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THE KLUITENBERG–VERHÁS MODEL OF VISCOELASTICITY AND ITS IRREVERSIBLE THERMODYNAMICAL CONSISTENCY VIA RHEOLOGICAL ENERGY

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Abstract. Rocks and plastics are two examples of classes of materials where the Kluitenberg–Verhás model of viscoelasticity has proved adequate for describing the observed mechanical behaviour. This model cannot be interpreted as a generalized Maxwell model, and requires in fact a novel rheological circuit element beyond the spring and the dashpot. Moreover, its thermodynamical consistency leads to nontrivial inequality conditions on the model coefficients, ignoring which would lead to violation of the second law of thermodynamics and would thus be a principal source of instability of numerical solutions. The present paper explores the role of the rheological energy contribution embedded in the Kluitenberg–Verhás model, in a didactic and application-oriented way. The explicit meaning of the involved irreversible thermodynamical additional state variable – internal degree of freedom – is identified. Neither thermal expansion nor heat conduction is neglected, enhancing the applicability of the constructed framework.

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1. Introduction

Materials such as rocks and plastics do not exhibit a prompt and reversible elastic response to changes but manifest a delayed and damped behaviour. As explored by the Anelastic Strain Recovery (ASR) method [1–4], various rocks necessitate the Kluitenberg–Verhás viscoelastic model for describing the observed strain histories, and the same can be found for plastics [5]. This model adds the time derivative of stress and the first *and second* time derivatives of strain to the Hooke connection between stress and strain. Unlike the special case of Poynting–Thomson–Zener standard rheology with first stress derivative and first strain derivative, such a relationship is beyond the range of generalized Maxwell models of rheology (for a proof see, *e.g.,* [6],

Appendix A). Moreover, the Kluitenberg–Verhás model cannot be realized via the two conventional rheological circuit elements, springs and dashpots, but also demands the so-called Verhás element [7] (see Figure 1).

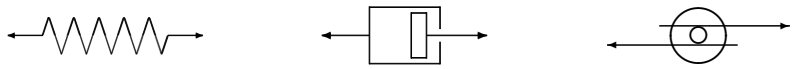

Figure 1. In addition to the two conventional rheological circuit elements, the spring (left) and the dashpot (middle), the Kluitenberg–Verhás model raises the need for a novel rheological circuit element, the Verhás element (right).

In [8], the Kluitenberg–Verhás model was derived via an additional *entropy* term, in the internal variable formalism of irreversible thermodynamics, and the inequality conditions on the model coefficiends imposed by the second law of thermodynamics have been identified and investigated.

One drawback of the treatment given in [8] is that the special case of constant model coefficients – the case most frequently demanded by concrete applications – can only be obtained in a quasi-isothermal approximation. As mentioned in [8] without any details (and considered in [9] as a forerunner to this paper), an alternative approach could be established via an additional *energy* term. Here, the latter option is worked out in detail, with unforeseen benefits: the case of constant coefficients does not require approximation of any kind, and the concrete meaning of the extra state variable – internal variable – can be revealed. This latter is particularly valuable given that, typically, the internal variable methodology cannot interpret the assumed internal variable itself but only derive its consequences on the rest, well-interpreted and measureable, quantities. Here it turns out that the rheological internal variable can be accessed as a combination of quantities that can be identified via measurement. This opens the possibility of finding analytical or numerical solutions for the whole set of state variables, including the internal variable as well. In parallel, while in [8] thermal expansion is neglected, it is not the case here. Hence, the framework presented here is suitable for applications like the ASR method of rock mechanics, which, for instance, was successfully utilized in a recent R&D project [10].

A final introductory remark is that, besides the internal-variable framework applied here, rheology utilizes various other approaches, too. Each has its virtues and limitations.

The simplest concept is that of rheological circuits. It offers a pleasant interpretation in terms of additive stresses (dynamical additivity) [realized as parallel connections] and additive strains (kinematic additivity) [realized as serial connections]. This interpretation is, however, non-unique and can thus be misleading: for example, the Poynting–Thomson–Zener model can be realized via two different three-element circuits (see, *e.g.,* [11]). Another shortcoming is that, following historical roots, only springs and dashpots are typically considered, so thermodynamically legitimate cases like the Verhás element cannot be covered.

A much more general approach is introducing history dependence using integral equations (see, for example, [12–15]). Here, one challenge is to find a suitable integral representation (cf. *e.g.,* page 310 of [13]). Another complication arises when introducing history dependence of energy and entropy (see, *e.g.,* page 78 of [14]). Finally, an integral kernel based description means infinitely many free parameters to fit from experimental data. This problematical richness is typically reduced by using prescribed patterns with a finite number of parameters, as in the form of a Prony series (cf. *e.g.,* page 315 of [15]). This introduces the limitation that, for example, the Kluitenberg–Verhás model remains out of scope (see, *e.g.,* the above-mentioned proof in Appendix A of [6]).

The internal-variable framework can also be done in an interpretation-friendlier but restrictive form when the considered internal variables are stress contributions themselves (like in [16], page 155; see also [17], page 343). This also fails to cover the Kluitenberg–Verhás model (as revealed below in the final formula of (61), where the internal variable is found to also contain a velocity gradient dependent part in addition to the nonelastic stress contribution).

When we do not bind the introduced internal variable to any particular interpretation, one drawback is the absence of any initial interpretation. On the other hand, one advantage is obtaining a model that is the most general model within that universality class. Consequently, the scene is open for any possible later interpretation – macroscopic, mesoscopic or microscopic. Fortunately, while applying the internal variable methodology, no particular interpretation is necessary: just general rules and assumptions of irreversible thermodynamics are to be followed, using mathematics only.

Moreover, in the concrete case presented in this article, even a macroscopic interpretation is found [at the already-mentioned final formula of (61)]. This enables us to identify concrete mechanisms in the future that can lead to such an additional degree of freedom.

2. The starting point

The continuum background necessary for starting is as follows (see, *e.g.,* [18]).

The velocity gradient tensor $\mathbf{L}$ is the derivative of the velocity field $\mathbf{v}$:

$$\mathbf{L} = \mathbf{v} \otimes \overleftarrow{\nabla} \qquad (\text{in Cartesian components: } L_{ij} = \partial_j \mathrm{v}_i)\,, \tag{1}$$

and the local/differential balance of internal energy is the relationship

$$\varrho \dot{e} = -\nabla \cdot \mathbf{j}_E + \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L}) = -\nabla \cdot \mathbf{j}_E + \mathrm{tr}\left(\boldsymbol{\sigma}\mathbf{L}^{\mathsf{S}}\right) \tag{2}$$

among (mass-)specific internal energy e, heat current density $\mathbf{j}_E$, the stress tensor $\boldsymbol{\sigma}$ (assumed to be symmetric), and the symmetric part of the velocity gradient, denoted as $\mathbf{L}^{\mathsf{S}}$.[1] (Volumetric energy source is not displayed here but could be added easily

[1] In this article, a product of tensors like $\boldsymbol{\sigma}\mathbf{L}$ denotes the composition of linear maps, hence, in Cartesian indices, $(\boldsymbol{\sigma}\mathbf{L})_{kl} = \sigma_{km} L_{ml}$. Another popular convention is to write this as $\boldsymbol{\sigma} \cdot \mathbf{L}$.

[19].) The local/differential balance of entropy is the connection

$$\varrho\dot{s} = -\nabla\cdot\mathbf{j}_S + \pi_S \tag{3}$$

between specific entropy s, entropy current density $\mathbf{j}_S$, and entropy production rate density π_S. Between entropy current density and heat current density, we consider the customary relationship

$$\mathbf{j}_S = \frac{1}{T}\mathbf{j}_E\,, \tag{4}$$

where $T > 0$ is absolute temperature.

In case of a reversible model,

$$\pi_S = 0 \tag{5}$$

along any allowed process, while for an irreversible model we investigate the inequality

$$\pi_S \geq 0 \tag{6}$$

as dictated by the second law of thermodynamics.

Heat conduction is going to be incorporated in Sect. 13; before that, in order to help focus on the mechanical irreversibility, we "switch heat conduction off":

$$\mathbf{j}_E = \mathbf{0}, \quad \mathbf{j}_S = \mathbf{0}\,. \tag{7}$$

Then, for a reversible model,

$$T\pi_S = 0, \tag{8}$$

require

$$T\pi_S = \varrho T\dot{s} \geq 0\,. \tag{9}$$

As a remark, the form $\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L})$ of mechanical power density enables the forthcoming considerations to be applicable not only for continuum theory but also for the space-independent/homogeneous (also called lumped/concentrated-parameter) description, in other words, temporal thermodynamics [18, 20, 21]. Namely, no space derivatives appear hereafter, and $\mathbf{L}^{\mathsf{S}}$ may also be treated as a degree of freedom in its own right, not only as a space derivative of some other degree of freedom. Its connection to other kinematic quantities may be derived not only from the continuum perspective [22–27] but can also be postulated (still motivated by the continuum description, naturally). Therefore, we can obtain a space-independent, finite degree-of-freedom, temporal dynamical model for elastic, viscoelastic, and thermal-expansion phenomena. This is a practically simple framework that is satisfactory for discussing a class of experimental setups (such as those mentioned in the Introduction regarding rocks and plastics).

We are closer to engineering practice if our thermal state variable is T rather than e or s. First, let us allow for thermal changes only:

$$e = e_{\mathrm{th}}(T), \quad s = s_{\mathrm{th}}(T), \tag{10}$$

among which the Gibbs relation

$$\mathrm{d}e = T\,\mathrm{d}s \tag{11}$$

(as a constitutive consistency condition) is ensured by imposing

$$\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T} = T\frac{\mathrm{d}s_{\mathrm{th}}}{\mathrm{d}T}\,. \tag{12}$$

[In the language of the variable T, concavity of $s(e)$ can be expressed as $\mathrm{d}e_{\mathrm{th}}/\mathrm{d}T > 0$, or, equivalently, $\mathrm{d}s_{\mathrm{th}}/\mathrm{d}T > 0$.] In particular,

$$e_{\mathrm{th}}(T) = cT, \qquad s_{\mathrm{th}}(T) = c\ln\frac{T}{T_{\mathrm{ref}}} + s_{\mathrm{ref}} \tag{13}$$

embodies a model with constant specific heat capacity c (here, s_{ref} denotes the value of s at an arbitrarily chosen reference temperature value T_{ref}.)

In addition to thermal changes, we are going to introduce elastic, viscoelastic and thermal-expansion phenomena gradually.

We restrict ourselves to small deformations of an isotropic material. In this approximation, mass density ϱ is constant.

We are going to have elastic and thermal-expansion deformations. Elastic deviation is measured locally and tensorially by the elastic deformedness tensor $\mathbf{D}$ – the full details[2] on the kinematic-type state quantity $\mathbf{D}$ and its relationship to $\mathbf{L}$ and the customarily used strain tensor $\boldsymbol{\varepsilon}$ can be found in [18, 22–27]. Even in small-$\mathbf{D}$ approximation elasticity, $\boldsymbol{\varepsilon} \not\equiv \mathbf{D}$ but $\boldsymbol{\varepsilon} = \mathbf{D} - \mathbf{D}_{\mathrm{initial}}$. Unlike it is done customarily, we do not assume $\mathbf{D}_{\mathrm{initial}} = \mathbf{0}$, *i.e.,* that the process starts from an undeformed state. This is particularly important for applications like the above-mentioned ASR method of rock mechanics, where the sample is initially in a compressed underground state. Accordingly, strain is not a thermodynamical state quantity but a change quantity, like heat and work. The elasticity-related thermodynamical state quantity is $\mathbf{D}$. Later [in formula (71) below] we will see that, while $\dot{\boldsymbol{\varepsilon}} = \mathbf{L}^{\mathsf{S}}$ is only a gross kinematic change rate, the thermodynamical state quantities $\mathbf{D}$ and T enable its decomposition into elastic and thermal-expansion change rates.

Regarding thermal expansion, the thermal expansion coefficient is assumed constant. Its temperature dependence would make even the small-deformation treatment nonlinear. Fortunately, for a large range of practical applications, the contribution of this higher-order effect would be very small.[3] On the other side, a constant but *nonzero* thermal expansion coefficient is important to consider. Its role has been found remarkable, for example, in the ASR method. Hence, in the ASR protocol, either air conditioning in a sealed room or a water or oil bath with a circulator is

[2] As a brief summary, at a material point on the material manifold, the instantaneous metric tensor $\mathbf{h}$ is the realized metric at an instant (providing the instantaneous distances between material points), while the relaxed metric tensor $\mathbf{g}$ is the metric the solid takes when being in a stressless relaxed state. The elastic deformedness tensor $\mathbf{D} = \ln\sqrt{\mathbf{g}^{-1}\mathbf{h}}$ measures the deviation between these two metrics. If $\mathbf{h} \approx \mathbf{g}$ then $\mathbf{D}$ is small ($\|\mathbf{D}\| \ll 1$). The time evolution of $\mathbf{D}$ is determined by the time evolution of $\mathbf{h}$ (which is of purely kinematic origin), while, for pure elasticity, $\mathbf{g}$ is constant, leading for small $\mathbf{D}$ to $\dot{\mathbf{D}} \approx \mathbf{L}^{\mathsf{S}} = \dot{\boldsymbol{\varepsilon}} \Rightarrow \mathbf{D} - \mathbf{D}_{\mathrm{initial}} = \boldsymbol{\varepsilon}$ via time integration (as $\boldsymbol{\varepsilon}_{\mathrm{initial}} = \mathbf{0}$ by definition). Thermal expansion is manifested in nonconstant, T-dependent, $\mathbf{g}$, and time-dependent T leads to time-dependent $\mathbf{g}$; theű analogous time integration for small deformations reveals an additive thermal expansion-related contribution in $\boldsymbol{\varepsilon}$.

[3] As an example, in the comprehensive book [28] on thermal stresses induced by thermal expansion, it is hard to find any mentioning of temperature dependence of the thermal expansion coefficient.

applied to keep the temperature of the measured sample as constant as possible [1–4]. In addition, they found it necessary to add a dummy specimen, which does not undergo any deformation except thermal expansion, for monitoring the deformation due to the remaining unavoidable (1−2) °C temperature fluctuation.

In order to focus on the thermodynamical aspects, first let us consider one space dimension – more properly, one kinematic degree of freedom. Note that such a treatment is not for uniaxial processes: that would be a self-deception because transversal changes are hardwired to the longitudinal ones – via Poisson's ratio – only for pure elasticity; rheology makes the transversal changes decouple from the longitudinal ones, see, *e.g.,* [18]. Instead, for isotropic materials, the deviatoric and spherical parts of the model both prove to be effectively scalar (o)ne kinematic degree of freedom type. Between the two parts, the spherical (volumetric) part will contain isotropic thermal expansion. The details will follow below. The full tensorial treatment of isotropic materials begins in Sect. 12.

3. The Hooke model

Elasticity is actually a reversible internal-variable extension of the above thermal-only model (10)–(12).

Namely, in addition to T and the related aspects (10)–(12), we consider an extra degree of freedom D. It plays a kinematic role, which will be expressed by its relationship to L as

$$\dot{D} = L\,. \tag{14}$$

Let stress, σ, be the linear function

$$\sigma = ED \tag{15}$$

of this state variable, with the positive coefficient E (which, in case of uniaxial interpretation, is Young's modulus). The mechanical power density σL of this stress is reversible and nondissipative if specific internal energy is shifted in the following quadratic convex way:

$$e(T,D) = e_{\mathrm{th}}(T) + \frac{E}{2\varrho}D^2 \tag{16}$$

[following from which $s(e,D)$ proves concave]. In the meantime, specific entropy is not shifted and is kept D-independent:

$$s = s_{\mathrm{th}}(T)\,. \tag{17}$$

Then

$$\begin{aligned} T\pi_S = \varrho T\dot{s} = \varrho T\frac{\mathrm{d}s_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = \varrho\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = \varrho\dot{e}_{\mathrm{th}} = \varrho\left(e - \frac{E}{2\varrho}D^2\right)^{\cdot} = \varrho\dot{e} - ED\dot{D} = \\ = \sigma L - ED\dot{D} = \sigma\dot{D} - ED\dot{D} = (\sigma - ED)\dot{D} = 0\,. \end{aligned} \tag{18}$$

Meanwhile, temperature proves constant since, according to (18), along any allowed process

$$\dot{e}_{\mathrm{th}} = 0, \tag{19}$$

thus

$$0 = \dot{e}_{\mathrm{th}} = \frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T}, \tag{20}$$

which, in light of $\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T} > 0$ (see above) leads to

$$\dot{T} = 0\,. \tag{21}$$

Hence, the model is reversible (*i.e.,* no entropy production) and nondissipative (*i.e.,* no change in $T \Rightarrow$ no change in the thermal term $e_{\mathrm{th}} \Rightarrow$ mechanical power modifies the elastic term $\frac{E}{2\varrho}D^2$ only $\Rightarrow$ no conversion of mechanical energy into thermal).

4. The Kelvin–Voigt model

Compared to the previous elastic model, let e and s remain unmodified. However, let an extra – viscosity-type – term be present in stress:

$$\sigma = ED + \hat{E}L = ED + \hat{E}\dot{D}\,, \tag{22}$$

with viscosity coefficient $\hat{E}$.[4]

This is the Kelvin–Voigt model. With the irreversible stress contribution

$$\hat{\sigma} = \sigma - ED\,, \tag{23}$$

which is now

$$\hat{\sigma} = \hat{E}\dot{D}\,, \tag{24}$$

$$\begin{aligned} T\pi_S = \varrho T\dot{s} = \varrho T\frac{\mathrm{d}s_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = \varrho\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = \varrho\dot{e}_{\mathrm{th}} = \varrho\left(e - \frac{E}{2\varrho}D^2\right)^{\cdot} = \varrho\dot{e} - ED\dot{D} = \\ = \sigma L - ED\dot{D} = \sigma\dot{D} - ED\dot{D} = \hat{\sigma}\dot{D} + ED\dot{D} - ED\dot{D} = \hat{E}\dot{D}^2 \geq 0 \end{aligned} \tag{25}$$

so the model is irreversible ($\hat{E} < 0$ is excluded by the second law of thermodynamics). In parallel, temperature increases (r decreases) monotonously (:)

$$\varrho\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = \varrho\dot{e}_{\mathrm{th}} = \hat{E}\dot{D}^2 \geq 0\,. \tag{26}$$

Let us also observe the relationship

$$T\pi_S = \hat{\sigma}\dot{D} = \hat{\sigma}L \tag{27}$$

because we find similar formulae below.

[4]The coefficient combination $\hat{E}/E$ is a time scale, and turns out to be the characteristic time for creep (which is when $\sigma = \mathrm{const.}$, hence, D increases in time exponentially towards the corresponding Hooke prediction σ/E).

5. The Poynting–Thomson–Zener model

For the purpose of the Poynting–Thomson–Zener (PTZ) model[5]

$$\sigma + \tau\dot{\sigma} = ED + \hat{E}L \qquad \text{or} \qquad \sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D}\,, \tag{28}$$

a new degree of freedom introducing irreversibility is needed. In paper [8] specific entropy is shifted by a term proportional to the square of the new degree of freedom ξ: temporarily switching to their notations,

$$\tilde{s}(e, D, \xi) = s(e, D) - \frac{1}{2}\xi^2\,, \tag{29}$$

while they do not modify e. This, however, is approximately the same as if we leave specific entropy unmodified but shift specific internal energy by a quadratic term of ξ:

$$e(\tilde{s}, D) = e\left(s + \tfrac{1}{2}\xi^2, D\right) \approx e(s, D) + \tfrac{\partial e}{\partial s}\big|_D \cdot \tfrac{1}{2}\xi^2 = e + T \cdot \tfrac{1}{2}\xi^2\,. \tag{30}$$

Here, we choose to shift specific internal energy by an extra term and keep s unmodified.[6]

In case of the PTZ model we can find the appropriate extra term "by hand". As the first step, let us rewrite (28) (in its second variant) with the aid of (23):

$$(\hat{\sigma} + ED) + \tau(\hat{\sigma} + ED)^{\cdot} = ED + \hat{E}\dot{D}\,, \tag{31}$$

$$\hat{\sigma} + \tau\dot{\hat{\sigma}} = \left(\hat{E} - \tau E\right)\dot{D} = \hat{I}\dot{D}, \tag{32}$$

where

$$\hat{I} := \hat{E} - \tau E \tag{33}$$

(named the *index of damping*). Both convexity of e (concavity of s) and positive definiteness of π_S excludes $\hat{I} < 0$ [see (36)], and $\hat{I} = 0$ means a redundant resonant [29] case: (28) is then Hooke's law (15) plus its time derivative multiplied by τ. Therefore, a meaningful PTZ model has $\hat{I} > 0$ (which we assume hereafter).

Next, let us consider the balance of internal energy:

$$\varrho\dot{e} = \sigma L = \sigma\dot{D} = ED\dot{D} + \hat{\sigma}\dot{D} = \left(\frac{E}{2}D^2\right)^{\cdot} + \hat{\sigma}\left[\frac{1}{\hat{I}}\left(\hat{\sigma} + \tau\dot{\hat{\sigma}}\right)\right] =$$

$$= \left(\frac{E}{2}D^2 + \frac{\tau}{2\hat{I}}\hat{\sigma}^2\right)^{\cdot} + \frac{\hat{\sigma}^2}{\hat{I}}\,. \tag{34}$$

[5]The coefficient τ is a time scale, and turns out to be the characteristic time for stress relaxation (which is when $D = \text{const.}$, hence, σ decreases in time exponentially towards the corresponding Hooke prediction ED).

[6]Assuming a new term in s or e in the internal-variable approach is a step corresponding to assuming a new stress contribution. Thermodynamical consistency requires it so that the resulting generalized model will be thermodynamically correct. *Interpreting* such a new term may be possible when the corresponding internal variable gets an interpretation. For complex materials like rocks the interpretation may be difficult; nevertheless, the model as an effective description can be valuable: just a few new parameters fitted from experiments can provide a reliable and efficient modeling of the time-dependent and irreversible processes of that complex material.

This points out that if

$$e = e_{\mathrm{th}} + \frac{E}{2\varrho} D^2 + \frac{\tau}{2\varrho \hat{I}} \hat{\sigma}^2 \tag{35}$$

is our specific internal energy then, analogously to what we have seen above,

$$\begin{aligned} T\pi_S = \cdots = \varrho \dot{e}_{\mathrm{th}} = \varrho \left[\dot{e} - \left(\frac{E}{2\varrho} D^2 + \frac{\tau}{2\varrho \hat{I}} \hat{\sigma}^2 \right)^{\cdot} \right] = \\ = \left(\frac{E}{2} D^2 + \frac{\tau}{2\hat{I}} \hat{\sigma}^2 \right)^{\cdot} + \frac{\hat{\sigma}^2}{\hat{I}} - \varrho \left(\frac{E}{2} D^2 + \frac{\tau}{2\hat{I}} \hat{\sigma}^2 \right)^{\cdot} = \frac{\hat{\sigma}^2}{\hat{I}} \geq 0 , \end{aligned} \tag{36}$$

i.e., we have obtained a model with positive definite entropy production rate density in consistency with the second law of thermodynamics.

In parallel, also analogously, the time dependence of temperature is monotonously nondecreasing again:

$$\varrho \frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T} \dot{T} = \varrho \dot{e}_{\mathrm{th}} = \frac{\hat{\sigma}^2}{\hat{I}} \geq 0 . \tag{37}$$

Therefore, in case of the PTZ model, the extra internal variable has been obtained directly: the beyond-Hooke stress contribution $\hat{\sigma}$ itself plays this role.

6. The inertial Hooke model

Also still as a preparatory phase, let us now try to find the appropriate shift of specific internal energy for a doubly incomplete Kluitenberg–Verhás model – the inertial Hooke model –

$$\sigma = ED + \hat{\hat{E}} \ddot{D} \ : \tag{38}$$

now the beyond-Hooke stress contribution is[7]

$$\hat{\sigma} = \hat{\hat{E}} \ddot{D} \tag{39}$$

and, based on

$$\varrho \dot{e} = \sigma \dot{D} = ED\dot{D} + \hat{\sigma} \dot{D} = \left(\frac{E}{2} D^2 + \frac{\hat{\hat{E}}}{2} \dot{D}^2 \right)^{\cdot} , \tag{40}$$

having

$$e = e_{\mathrm{th}} + \frac{E}{2\varrho} D^2 + \frac{\hat{\hat{E}}}{2\varrho} \dot{D}^2 \tag{41}$$

[7]This rheological term describes an inertia-type effect coming possibly from a mesoscopic or microscopic level (for instance, micro-translations or micro-rotations), appearing macroscopically with a coefficient $\hat{\hat{E}}$. This is the effect that requires introducing the Verhás element (see Figure 1). Probably because of its conceptual novelty, this element is not yet a typical target for experimental or microscopic/mesoscopic justifications, although related ideas supporting this go back to [30].

gives

$$T\pi_S = \cdots = \varrho\dot{e}_{\mathrm{th}} = \varrho\left[\dot{e} - \left(\frac{E}{2\varrho}D^2 + \frac{\hat{\hat{E}}}{2\varrho}\dot{D}^2\right)^{\cdot}\right] = 0\,, \tag{42}$$

that is, the model is reversible and the temperature is constant.

For future benefit, it is worth rewriting (41) as

$$e = e_{\mathrm{th}} + \frac{E}{2\varrho}D^2 + \frac{1}{2\varrho\hat{\hat{E}}}\left(\hat{\hat{E}}L\right)^2 \tag{43}$$

– the benefit of this will turn out later.

Note that, in this case, not $\hat{\sigma}$ but L plays the role of the extra degree of freedom. The two cases seem to have nothing in common. This is a puzzling situation. (A synthesis will arrive, nevertheless, in Sect. 8.)

7. The relaxation-free Kluitenberg–Verhás model

Adding to the previous model (38) the Kelvin–Voigt viscous stress term results in the relaxation-free Kluitenberg–Verhás model

$$\sigma = ED + \hat{E}\dot{D} + \hat{\hat{E}}\ddot{D}\,, \tag{44}$$

in which case the same specific internal energy

$$e = e_{\mathrm{th}} + \frac{E}{2\varrho}D^2 + \frac{\hat{\hat{E}}}{2\varrho}\dot{D}^2, \qquad \text{or} \qquad e_{\mathrm{th}} + \frac{E}{2\varrho}D^2 + \frac{1}{2\varrho\hat{\hat{E}}}\left(\hat{\hat{E}}L\right)^2 \tag{45}$$

yields, again analogously,

$$T\pi_S = \cdots = \varrho\dot{e}_{\mathrm{th}} = \cdots = \hat{E}\dot{D}^2 \geq 0. \tag{46}$$

so both irreversibility and the time dependence of temperature agree with what we have seen for the Kelvin–Voigt model.

8. The full Kluitenberg–Verhás model

After our successes with all the special subfamilies, it may be surprising (and probably disappointing) that no similar direct construction is viable for the full Kluitenberg–Verhás model, that is, for

$$\sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D} + \hat{\hat{E}}\ddot{D} \tag{47}$$

with all coefficients being nonzero. Looking back from the end of this Section, this is no wonder: the full Kluitenberg–Verhás model carries a richness that is not present in any of the subfamilies. Instead of trying to find the thermodynamical model around (47), let us adapt the methodology seen in [8] but, here, use it for shifting specific internal energy rather than specific entropy.

Accordingly, let us now proceed in the opposite direction, where (47) is not an input but an output; our initial demand is only that

$$\hat{\sigma} \equiv \sigma - ED \tag{48}$$

(the rheological stress contribution) should be nonzero.

Let us shift the Hooke-level specific internal energy (16) by a term depending on a – yet uninterpreted – extra internal variable η. In order to maintain convexity (and thus concavity of s), the simplest choice to try is

$$e(T, D, \eta) = e_{\mathrm{th}}(T) + \frac{E}{2\varrho} D^2 + \frac{\kappa}{2\varrho} \eta^2 \tag{49}$$

with a non-negative κ that could in general be dependent on the state variables but here, aiming for a model practical for applications, we wish it to be constant.

As in the earlier cases, we do not modify specific entropy. In addition, let L still be equal to $\dot{D}$ [cf. (14)] (thus thermal expansion is still neglected, until Sect. 10). Accordingly, we can proceed analogously as before:

$$\begin{aligned} T\pi_S = \cdots = \varrho \dot{e}_{\mathrm{th}} = \varrho \left[\dot{e} - \left(\frac{E}{2\varrho} D^2 + \frac{\kappa}{2\varrho} \eta^2 \right)^{\cdot} \right] = \sigma L - ED\dot{D} - \kappa \eta \dot{\eta} = \\ = \hat{\sigma} L + EDL - ED\dot{D} - \kappa \eta \dot{\eta} = \hat{\sigma} L - \kappa \eta \dot{\eta} \,. \end{aligned} \tag{50}$$

The resulting expression is not *ab ovo* positive definite – let us ensure its positive definiteness choosing Onsagerian equations in our model, in the following form:

$$\hat{\sigma} = \mu_{11} L + \mu_{12} \kappa \dot{\eta} \,, \tag{51}$$

$$-\eta = \mu_{21} L + \mu_{22} \kappa \dot{\eta} \,, \tag{52}$$

and let us restrict ourselves to constant Onsagerian coefficients μ_{11}, μ_{12}, μ_{21}, μ_{22}. The rightmost-hand side of (50) is then positive definite if and only if the symmetric part of the 2×2 matrix μ (*i.e.,* μ^{S}) is positive definite, which is equivalent (see, *e.g.,* [8, 18]) to the set of conditions

$$\mu_{11} \geq 0 \,, \qquad \mu_{22} \geq 0 \,, \qquad \mu_{11}\mu_{22} - \left(\frac{\mu_{12} + \mu_{21}}{2} \right)^2 \geq 0 \,. \tag{53}$$

Let us eliminate the internal variable η: first, let us rewrite (52) into

$$-\left(1 + \mu_{22} \kappa \partial_t\right) \eta = \mu_{21} L \,, \tag{54}$$

let us act by the here-recognized differential operator $1 + \mu_{22}\kappa\partial_t$ on (51):

$$\begin{aligned} \hat{\sigma} + \underbrace{\mu_{22}\kappa}_{=:\tau} \dot{\hat{\sigma}} = \underbrace{\mu_{11}}_{=:\hat{I}} L + \mu_{11}\mu_{22}\kappa \dot{L} + \left(1 + \mu_{22}\kappa\partial_t\right) \mu_{12} \kappa \partial_t \eta = \\ = \hat{I} L + \kappa \mu_{11}\mu_{22} \dot{L} + \mu_{12}\kappa\partial_t \left(1 + \mu_{22}\kappa\partial_t\right) \eta = \\ = \hat{I} L + \kappa \mu_{11}\mu_{22} \dot{L} + \mu_{12}\kappa\partial_t \left[-\mu_{21} L\right] = \hat{I} L + \underbrace{\kappa \det \mu}_{=:\hat{\hat{E}}} \dot{L} \,, \end{aligned} \tag{55}$$

and substituting (48) (and applying $L = \dot{D}$ for combining two terms and rewriting a further term) the result is of the form (47):

$$\sigma - ED + \tau\left(\dot{\sigma} - E\dot{D}\right) = \hat{I}\dot{D} + \hat{\hat{E}}\ddot{D}\,, \tag{56}$$

$$\sigma + \tau\dot{\sigma} = ED + \big(\underbrace{\hat{I} + \tau E}_{=:\hat{E}}\big)\,\dot{D} + \hat{\hat{E}}\ddot{D}\,. \tag{57}$$

For the coefficients appearing here, we find in virtue of (53)

$$\hat{I} \geq 0\,, \qquad \tau \geq 0\,, \qquad \hat{E} \geq \tau E \tag{58}$$

and, as consequence of

$$0 \leq \mu_{11}\mu_{22} - \left(\frac{\mu_{12}+\mu_{21}}{2}\right)^2 = \mu_{11}\mu_{22} - \frac{\mu_{12}^2}{4} - \frac{\mu_{12}\mu_{21}}{2} - \frac{\mu_{21}^2}{4} =$$

$$= \det\mu - \left(\frac{\mu_{12}-\mu_{21}}{2}\right)^2 \leq \det\mu\,, \tag{59}$$

$$\hat{\hat{E}} \geq 0\,. \tag{60}$$

Inspecting equations (51)–(52), we can see that, for $\mu_{12} = 0$, (51) establishes a Kelvin–Voigt relationship between $\hat{\sigma}$ and L, while η is decoupled. We have seen previously that the Kelvin–Voigt case does not require a rheological extra energy term. (Even if there were an extra energy term, the Kelvin–Voigt mechanics would not make it change in time so it would be a "sleeping", irrelevant energy type.) Here, we are interested in the thermodynamically consistent background behind the beyond-Hooke stress contribution so let us now see what we can tell about the extra energy term for $\mu_{12} \neq 0$ (and, naturally, with $\kappa \neq 0$). We can start using (52), and then we can substitute (51):

$$\frac{\kappa}{2\varrho}\eta^2 = \frac{\kappa}{2\varrho}\left[-\big(\mu_{21}L + \mu_{22}\kappa\dot{\eta}\big)\right]^2 = \frac{\kappa}{2\varrho}\left[\mu_{21}L + \mu_{22}\kappa\frac{\hat{\sigma} - \mu_{11}L}{\mu_{12}\kappa}\right]^2 =$$

$$= \frac{\kappa}{2\varrho\mu_{12}^2\kappa^2}\left[\mu_{21}\mu_{12}\kappa L + \mu_{22}\kappa\hat{\sigma} - \mu_{11}\mu_{22}\kappa L\right]^2 = \frac{1}{2\varrho\mu_{12}^2\kappa}\left[\tau\hat{\sigma} - \hat{\hat{E}}L\right]^2 =$$

$$= \frac{\chi}{2\varrho}\Lambda^2\,, \qquad \text{where} \quad \chi = \frac{1}{\kappa\mu_{12}^2}\,, \quad \Lambda = \tau\hat{\sigma} - \hat{\hat{E}}L\,. \tag{61}$$

Hence, via rescaling of the internal variable (which is a legitimate step since κ was a free parameter and there was no convention fixing on η, either), it can be understood as a combination of the kinematically well-interpretable L, the mechanically well-interpretable $\hat{\sigma}$, and the (experimentally fittable) coefficients τ, $\hat{\hat{E}}$ of the Kluitenberg–Verhás model. Let us identify this rescaling explicitly: expressing $\dot{\eta}$ from (51), and substituting it into (52),

$$\eta = \frac{-1}{\kappa\mu_{12}}\Lambda\,. \tag{62}$$

Let us investigate the thermodynamical restrictions on χ: in terms of the *index of inertia*

$$\hat{\hat{I}} := \hat{\hat{E}} - \tau\hat{I} = \kappa \det \mu - \kappa\mu_{22}\mu_{11} = -\kappa\mu_{12}\mu_{21}, \tag{63}$$

we can deduce

$$0 \le \kappa \det \mu^{\mathsf{S}} = \kappa\mu_{11}\mu_{22} - \kappa\left(\frac{\mu_{12}+\mu_{21}}{2}\right)^2 = \tau\hat{I} - \kappa\frac{\mu_{12}\mu_{21}}{2} - \frac{\kappa}{4}\left(\mu_{12}^2+\mu_{21}^2\right), \tag{64}$$

$$\frac{\kappa}{4}\left(\mu_{12}^2+\mu_{21}^2\right) \le \tau\hat{I} + \frac{\hat{\hat{I}}}{2}, \qquad \frac{\kappa}{4}\left(\mu_{12}^2 + \frac{\hat{\hat{I}}^2}{\kappa^2\mu_{12}^2}\right) \le \tau\hat{I} + \frac{\hat{\hat{I}}}{2},$$
$$\kappa^2\mu_{12}^4 + \hat{\hat{I}}^2 \le 2\left(2\tau\hat{I} + \hat{\hat{I}}\right)\kappa\mu_{12}^2, \tag{65}$$

$$\left(\kappa\mu_{12}^2\right)^2 - 2\left(2\tau\hat{I} + \hat{\hat{I}}\right)\left(\kappa\mu_{12}^2\right) + \hat{\hat{I}} \le 0. \tag{66}$$

This is a second-order algebraic inequality, the solution of which is

$$\left(2\tau\hat{I} + \hat{\hat{I}}\right) - \sqrt{\left(2\tau\hat{I} + \hat{\hat{I}}\right)^2 - \hat{\hat{I}}^2} \le \kappa\mu_{12}^2 \le \left(2\tau\hat{I} + \hat{\hat{I}}\right) + \sqrt{\left(2\tau\hat{I} + \hat{\hat{I}}\right)^2 - \hat{\hat{I}}^2}. \tag{67}$$

This, replacing $\hat{\hat{I}}$ with $\hat{\hat{E}} - \tau\hat{I}$, can be rewritten after simple algebraic steps into the following informative form:

$$\left(\sqrt{\tau\hat{I}} - \sqrt{\hat{\hat{E}}}\right)^2 \le \kappa\mu_{12}^2 \le \left(\sqrt{\tau\hat{I}} + \sqrt{\hat{\hat{E}}}\right)^2. \tag{68}$$

These conditions become irrelevant if $\kappa = 0$ or $\mu_{12} = 0$. If neither of them is zero, then

$$\frac{1}{\left(\sqrt{\tau\hat{I}} + \sqrt{\hat{\hat{E}}}\right)^2} \le \chi \le \frac{1}{\left(\sqrt{\tau\hat{I}} - \sqrt{\hat{\hat{E}}}\right)^2} \tag{69}$$

emerges from the second law-imposed inequalities. [τ, $\hat{I}$, $\hat{\hat{E}}$ are non-negative so the lower bound here is always well-defined. In parallel, for $\hat{\hat{E}} = \tau\hat{I}$ (*i.e.,* for $\hat{\hat{I}} = 0$), the upper bound is ∞, with $<$ instead of $\le$ before it.]

If at least one of τ, $\hat{I}$, or $\hat{\hat{E}}$ is zero then the lower and upper bounds are equal, consequently, χ is uniquely determined. It is easy to check that then we happen to reproduce the extra energy term obtained directly in Sects. 5–7. On the other side, in the case of a full Kluitenberg–Verhás model, χ moves freely within an interval. Its value can be experimentally fitted only from caloric information – most easily from measuring temperature. Namely, the model predicts the time derivative of temperature as follows: similarly as above, we use (50), replacing η with its form (62), and substituting $\dot{\eta}$ expressed from (61), finding

$$\varrho\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = \varrho\dot{e}_{\mathrm{th}} = \hat{\sigma}L - \kappa\eta\dot{\eta} = \hat{\sigma}L + \chi\left(\hat{\sigma} - \hat{I}L\right)\Lambda, \tag{70}$$

and this contains χ and kinematically and mechanically accessible quantities, consequently, χ can be determined from experimental data.

9. Comparison of the presented rheological models

Before proceeding, it is instructive to visualize the relationships among the rheological models introduced so far, and to compare the typical behaviours of these models. Figure 2 displays the relationships.

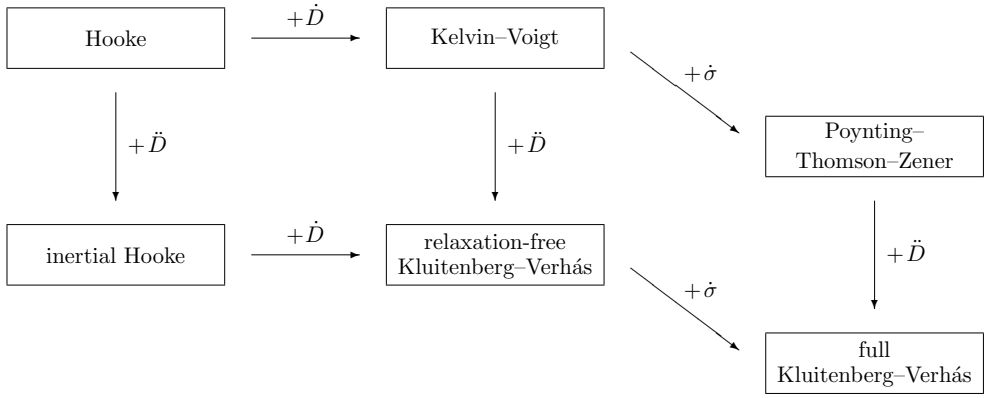

Figure 2. From Hooke to Kluitenberg–Verhás: relationships among the presented rheological models.

Next, Figure 3 shows the different time dependencies $D(t)$ in different rheological models for the same prescribed stress history. Some remarkable observations are as follows.

- As expected, the Hooke model ($\sigma = ED$) mimics what happens to stress.
- The inertial Hooke model $\left(\sigma = ED + \hat{\hat{E}}\ddot{D}\right)$ oscillation around the Hooke prediction.
- The Poynting–Thomson–Zener model $\left(\sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D}\right)$ also presents a corner when stress has a corner, which can be illustrated as follows: for fast changes, among the σ terms only the highest time derivative term is relevant, and among the D terms only the highest time derivative term is relevant, hence, $\tau\dot{\sigma} \approx \hat{E}\dot{D}$, and integrating this along a small enough time interval gives $\tau\Delta\sigma \approx \hat{E}\Delta D$, $\Delta\sigma/\Delta D \approx \hat{E}/\tau$ (dynamic Young's modulus), meaning a Hooke-like fast response. Finally, for corners, this reasoning can be generalized from functions to distributions. (Note that a sharp corner can be conceived as a limit of a sequence of smooth turns, sharper-and-sharper ones.)

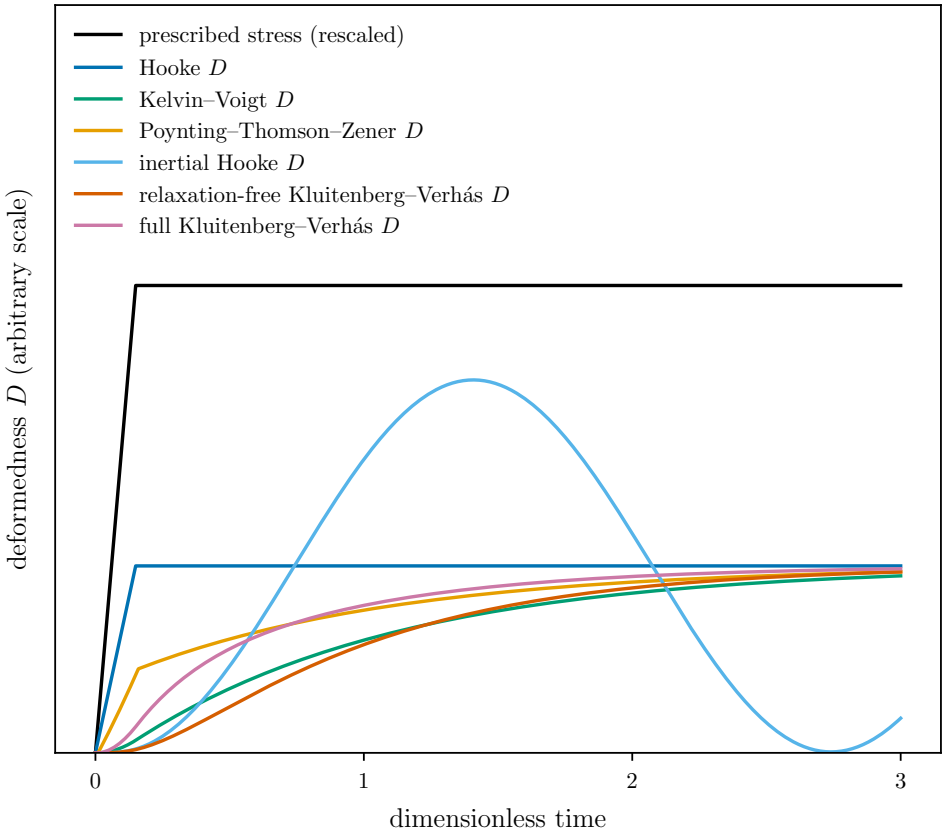

Figure 3. Behaviour of the Kluitenberg–Verhás model and its special subfamilies for $\tau = 0.4t_{\text{unit}}$ and $\hat{\hat{E}} = 0.8t_{\text{unit}}^2 E$, using the time scale $\hat{E}/E$ as the time unit t_{unit} (the vertical scale being arbitrary since all these models are linear), when stress is increased linearly in time until instant $0.15t_{\text{unit}}$ and then it is kept constant.

- The Poynting–Thomson–Zener model $\left(\sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D}\right)$ also presents a corner when stress has a corner, which can be illustrated as follows: for fast changes, among the σ terms only the highest time derivative term is relevant, and among the D terms only the highest time derivative term is relevant, hence, $\tau\dot{\sigma} \approx \hat{E}\dot{D}$, and integrating this along a small enough time interval gives $\tau\Delta\sigma \approx \hat{E}\Delta D$, $\Delta\sigma/\Delta D \approx \hat{E}/\tau$ (dynamic Young's modulus), meaning a Hooke-like fast response. Finally, for corners, this reasoning can be generalized from functions to distributions. (Note that

a sharp corner can be conceived as a limit of a sequence of smooth turns, sharper-and-sharper ones.)

- The rest of the considered rheological models [Kelvin–Voigt $\left(\sigma = ED + \hat{E}\dot{D}\right)$, relaxation-free Kluitenberg–Verhás $\left(\sigma = ED + \hat{E}\dot{D} + \hat{\hat{E}}\ddot{D}\right)$, full Kluitenberg–Verhás $\left(\sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D} + \hat{\hat{E}}\ddot{D}\right)$] have a higher order time derivative of D than that of σ so these further models react more smoothly to an abrupt stress change.
- Except for Hooke and inertial Hooke, all models tend irreversibly to the stationary Hooke prediction.

10. Thermal expansion: the initial model

In the presence of thermal expansion, L is no longer equal to $\dot{D}$: the kinematic changes are not only elastic deviation type but thermal expansion type as well. Closer (and tensorial) formulation is provided in [18, 22–27], here, only the small-deformation (and first only "one-dimensional" – one kinematic degree of freedom) formula

$$L = \dot{D} + \alpha\dot{T} \tag{71}$$

is needed, in which the linear thermal expansion coefficient α will be considered constant, for better readability and ease in application. The works [18, 22–27] also provide the consistent thermomechanical background of elastic plus thermal-expansion phenomena. This Hooke plus thermal-expansion model will be the initial model – the starting point – for deriving the Kluitenberg–Verhás plus thermal expansion model. Stress remains unmodified as a function of D,

$$\sigma = ED \tag{72}$$

(as the cited works show, if $D_{\text{initial}} = 0$ then replacing D with ε yields the classic Duhamel–Neumann formula). On the other side, e and s become generalized:

$$e(T, D) = e_{\text{th}}(T) + \frac{\alpha E}{\varrho} TD + \frac{E}{2\varrho} D^2 , \tag{73}$$

$$s(T, D) = s_{\text{th}}(T) + \frac{\alpha E}{\varrho} D . \tag{74}$$

This initial model is reversible and nondissipative:

$$\begin{aligned} T\pi_S = \varrho T\dot{s} = \varrho T \frac{\mathrm{d}s_{\text{th}}}{\mathrm{d}T}\dot{T} + \varrho T \frac{\alpha E}{\varrho}\dot{D} = \varrho \frac{\mathrm{d}e_{\text{th}}}{\mathrm{d}T}\dot{T} + \varrho T \frac{\alpha E}{\varrho}\dot{D} = \\ = \varrho\dot{e}_{\text{th}} + T \cdot \alpha E\dot{D} = \varrho\left(e - \frac{\alpha E}{\varrho} TD - \frac{E}{2\varrho} D^2\right)^{\cdot} + \alpha ET\dot{D} = \\ = \sigma L - \alpha E\left(\dot{T}D + T\dot{D}\right) - ED\dot{D} + \alpha ET\dot{D} = \\ = ED\left(\dot{D} + \alpha\dot{T}\right) - \alpha E\dot{T}D - \alpha ET\dot{D} - ED\dot{D} + \alpha ET\dot{D} = 0 . \end{aligned} \tag{75}$$

As a consequence, temperature is no longer time-independent:

$$0 = T\pi_S = \cdots = \varrho \frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} + \varrho T \frac{\alpha E}{\varrho}\dot{D} \qquad \Longrightarrow \qquad \frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = -T\frac{\alpha E}{\varrho}\dot{D}. \tag{76}$$

11. Extension of the Kluitenberg–Verhás model to include thermal expansion

We assume again an extra stress contribution

$$\hat{\sigma} = \sigma - ED\,, \tag{77}$$

and we shift e by a term as before:

$$e(T, D, \eta) = e_{\mathrm{th}}(T) + \frac{\alpha E}{\varrho}TD + \frac{E}{2\varrho}D^2 + \frac{\kappa}{2\varrho}\eta^2\,, \tag{78}$$

while s is kept untouched. (75) becomes extended due to the mechanical power contribution $\hat{\sigma}L$ and via the time derivative of the extra (rheological) specific energy term:

$$T\pi_S = \cdots = \hat{\sigma}L - \kappa\eta\dot{\eta}\,. \tag{79}$$

Starting from here, formulae (51)–(55) and (58)–(69) all remain valid. (Not by coincidence: as a precaution, they have been formulated in such a way that they are not influenced by the appearance of thermal expansion.) The relationship $L = \dot{D}$ was used in (56)–(57), their generalization for thermal expansion is

$$\sigma - ED + \tau\left(\dot{\sigma} - E\dot{D}\right) = \hat{I}L + \hat{\hat{E}}\dot{L}\,, \tag{80}$$

$$\sigma + \tau\dot{\sigma} = ED + \tau E\dot{D} + \hat{I}\left(\dot{D} + \alpha\dot{T}\right) + \hat{\hat{E}}\left(\ddot{D} + \alpha\ddot{T}\right)\,, \tag{81}$$

$$\sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D} + \hat{\hat{E}}\ddot{D} + \hat{I}\alpha\dot{T} + \hat{\hat{E}}\alpha\ddot{T}\,. \tag{82}$$

In addition, for expressing the time derivative of temperature, (70) is to be generalized: (76) remains valid to the extent that

$$T\pi_S = \cdots = \varrho \frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} + \varrho T \frac{\alpha E}{\varrho}\dot{D}\,. \tag{83}$$

In parallel, (79) is also rewritable as in (70):

$$T\pi_S = \hat{\sigma}L - \kappa\eta\dot{\eta} = \hat{\sigma}L + \chi\left(\hat{\sigma} - \hat{I}L\right)\Lambda\,, \tag{84}$$

therefore,

$$\varrho \frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T} = -\varrho T \frac{\alpha E}{\varrho}\dot{D} + \hat{\sigma}L + \chi\left(\hat{\sigma} - \hat{I}L\right)\Lambda\,. \tag{85}$$

12. The three-dimensional tensorial variant

In the three-dimensional (3D) treatment, we restrict the discussion to isotropic materials. Also, we remain in the small-deformation regime.[8] $\boldsymbol{\sigma}$ and $\mathbf{D}$ are symmetric tensors so the internal variable $\boldsymbol{\eta}$ with a mechanical role is also chosen to be a symmetric tensor. Knowledge of the decomposition of symmetric tensors to deviatoric ($^{\mathrm{dev}}$) and spherical ($^{\mathrm{sph}}$) parts and the corresponding basic properties (*e.g.,* that the trace of the product of a deviatoric tensor and a spherical tensor is zero) will be assumed (see, *e.g.,* [18], and [8] is also useful for the aspects in the present context). Because of this decoupling of deviatoric and symmeric parts, the deviatoric part can be viewed as a model with one kinematic degree of freedom ("1D") (a)nd free of thermal expansion because thermal expansion is also isotropic, while the spherical part is also effectively 1D (but contains thermal expansion).[9]

The 3D Kluitenberg–Verhás model extended to include thermal expansion is as follows. Kinematically, the relationship among the kinematically involved quantities is

$$\mathbf{L}^{\mathsf{S}} = \dot{\mathbf{D}} + \alpha \dot{T} \mathbf{1} \tag{86}$$

[18, 22–27]. The deviation $\hat{\boldsymbol{\sigma}}$ of stress from the Hooke–Duhamel–Neumann elastic one is

$$\hat{\boldsymbol{\sigma}}^{\mathrm{dev}} = \boldsymbol{\sigma}^{\mathrm{dev}} - E^{\mathrm{dev}} \mathbf{D}^{\mathrm{dev}}, \qquad \hat{\boldsymbol{\sigma}}^{\mathrm{sph}} = \boldsymbol{\sigma}^{\mathrm{sph}} - E^{\mathrm{sph}} \mathbf{D}^{\mathrm{sph}}. \tag{87}$$

Specific internal energy is

$$e(T, \mathbf{D}, \boldsymbol{\eta}) = e_{\mathrm{th}}(T) + \frac{\alpha E^{\mathrm{sph}}}{\varrho} T \operatorname{tr} \mathbf{D} + \frac{E^{\mathrm{dev}}}{2\varrho} \operatorname{tr}\left(\mathbf{D}^{\mathrm{dev}} \mathbf{D}^{\mathrm{dev}}\right) + \frac{E^{\mathrm{sph}}}{2\varrho} \operatorname{tr}\left(\mathbf{D}^{\mathrm{sph}} \mathbf{D}^{\mathrm{sph}}\right) + \\ + \frac{\kappa^{\mathrm{dev}}}{2\varrho} \operatorname{tr}\left(\boldsymbol{\eta}^{\mathrm{dev}} \boldsymbol{\eta}^{\mathrm{dev}}\right) + \frac{\kappa^{\mathrm{sph}}}{2\varrho} \operatorname{tr}\left(\boldsymbol{\eta}^{\mathrm{sph}} \boldsymbol{\eta}^{\mathrm{sph}}\right), \tag{88}$$

and specific entropy is

$$s(T, \mathbf{D}) = s_{\mathrm{th}}(T) + \frac{\alpha E^{\mathrm{sph}}}{\varrho} \operatorname{tr} \mathbf{D}, \tag{89}$$

where (12) is still valid. Analogously to (75),

$$T \pi_S = \dots = \tag{90}$$
$$= \operatorname{tr}\left(\hat{\boldsymbol{\sigma}}^{\mathrm{dev}} \mathbf{L}^{\mathsf{S}^{\mathrm{dev}}}\right) - \kappa^{\mathrm{dev}} \operatorname{tr}\left(\boldsymbol{\eta}^{\mathrm{dev}} \dot{\boldsymbol{\eta}}^{\mathrm{dev}}\right) + \operatorname{tr}\left(\hat{\boldsymbol{\sigma}}^{\mathrm{sph}} \mathbf{L}^{\mathsf{S}^{\mathrm{sph}}}\right) - \kappa^{\mathrm{sph}} \operatorname{tr}\left(\boldsymbol{\eta}^{\mathrm{sph}} \dot{\boldsymbol{\eta}}^{\mathrm{sph}}\right),$$

[8] This enables us to have a linear and mathematically easily treatable framework, which is nevertheless suitable for a broad range of applications. Extension to arbitrary deformations is already a work in progress.

[9] Let us emphasize here that the forthcoming decoupling of the *model* into deviatoric and spherical submodels [(91)–(92)] is a consequence of isotropy, and does not hold for anisotropic materials in general.

the deviatoric and spherical parts are Onsagerically decomposed (due to the isotropy of the material), and $T\pi_S \geq 0$ is ensured by

$$\hat{\boldsymbol{\sigma}}^{\mathrm{dev}} = \mu_{11}^{\mathrm{dev}} \mathbf{L}^{\mathsf{S}^{\mathrm{dev}}} + \mu_{12}^{\mathrm{dev}} \kappa^{\mathrm{dev}} \dot{\boldsymbol{\eta}}^{\mathrm{dev}}, \qquad \hat{\boldsymbol{\sigma}}^{\mathrm{sph}} = \mu_{11}^{\mathrm{sph}} \mathbf{L}^{\mathsf{S}^{\mathrm{sph}}} + \mu_{12}^{\mathrm{sph}} \kappa^{\mathrm{sph}} \dot{\boldsymbol{\eta}}^{\mathrm{sph}}, \tag{91}$$

$$-\boldsymbol{\eta}^{\mathrm{dev}} = \mu_{21}^{\mathrm{dev}} \mathbf{L}^{\mathsf{S}^{\mathrm{dev}}} + \mu_{22}^{\mathrm{dev}} \kappa^{\mathrm{dev}} \dot{\boldsymbol{\eta}}^{\mathrm{dev}}, \qquad -\boldsymbol{\eta}^{\mathrm{sph}} = \mu_{21}^{\mathrm{sph}} \mathbf{L}^{\mathsf{S}^{\mathrm{sph}}} + \mu_{22}^{\mathrm{sph}} \kappa^{\mathrm{sph}} \dot{\boldsymbol{\eta}}^{\mathrm{sph}}, \tag{92}$$

where the coefficients μ_{ij}^{dev}, μ_{ij}^{sph} are constants and fulfil the inequalities

$$\mu_{11}^{\mathrm{dev}} \geq 0, \qquad \mu_{22}^{\mathrm{dev}} \geq 0, \qquad \mu_{11}^{\mathrm{dev}} \mu_{22}^{\mathrm{dev}} - \left(\frac{\mu_{12}^{\mathrm{dev}} + \mu_{21}^{\mathrm{dev}}}{2} \right)^2 \geq 0, \tag{93}$$

$$\mu_{11}^{\mathrm{sph}} \geq 0, \qquad \mu_{22}^{\mathrm{sph}} \geq 0, \qquad \mu_{11}^{\mathrm{sph}} \mu_{22}^{\mathrm{sph}} - \left(\frac{\mu_{12}^{\mathrm{sph}} + \mu_{21}^{\mathrm{sph}}}{2} \right)^2 \geq 0. \tag{94}$$

From these Onsagerian coefficients, we define the derived coefficients

$$\tau^{\mathrm{dev}} := \kappa^{\mathrm{dev}} \mu_{22}^{\mathrm{dev}} \geq 0, \quad \hat{I}^{\mathrm{dev}} := \mu_{11}^{\mathrm{dev}} \geq 0, \quad \hat{E}^{\mathrm{dev}} := \hat{I}^{\mathrm{dev}} + \tau^{\mathrm{dev}} E^{\mathrm{dev}} \geq \tau^{\mathrm{dev}} E^{\mathrm{dev}}, \tag{95}$$

$$\tau^{\mathrm{sph}} := \kappa^{\mathrm{sph}} \mu_{22}^{\mathrm{sph}} \geq 0, \quad \hat{I}^{\mathrm{sph}} := \mu_{11}^{\mathrm{sph}} \geq 0, \quad \hat{E}^{\mathrm{sph}} := \hat{I}^{\mathrm{sph}} + \tau^{\mathrm{sph}} E^{\mathrm{sph}} \geq \tau^{\mathrm{sph}} E^{\mathrm{sph}}, \tag{96}$$

$$\hat{\hat{E}}^{\mathrm{dev}} := \kappa^{\mathrm{dev}} \det \mu^{\mathrm{dev}} \geq 0, \qquad \hat{\hat{I}}^{\mathrm{dev}} := \hat{\hat{E}}^{\mathrm{dev}} - \tau^{\mathrm{dev}} \hat{I}^{\mathrm{dev}}, \tag{97}$$

$$\hat{\hat{E}}^{\mathrm{sph}} := \kappa^{\mathrm{sph}} \det \mu^{\mathrm{sph}} \geq 0, \qquad \hat{\hat{I}}^{\mathrm{sph}} := \hat{\hat{E}}^{\mathrm{sph}} - \tau^{\mathrm{sph}} \hat{I}^{\mathrm{sph}}. \tag{98}$$

These coefficients appear when we remove the internal variable, obtaining

$$\boldsymbol{\sigma}^{\mathrm{dev}} + \tau^{\mathrm{dev}} \dot{\boldsymbol{\sigma}}^{\mathrm{dev}} = E^{\mathrm{dev}} \mathbf{D}^{\mathrm{dev}} + \hat{E}^{\mathrm{dev}} \dot{\mathbf{D}}^{\mathrm{dev}} + \hat{\hat{E}}^{\mathrm{dev}} \ddot{\mathbf{D}}^{\mathrm{dev}}, \tag{99}$$

$$\boldsymbol{\sigma}^{\mathrm{sph}} + \tau^{\mathrm{sph}} \dot{\boldsymbol{\sigma}}^{\mathrm{sph}} = E^{\mathrm{sph}} \mathbf{D}^{\mathrm{sph}} + \hat{E}^{\mathrm{sph}} \dot{\mathbf{D}}^{\mathrm{sph}} + \hat{\hat{E}}^{\mathrm{sph}} \ddot{\mathbf{D}}^{\mathrm{sph}} + \hat{I}^{\mathrm{sph}} \alpha \dot{T} \mathbf{1} + \hat{\hat{E}}^{\mathrm{sph}} \alpha \ddot{T} \mathbf{1}. \tag{100}$$

In terms of the combinations

$$\boldsymbol{\Lambda}^{\mathrm{dev}} = \tau^{\mathrm{dev}} \hat{\boldsymbol{\sigma}}^{\mathrm{dev}} - \hat{\hat{E}}^{\mathrm{dev}} \mathbf{L}^{\mathsf{S}^{\mathrm{dev}}}, \qquad \boldsymbol{\Lambda}^{\mathrm{sph}} = \tau^{\mathrm{sph}} \hat{\boldsymbol{\sigma}}^{\mathrm{sph}} - \hat{\hat{E}}^{\mathrm{sph}} \mathbf{L}^{\mathsf{S}^{\mathrm{sph}}}, \tag{101}$$

the extra/rheological energy terms are nonzero if at least one of τ^{dev}, $\hat{I}^{\mathrm{dev}}$, $\hat{\hat{E}}^{\mathrm{dev}}$ is nonzero and if at least one of τ^{sph}, $\hat{I}^{\mathrm{sph}}$, $\hat{\hat{E}}^{\mathrm{sph}}$ is nonzero, respectively, and then these extra terms are

$$\frac{\chi^{\mathrm{dev}}}{2\varrho} \operatorname{tr}\left(\boldsymbol{\Lambda}^{\mathrm{dev}} \boldsymbol{\Lambda}^{\mathrm{dev}} \right) \qquad \text{resp.} \qquad \frac{\chi^{\mathrm{sph}}}{2\varrho} \operatorname{tr}\left(\boldsymbol{\Lambda}^{\mathrm{sph}} \boldsymbol{\Lambda}^{\mathrm{sph}} \right), \tag{102}$$

where

$$\frac{1}{\left(\sqrt{\tau^{\mathrm{dev}}\hat{I}^{\mathrm{dev}}}+\sqrt{\hat{\hat{E}}^{\mathrm{dev}}}\right)^2}\leq\chi^{\mathrm{dev}}\leq\frac{1}{\left(\sqrt{\tau^{\mathrm{dev}}\hat{I}^{\mathrm{dev}}}-\sqrt{\hat{\hat{E}}^{\mathrm{dev}}}\right)^2}\,, \tag{103}$$

$$\frac{1}{\left(\sqrt{\tau^{\mathrm{sph}}\hat{I}^{\mathrm{sph}}}+\sqrt{\hat{\hat{E}}^{\mathrm{sph}}}\right)^2}\leq\chi^{\mathrm{sph}}\leq\frac{1}{\left(\sqrt{\tau^{\mathrm{sph}}\hat{I}^{\mathrm{sph}}}-\sqrt{\hat{\hat{E}}^{\mathrm{sph}}}\right)^2}\,. \tag{104}$$

The equation expressing the time evolution of temperature is

$$\varrho\frac{\mathrm{d}e_{\mathrm{th}}}{\mathrm{d}T}\dot{T}=-\varrho T\frac{\alpha E^{\mathrm{sph}}}{\varrho}\,\mathrm{tr}\,\dot{\mathbf{D}}+\mathrm{tr}\left(\hat{\boldsymbol{\sigma}}^{\mathrm{dev}}\mathbf{L}^{\mathsf{S}^{\mathrm{dev}}}\right)+\chi^{\mathrm{dev}}\,\mathrm{tr}\left[\left(\hat{\boldsymbol{\sigma}}^{\mathrm{dev}}-\hat{I}^{\mathrm{dev}}\mathbf{L}^{\mathsf{S}^{\mathrm{dev}}}\right)\boldsymbol{\Lambda}^{\mathrm{dev}}\right]+$$
$$+\,\mathrm{tr}\left(\hat{\boldsymbol{\sigma}}^{\mathrm{sph}}\mathbf{L}^{\mathsf{S}^{\mathrm{sph}}}\right)+\chi^{\mathrm{sph}}\,\mathrm{tr}\left[\left(\hat{\boldsymbol{\sigma}}^{\mathrm{sph}}-\hat{I}^{\mathrm{sph}}\mathbf{L}^{\mathsf{S}^{\mathrm{sph}}}\right)\boldsymbol{\Lambda}^{\mathrm{sph}}\right]. \tag{105}$$

13. Heat conduction

For an isotropic material, heat conduction can be incorporated in a straightforward way in the continuum description: Fourier's law

$$\mathbf{j}_E=-\lambda\nabla T \tag{106}$$

is to be added to the set of equations, and the corresponding energy balance (2) is to be considered (from which the time derivative of temperature can be obtained). No complication can arise since Onsagerian coupling is not possible between the tensorial mechanical quantities and the vectorial heat current density [7, 19].

For the special subfamily of Poynting–Thomson–Zener standard rheology coupled to heat conduction via thermal expansion, numerical solutions in one space dimension have been presented in [31]. So far, no corresponding result exists (at least to the author's knowledge) for the Kluitenberg–Verhás generalization.

14. Conclusion

Analyzing the rheology-related additional energy term has provided valuable additional insight into the Kluitenberg–Verhás model and its internal variable, and increased its applicability. The present approach opens the way for generalizing analytical solutions like [6, 32] to include the rheological internal variable as well, and for generalizing numerical solutions of Poynting–Thomson–Zener standard rheology coupled to heat conduction via thermal expansion [31] to the full Kluitenberg–Verhás level.

Generalization of the treatment presented here to *fluids* with any thermodynamically allowed nonlinear constitutive characteristics appears feasible, and is expected to yield a practical non-Newtonian model family {a Jeffreys model for the rheological part of the stress (or pressure) tensor [7]}. In addition, this generalization could be the first step towards the corresponding model family for *solids* with *arbitrary* thermodynamically allowed nonlinear consitutive characteristics *without* the small-deformation

approximation. Hopefully, together with the spacetime-compatible finite-deformation kinematics of solids [22–27], this amalgamation can be done in the near future.

Appendix A. Kluitenberg–Verhás model coefficients fitted on experimental data for some plastic and rock materials

coefficient	value
$\tau^{\mathrm{dev}}/\mathrm{s}$	0.3600
$E^{\mathrm{dev}}/\mathrm{GPa}$	0.8612
$\hat{E}^{\mathrm{dev}}/(\mathrm{GPa}\cdot\mathrm{s})$	0.4724
$\hat{\hat{E}}^{\mathrm{dev}}/\left(\mathrm{GPa}\cdot\mathrm{s}^2\right)$	0.0029
$\tau^{\mathrm{sph}}/\mathrm{s}$	0.2329
$E^{\mathrm{sph}}/\mathrm{GPa}$	4.5708
$\hat{E}^{\mathrm{sph}}/(\mathrm{GPa}\cdot\mathrm{s})$	1.8566
$\hat{\hat{E}}^{\mathrm{sph}}/\left(\mathrm{GPa}\cdot\mathrm{s}^2\right)$	0.0013

Table 1. Fitted rheological coefficients for experimental data obtained on a polyamide-6 sample. Results taken from [5].

coefficient	Granite	Marble	Sandstone
$\tau^{\mathrm{dev}}/\mathrm{h}$	13.15	13.24	16.00
$E^{\mathrm{dev}}/\mathrm{MPa}$	0.82	0.61	0.23
$\hat{E}^{\mathrm{dev}}/(\mathrm{MPa}\cdot\mathrm{h})$	16.46	13.82	6.78
$\hat{\hat{E}}^{\mathrm{dev}}/\left(\mathrm{MPa}\cdot\mathrm{h}^2\right)$	6.38	3.42	10.96
$\tau^{\mathrm{sph}}/\mathrm{h}$	13.97	6.42	2.07
$E^{\mathrm{sph}}/\mathrm{MPa}$	1.65	2.35	0.46
$\hat{E}^{\mathrm{sph}}/(\mathrm{MPa}\cdot\mathrm{h})$	28.44	18.63	5.58
$\hat{\hat{E}}^{\mathrm{sph}}/\left(\mathrm{MPa}\cdot\mathrm{h}^2\right)$	7.66	3.32	2.73

Table 2. Determined rheological coefficients from experimental data obtained on three different types of rock, in 48-hour measurements. Results derived from [4], using the fact that their interpretation in terms of the rheological circuit $\left(E_1 \,||\, \hat{E}_1\right) \sim \left(E_2 \,||\, \hat{E}_2\right)$ ($^{\mathrm{dev}}$ or $^{\mathrm{sph}}$, respectively; $||$ denoting parallel connection and $\sim$ standing for serial connection) is equivalent to $\sigma + \tau\dot{\sigma} = ED + \hat{E}\dot{D} + \hat{\hat{E}}\ddot{D}$ with $\tau = \frac{\hat{E}_1+\hat{E}_2}{E_1+E_2}$, $E = \frac{E_1E_2}{E_1+E_2}$, $\hat{E} = \frac{E_1\hat{E}_2+E_2\hat{E}_1}{E_1+E_2}$, $\hat{\hat{E}} = \frac{\hat{E}_1\hat{E}_2}{E_1+E_2}$ [as is straightforward to find; see, *e.g.,* [11], page 103, Eq. (85)].

In each case (Table 1, Table 2), we can find that $\hat{I} > 0$, fulfilling the second law of thermodynamics. In addition, the $\hat{\hat{I}}$ values turn out to be negative (overdamped case).

For a given material type, the values of these coefficients – and especially the rheological time scales identified from them – provide insight into when the various beyond-Hooke terms in the Kluitenberg–Verhás model play a remarkable role or when their contribution can be neglected. In view of the fact that the rheological time scales can be seconds (see Table 1), hours (as in Table 2), or years [33], it is a good strategy to always be prepared for rheological effects.

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