left[\frac{3}{2}(q_2 - q_1)\frac{\bar{d}'_{N2}}{(d'_{N2})^2} + \frac{1}{2}(q_2 - q_1)^2 \frac{\bar{\bar{d}}'_{N2}}{(d'_{N2})^3} \right] . \tag{24}$$

As can be seen from the previous equations, the full non-linear stiffness matrix of a bar element with variation of the cross-sectional area contains transfer constant d'_{N2} and two new modified transfer constants $\bar{d}'_{N2}$ and $\bar{\bar{d}}'_{N2}$. Numerical computation of transfer constant d'_{N2} is described in the Appendix, the modified transfer constants $\bar{d}'_{N2}$ and $\bar{\bar{d}}'_{N2}$ are the same transfer constants as d'_{N2}, but the polynomial $\eta_A(x)$ is changed to $(\eta_A(x))^2$ and $(\eta_A(x))^3$, respectively.

3.4. **Full non-linear tangent matrix.** The system of equations (7) with stiffness matrix in form (22) is non-linear, which is usually solved using the Newton-Raphson method. This iteration method makes use of derivatives of single functions of a system whose solution is being found, and that is why the full tangent stiffness matrix is required to be built.

In the formal way, the tangent stiffness matrix can be derived as

$$\begin{aligned}\mathbf{K}_T(\mathbf{q}) &= \frac{\partial \mathbf{F}}{\partial \mathbf{q}} = \frac{\partial \mathbf{K}(\mathbf{q})}{\partial \mathbf{q}}\mathbf{q} + \mathbf{K}(\mathbf{q}) = \frac{\partial \mathbf{K}^{NL}(\mathbf{q})}{\partial \mathbf{q}}\mathbf{q} + \mathbf{K}^{NL}(\mathbf{q}) + \mathbf{K}^L \\ &= \mathbf{K}^L + \mathbf{K}^{NLT}(\mathbf{q})\end{aligned} \tag{25}$$

Using equations (22), (23), (24) and (25) we can write the full non-linear stiffness matrix for a bar element with variation of cross-sectional area as

$$\mathbf{K}_T(\mathbf{q}) = (k^L + k^{NLT}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{26}$$

where k^L is defined by equation (23) and

$$k^{NLT} = k^L \left[3(q_2 - q_1)\frac{\bar{d}'_{N2}}{(d'_{N2})^2} + \frac{3}{2}(q_2 - q_1)^2 \frac{\bar{\bar{d}}'_{N2}}{(d'_{N2})^3} \right] \tag{27}$$

For the evaluation of the efficiency of the iteration procedure, we use the Euclidean norm of residual forces, which is compared with the norm of external nodal forces multiplied by a very small coefficient ϵ.

4. Implementation

The whole process of a bar element with variation of cross-sectional area for geometric non-linear problems was prepared in Fortran language. Solution for the transfer functions were taken from our program RAM3D [7], which was designed for linear problems and handled the beam element with variation of cross-sectional characteristics.

For the evaluation of the efficiency of the iteration procedure, we use the Euclidean norm of residual forces, which is compared with the norm of external nodal forces multiplied by the coefficient ϵ. More details about internal forces, residual forces, norms of residual forces and iteration process are published in [3].

5. Numerical experiments

The convenience of our new bar element with variation of cross-sectional area for geometric non-linear problems is illustrated in the next example. Figure 2 shows a simple arch structure assembled of two bar elements with variation of cross-sectional area - a well-known snap-through problem.

The material properties of the bars are defined by the Young modulus of elasticity $E = 2.1 \times 10^{11}$ Pa. The geometry is given by the parameters $L^0 = 1$ m and $\alpha = 15^{\circ}$.

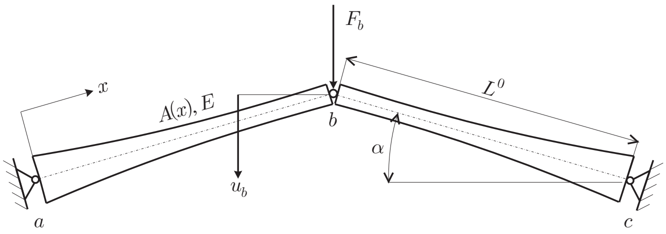

Figure 2. Snap-through problem

We considered four types of cross-sectional area $A(x)$ - from linear polynomial to fourth order polynomial. These have the following forms

type A

$$A(x) = 0.005 - 0.0047x \ ,$$

type B

$$A(x) = 0.005 - 0.0094x + 0.0047x^2 \ ,$$

type C

$$A(x) = 0.005 - 0,0141x + 0.0141x^2 - 0.0047x^3 \ ,$$

type D

$$A(x) = 0.005 - 0.0188x + 0.0282x^2 - 0.0188x^3 + 0.0047x^4 \ .$$

All cross-sectional areas are shown in Figure 3.

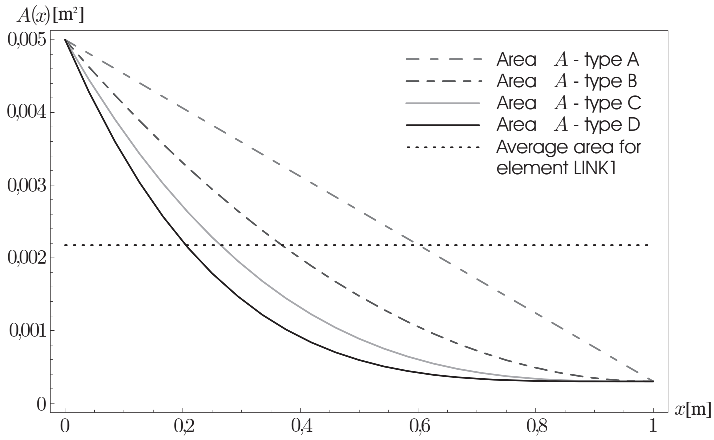

Figure 3. All types of cross-sectioal areas $A(x)$ considered

The goal is to find dependence between displacement u_b and load F_b.

The solution was obtained by our program NelinPrut, in which the new bar element is implemented, and comparative results were obtained from program ANSYS.

In our NelinPrut program, each bar was represented by one element only, but the variation of cross-sectional area was described exactly. In program ANSYS, there are two suitable elements - the classical bar element LINK1 or the beam element with variation of cross-sectional characteristics BEAM54. But the bar element LINK1 is developed for constant cross-sectional area and that is why we should compute some average area according to [6] (see Figure 3). In LINK1 we use also one element only. Element BEAM54 is more suitable, because it is developed for varying cross-sections and because we could refine mesh. But this element does not describe variation of the cross-section exactly, either.

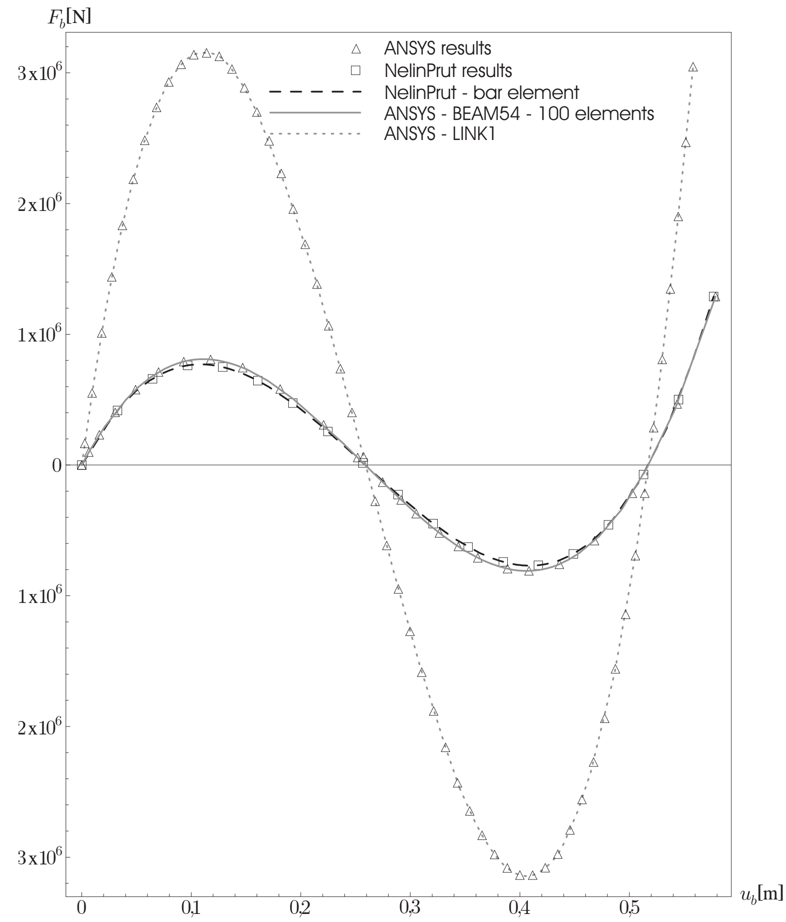

Figure 4. Snap-through problem: dependence between u_b and F_b for type D

Figure 4 shows results for cross-sectional area type D. In this Figure, there are 3 equilibrium paths. The equilibrium paths of the new bar element and BEAM54 with 100 elements are very similar but the equilibrium path of LINK1 is different. Differences between our results and BEAM54 and LINK1 results are caused by linearization of the Green-Lagrange strain tensor and also by variation of cross-sectional area. While our new bar element has shape functions which respect variation of cross-sectional area, BEAM54 and LINK1 use classical shape functions, which do not respect variation of cross-section. BEAM54 results are more accurate than LINK1 results because BEAM54 allows refinement: there are 100 elements and then variation of cross-sectional area is described more suitable than in LINK1.

Variation	NelinPrut		ANSYS-BEAM54 100 elements	
Type	u_b [mm]	IT	u_b [mm]	IT
A	6.6510	7	6.6171	8
B	12.872	8	12.750	10
C	17.958	9	17.744	11
D	21.872	10	21.574	12

Table 1. Snap-through problem: displacement u_b and number of iterations IT for load $F_b = 0,3 \times 10^6$ N for single types of cross-sectional areas A

Variation	NelinPrut		ANSYS-BEAM54 100 elements	
Type	u_b [mm]	IT	u_b [mm]	IT
A	13.927	9	13.777	10
B	28.490	12	27.854	13
C	42.165	15	40.811	17
D	54.635	19	52.290	21

Table 2. Snap-through problem: displacement u_b and number of iterations IT for load $F_b = 0,6 \times 10^6$ N for single types of cross-sectional areas A

Tables 1 and 2 show displacement u_b as function of F_b for all four types of cross-section variation for the new bar element and BEAM54. Table 1 shows results for load $F_b = 0,3 \times 10^6$ N and Table 2 for load $F_b = 0,6 \times 10^6$ N. As can be seen from the Tables, the difference between our and ANSYS results grows with increasing load, where the linearization of the Green-Lagrange strain tensor has more influence on result accuracy. The numbers of iterations IT are nearly equal.

Influence of mesh refinement is shown in Figure 5. We can see that for cross-sectional area type D – this is the most complicated variation –, ANSYS results for BEAM54 are not exactly the same as our new bar element results, neither for refinement. The difference is caused by linearization of the strain tensor mentioned above.

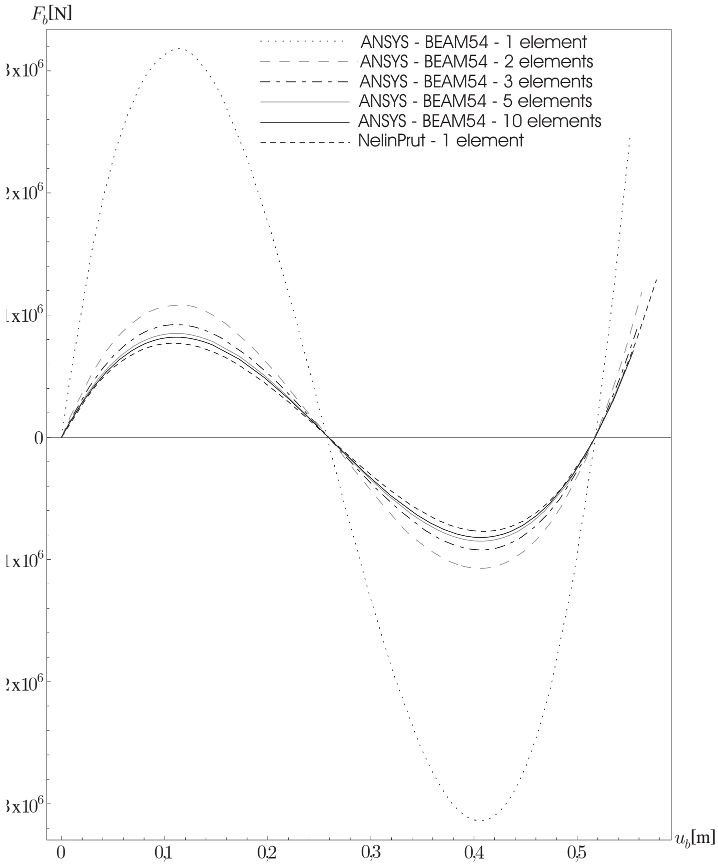

Figure 5. Snap through problem: influence of mesh refinement of BEAM54 element, cross-sectional area type - D

ANSYS - BEAM54		
N_{elem}	u_b [mm]	IT
1	5.0394	5
2	15.648	7
4	19.808	9
6	20.754	9
8	21.106	10
10	21.273	10
20	21.500	10
50	21.565	11
100	21.574	12

NelinPrut		
N_{elem}	u_b [mm]	IT
1	21.872	10

Table 3. Snap-through problem: displacement u_b and the number of iterations IT for load $0F_b = 0.3 \times 10^6$ N for type D as mesh density function

ANSYS - BEAM54		
N_{elem}	u_b [mm]	IT
1	10.383	6
2	35.157	11
4	46.778	15
6	49.677	16
8	50.786	17
10	51.320	18
20	52.051	18
50	52.260	21
100	52.290	21

NelinPrut		
N_{elem}	u_b [mm]	IT
1	54.635	19

Table 4. Snap-through problem: displacement u_b and number of iterations IT for load $F_b = 0,6 \times 10^6$ N for type D as mesh density function

Furthermore Tables 3 and 4 show a difference between our results and ANSYS results in the increasing number of BEAM54 elements in ANSYS.

6. **Conclusion**

The bar element with variation of cross-sectional area presented was derived without any linearization of the Green-Lagrange strain tensor or constitutive law. The method of solution is non-incremental.

As can be seen from the results, the linearization of the terms mentioned has influence on result accuracy also in refinement of mesh, because the main equations, which are used as equilibrium equations, are still linearized: the difference between our results for one bar element and 100 BEAM54 elements of ANSYS. Shape functions also have an influence on results: difference between our results for one bar element with the new shape functions and one bar LINK1 element of ANSYS, but this influence can be eliminated by refinement.

Numerical experiments confirm the applicability of the new bar element with variation of cross-sectional area with new shape functions for non-linear problems and it could be an alternative to the classical bar element with linear shape functions.

We remark that the paper was presented at the 9th International Conference on Numerical Methods in Continuum Mechanics, Zilina, Slovakia, 9-12 September 2003 and its shorter version was published in the Conference CD Proceedings.

REFERENCES

1. Washizu, K.: *Variational Methods in Elasticity and Plasticity.* Pergamon Press, Oxford, 1975.
2. Murín, J.: Implicit non-incremental FEM equations for non-linear continuum. *Strojnícky časopis,* **52**, (2001), 147-158.

3. Kutiš V.: Beam element with variation of cross-section which satisfies all local and global equilibrium conditions. *PhD Thesis*, FEI STU, Bratislava, 2001.

4. Murín, J., and Kutiš, V.: 3D - beam element with continues variation of the cross-sectional area. *Computers and Structures,* **80**, (2002), 329-338.

5. Rubin, H.: Analytische Lösung linearer Differentialgleichungen mit veränderlichen Koeffizienten und baustatische Anwendung. *Bautechnik,* **76**, 1999.

6. ANSYS 5.6 - *Theory Manual.* SAS IP, 1999.

7. Kutiš, V., and Murín, J.: RAM3D - program for statical analysis of frame structures. *Program code*, FEI STU, Bratislava, 2003.

APPENDIX

Determination of the transfer functions and transfer constants occurring in the stiffness matrix and shape functions is based on the following expression

$$d''_{Nj+2}(x) = \frac{a_j(x)}{\eta_A(x)} \ ,$$

where the function $a_j(x) = \frac{x^j}{j!}$ for $j \geq 0$, and for $j \leq 0$, $a_0 = 1$, $a_j = 0$. Closed solutions for the 1st and 2nd integrals of the function $d''_{Nj+2}(x)$ are known only for lower degree polynomials $\eta_A(x)$. For their numerical solution, which is more general, a recurrance rule was derived

$$d^{(n)}_{Nj}(x) = a_{j-n}(x) - \sum_{k=1}^{m} \eta_{Ak} \frac{(j-2+k)!}{(j-2)!} \, d^{(n)}_{Nj+k}(x) \qquad \text{for} \quad j \geq 2 \ , \ n = 0 \text{ a } 1 \ .$$

After some manipulation we get

$$d^{(n)}_{Nj}(x) = a_{j-n}(x) \, \sum_{t=0}^{\infty} \beta_{t,0}(x) \ ,$$

where $\beta_{t,0}(x)$ is expressed by

$$\beta_{t,0}(x) = -\sum_{k=1}^{m} \left[\eta_{Ak} \beta_{t,k}(x) \prod_{r=-k}^{-1} (s-1+r) \right]$$

with parameters

$$s = 1 + t \qquad e = \frac{x}{s-n} \qquad \beta_{t,k} = e\beta_{t-1,k-1} \qquad \text{for} \quad k = 1, ...m$$

and initial values

$$\beta_{0,0} = 1 \qquad \beta_{0,k} = 0 \qquad \text{for} \quad k = 1, ...m \ .$$