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GOVERNING EQUATIONS AND FUNDAMENTAL SOLUTIONS FOR AN ISOTROPIC MICROPOLAR THERMOELASTIC MEDIUM WITH TRIPLE POROSITY AND MICROTEMPERATURES

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Abstract. This paper aims to establish the fundamental governing equations and develop corresponding fundamental solutions for a medium exhibiting isotropic micropolar thermoelastic properties, encompassing triple porosity and microtemperatures. By considering the unique characteristics of the material, we formulate a comprehensive framework to describe its behavior under various conditions, including steady-state, pseudo-static, quasi-static oscillations, and equilibrium.

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1. Introduction

The micropolar theory of elasticity provides a valuable conceptual framework for understanding the microstructural motion of diverse materials, including polycrystalline materials and those with fibrous structures. By incorporating both displacement and microrotation vectors, this theory offers a comprehensive description of solid motion. Building upon this foundation, several researchers (see [1–6]) have extended the micropolar theory to incorporate thermal effects, leading to the development of the micropolar theory of thermoelasticity. This expanded theory introduces a coupling between mechanical deformation and temperature changes in materials, enabling a deeper understanding of their behavior.

The three levels of porosity in triple porosity can be categorized based on size or connectivity. Primary porosity refers to the macroscopic or larger-scale voids within the material, such as large pores or fractures. Secondary porosity corresponds to intermediate-scale voids, which may include smaller fractures or microfractures. Tertiary porosity represents the smallest-scale voids, often at the microscale, such as pores within the grains or microcracks.

Each level of porosity contributes to the overall permeability and fluid flow characteristics of the material. The presence of multiple levels of porosity can significantly affect the behavior of porous media, including their mechanical, thermal, and transport properties. Triple porosity models are commonly employed in various fields such as geomechanics, hydrogeology, petroleum engineering, and material science, where accurate characterization of porous materials is essential for understanding their performance and behavior.

The works of Svanadze [7], Straughan [8] and Kansal [9] have contributed to the establishment of governing equations in the field of elasticity and thermoelasticity, specifically addressing the concept of triple porosity. Svanadze [10–15] further investigated boundary value problems related to elastic solids and thermoelastic solids with triple porosity. The motivation behind developing theories for multiple porosity elasticity lies in the wide and expanding range of applications that require such frameworks. Straughan [16] discussed various applications of multiple porosity in his book, shedding light on the practical significance of these theories across different disciplines.

Grot [17] extended the theory of thermodynamics of elastic bodies with microstructure by considering microelements with different temperatures. He modified the Clausius-Duhem inequality to incorporate microtemperatures and introduced first-order moment of energy equations to the basic balance laws. These modifications aimed to determine the microtemperatures of a continuum. Iesan and Quintanilla [18] constructed a linear theory for elastic materials with an inner structure that includes microtemperatures in addition to the classical displacement and temperature fields. They focused on the existence of solutions for initial boundary value problems using semigroup theory. They proved an existence theorem and established the continuous dependence of solutions on the initial data and body loads. Iesan [19] established the field equations of a theory of microstretch thermoelastic bodies with microtemperatures. He specifically worked on the dynamic theory of anisotropic materials and proved a uniqueness theorem. This theorem demonstrates the uniqueness of solutions within the dynamic theory of anisotropic materials, considering the presence of microtemperatures. Iesan [20] also derived a linear theory of microstretch elastic solids with microtemperatures, where each microelement of a continuum possesses mechanical degrees of freedom for rigid rotations and microdilatation, in addition to the classical translation degrees of freedom. Furthermore, he established a uniqueness result within the dynamical theory of anisotropic bodies, providing insights into the uniqueness of solutions in this context.

Indeed, fundamental solutions play a crucial role in solving boundary value problems, particularly in the context of elasticity and thermoelasticity. The use of fundamental solutions allows for an integral representation of the solution to a boundary value problem, which is often more suitable to investigation by numerical methods than directly solving a differential equation with specified boundary and initial conditions. In the investigation of boundary value problems within the theories of elasticity and thermoelasticity using the potential method, it becomes necessary to construct fundamental solutions for the corresponding systems of partial differential equations and establish their basic properties. These fundamental solutions serve as building blocks for solving more complex problems and provide insight into the behavior of the underlying physical phenomena. Kansal [9, 21] and Svanadze [7, 22] have made contributions in this regard by constructing fundamental solutions using elementary functions in the theory of elasticity and thermoelasticity with triple porosity. By utilizing elementary functions, the solutions become more accessible to analysis and numerical techniques, facilitating the study and solution of boundary value problems within these theories.

This paper presents a comprehensive analysis of anisotropic generalized micropolar thermoelasticity with triple porosity and microtemperatures. The focus is on deriving the constitutive relations and field equations for this complex material system. Section 2 outlines the derivation of the constitutive equations and simplification of the anisotropic system into an isotropic system, facilitating analysis and interpretation. Sections 3 and 4 present the construction of the fundamental solution for steady oscillations, expressed in terms of elementary functions for ease of analysis. Section 5 extends the analysis to pseudo-static, quasi-static oscillations, and equilibrium, providing a comprehensive understanding of the material's behavior under different conditions. Section 6 establishes fundamental properties of the fundamental matrix to enhance understanding of the solutions obtained and their implications.

2. Basic Equations

Following [1–5], the equations of motions of linear micropolar elasticity are

$$\sigma_{ji,j} + \rho\tilde{F}_i = \rho\ddot{u}_i, \tag{2.1}$$

$$\varepsilon_{ijq}\sigma_{jq} + \mu_{ji,j} + \rho\tilde{G}_i = \rho J_{ij}\ddot{\varphi}_j. \tag{2.2}$$

Following [23], the balance of first moment of energy is given by

$$\rho\dot{\varepsilon}_i = -Q_{ji,j} - q_i + Q_i, \tag{2.3}$$

where σ_{ji} is the stress tensor, μ_{ji} is the couple stress tensor, ρ is the density, u_i are the components of the displacement vector $\mathbf{u}$, $\tilde{F}_i$ is the external force per unit mass, $\tilde{G}_i$ is the external applied couple per unit mass, ε_{ijq} is the alternating tensor, J_{ij} is the microinertia tensor, φ_j is the microrotation vector $\boldsymbol{\varphi}$, ε_i is the first moment of the energy vector, q_i is the heat flux vector, Q_{ji} is the first heat flux moment tensor, Q_i is the micro heat flux average vector.

The law of conservation of energy for an arbitrary material volume V bounded by a surface A at time t can be written as

$$\int_V \rho[\dot{u}_i\ddot{u}_i + J_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + k_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}]\mathrm{d}V =$$

$$\int_V \rho[\tilde{F}_i\dot{u}_i + \tilde{G}_i\dot{\varphi}_i + \Lambda_i\,\dot{\nu}_i]\mathrm{d}V + \int_A [f_i\dot{u}_i + \mu_i\dot{\varphi}_i + +\Omega_{ij}\varpi_j\dot{\nu}_i - q_i\varpi_i]\mathrm{d}A, \tag{2.4}$$

where U is the internal energy per unit mass, f_i is the surface traction vector $\mathbf{f}$ occurring on the surface A, μ_i is the couple stress vector $\boldsymbol{\mu}$ occurring on the surface A, ν_i are the volume fraction fields corresponding to macro-, meso-, micro-pores respectively, k_i is the coefficient of equilibrated inertia, Λ_i is extrinsic equilibrated body force per unit mass associated to macro-, meso-, or micro-pores, Ω_{ij} is the equilibrated stress vector corresponding to ν_i measured per unit area of surface A, and ϖ_i is the outward unit normal vector $\boldsymbol{\varpi}$ to the surface A.

The components f_i and μ_i are, respectively, connected to stress and couple stress vectors by the relations

$$f_i = \sigma_{ji}\varpi_j,\ \mu_i = \mu_{ji}\varpi_j. \tag{2.5}$$

Using equation (2.5) in equation (2.4) and applying the divergence theorem, we get

$$\int_V \rho[\dot{u}_i\ddot{u}_i + J_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + k_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}]\mathrm{d}V = \int_V \rho[\tilde{F}_i\dot{u}_i + \tilde{G}_i\dot{\varphi}_i + \Lambda_i\,\dot{\nu}_i]\mathrm{d}V +$$
$$\int_V [\sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \mu_{ji,j}\dot{\varphi}_i + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{ij,j}\dot{\nu}_i + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i}]\mathrm{d}V. \tag{2.6}$$

Since equation (2.6) is valid for every part of the body, the local form of conservation of energy is obtained as

$$\rho[\dot{u}_i\ddot{u}_i + J_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + k_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}] = \rho[\tilde{F}_i\dot{u}_i + \tilde{G}_i\dot{\varphi}_i + \Lambda_i\,\dot{\nu}_i] +$$
$$\sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \mu_{ji,j}\dot{\varphi}_i + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{ij,j}\dot{\nu}_i + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i}. \tag{2.7}$$

Equation (2.7) with the assistance of equations (2.1) and (2.2) yields a simplified form of conservation of energy

$$\rho\dot{U} = \sigma_{ji}\dot{\varepsilon}_{ji} + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{ij}\dot{\nu}_{i,j} - q_{i,i} - \Upsilon_i\dot{\nu}_i, \tag{2.8}$$

where ε_{ji} and Υ_i, $i = 1, 2, 3$ satisfy the relations

$$\varepsilon_{ji} = u_{i,j} - \varepsilon_{qji}\varphi_q, \Omega_{1j,j} + \Upsilon_1 + \rho\Lambda_1 = \rho k_1\ddot{\nu}_1,$$
$$\Omega_{2j,j} + \Upsilon_2 + \rho\Lambda_2 = \rho k_2\ddot{\nu}_2, \Omega_{3j,j} + \Upsilon_3 + \rho\Lambda_3 = \rho k_3\ddot{\nu}_3. \tag{2.9}$$

The local form of the principle of entropy can be expressed in the form of an inequality called the Clausius-Duhem inequality

$$\rho\dot{S} + \frac{q_{i,i}}{T} - \frac{q_i}{T^2}T_{,i} + \frac{Q_{ji,j}}{T}T_i - \frac{Q_{ji}}{T^2}T_{,j}T_i + \frac{Q_{ji}}{T}T_{i,j} \ \geq\ 0, \tag{2.10}$$

where S is entropy per unit mass, T is absolute temperature, and T_i is the microtemperature vector. Equation (2.8) with the use of equation (2.3) and inequality (2.10) can be expressed as

$$\rho[T\dot{S} - \dot{U} - T_i\dot{\varepsilon}_i] + \sigma_{ji}\dot{\varepsilon}_{ji} + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{ij}\dot{\nu}_{i,j} - \Upsilon_i\dot{\nu}_i - \frac{q_i}{T}T_{,i} -$$
$$- \frac{Q_{ji}}{T}T_{,j}T_i + Q_{ji}T_{i,j} - (q_i - Q_i)T_i \ \geq\ 0. \tag{2.11}$$

The Helmholtz free energy function Γ is defined as

$$\Gamma = U + T_i\varepsilon_i - TS.$$

Then, relation (2.11) in the context of linear theory becomes

$$-\rho[\dot{\Gamma} + \dot{T}S - \dot{T}_i\varepsilon_i] + \sigma_{ji}\dot{\varepsilon}_{ji} + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{ij}\dot{\nu}_{i,j} -$$
$$- \Upsilon_i\dot{\nu}_i - \frac{q_i}{T}T_{,i} + Q_{ji}T_{i,j} - (q_i - Q_i)T_i \ \geq\ 0. \tag{2.12}$$

The function Γ can be expressed in terms of the independent variables

$$\varepsilon_{ji}, \varphi_{i,j}, \nu_i, \nu_{i,j}, T, T_{,i}, T_i \text{ and } T_{i,j}.$$

Therefore, we have

$$\dot{\Gamma} = \frac{\partial\Gamma}{\partial\varepsilon_{ji}}\dot{\varepsilon}_{ji} + \frac{\partial\Gamma}{\partial\varphi_{i,j}}\dot{\varphi}_{i,j} + \frac{\partial\Gamma}{\partial\nu_i}\dot{\nu}_i + \frac{\partial\Gamma}{\partial\nu_{i,j}}\dot{\nu}_{i,j} + \frac{\partial\Gamma}{\partial T}\dot{T} + \frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j}. \quad (2.13)$$

Inequality (2.12) with the help of equation (2.13) becomes

$$\left[\sigma_{ji} - \rho\frac{\partial\Gamma}{\partial\varepsilon_{ji}}\right]\dot{\varepsilon}_{ji} + \left[\mu_{ji} - \rho\frac{\partial\Gamma}{\partial\varphi_{i,j}}\right]\dot{\varphi}_{i,j} + \left[\Omega_{ij} - \rho\frac{\partial\Gamma}{\partial\nu_{i,j}}\right]\dot{\nu}_{i,j} - \left[\Upsilon_i + \rho\frac{\partial\Gamma}{\partial\nu_i}\right]\dot{\nu}_i$$

$$-\rho[S + \frac{\partial\Gamma}{\partial T}]\dot{T} + \rho[\varepsilon_i - \frac{\partial\Gamma}{\partial T_i}]\dot{T}_i - \rho\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} - \rho\frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j}$$

$$-\frac{q_i}{T}T_{,i} + Q_{ji}T_{i,j} - (q_i - Q_i)T_i \ \geq\ 0.$$

Let us introduce the notations

$$\boldsymbol{\phi} = \boldsymbol{\nu} - \boldsymbol{\nu}_0, \quad \theta = T - T_0,$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, T_0 is the reference temperature of the body chosen such that $|\frac{\theta}{T_0}| \ll 1$, $\boldsymbol{\nu}_0$ are the volume fraction fields in reference configuration.

In the linear theory, the independent variables are $\varepsilon_{ji}, \varphi_{i,j}, \phi_i, \phi_{i,j}, \theta$ and T_i. The above inequality should be convinced for all rates $\dot{\varepsilon}_{ji}, \dot{\varphi}_{i,j}, \dot{\phi}_i, \dot{\phi}_{i,j}, \dot{\theta}, \dot{\theta}_{,i}, \dot{T}_i$ and $\dot{T}_{i,j}$. Hence the coefficients of above variables must vanish, that is,

$$\sigma_{ji} = \rho\frac{\partial\Gamma}{\partial\varepsilon_{ji}}, \mu_{ji} = \rho\frac{\partial\Gamma}{\partial\varphi_{i,j}}, \Omega_{ij} = \rho\frac{\partial\Gamma}{\partial\phi_{i,j}}, \Upsilon_i = -\rho\frac{\partial\Gamma}{\partial\phi_i},$$

$$S = -\frac{\partial\Gamma}{\partial\theta}, \varepsilon_i = \frac{\partial\Gamma}{\partial T_i}, \frac{\partial\Gamma}{\partial T_{,i}} = \frac{\partial\Gamma}{\partial T_{i,j}} = 0,$$

$$-q_i\theta_{,i} + T_0 Q_{ji}T_{i,j} - T_0(q_i - Q_i)T_i \ \geq\ 0. \quad (2.14)$$

It is assumed that the undeformed body is free from stresses and has zero intrinsic equilibrated body forces and entropy. If the body has a center of symmetry, then we have

$$\begin{aligned}2\rho\Gamma = {} & c_{jipl}\varepsilon_{ji}\varepsilon_{pl} + d_{jipl}\varphi_{j,i}\varphi_{p,l} - 2a_{ji}\varepsilon_{ji}\theta + 2b_{ji}\varepsilon_{ji}\phi_1 + 2c_{ji}\varepsilon_{ji}\phi_2 \\ & + 2d_{ji}\varepsilon_{ji}\phi_3 + \alpha_i\phi_i^2 + 2\alpha_4\phi_1\phi_2 + 2\alpha_5\phi_2\phi_3 + 2\alpha_6\phi_3\phi_1 \\ & + A_{ij}\phi_{1,i}\phi_{1,j} + B_{ij}\phi_{2,i}\phi_{2,j} + C_{ij}\phi_{3,i}\phi_{3,j} + 2D_{ij}\phi_{1,i}\phi_{2,j} + 2E_{ij}\phi_{2,i}\phi_{3,j} \\ & + 2F_{ij}\phi_{3,i}\phi_{1,j} - 2\ell_i\phi_i\theta - \frac{a\theta^2}{T_0} - s_{ij}T_iT_j - 2r_{ij}\phi_{1,j}T_i - 2t_{ij}\phi_{2,j}T_i - 2v_{ij}\phi_{3,j}T_i.\end{aligned}$$

Here c_{jipl} is the tensor of elastic constants, d_{jipl} is the tensor of microrotation constants, a_{ji} is the tensor of thermal expansion, $b_{ji}, c_{ji}, d_{ji}, A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}$ and α_i, $i = 1, ..., 6$ if you write i=1,2,....6 then we know the increments used are functions which are typical in porous theories, ℓ_i is the coupling constant, r_{ij}, t_{ij}, and v_{ij} are coupling tensors and s_{ij} is the tensor of the microtemperature effect. The constitutive coefficients have the following symmetries

$$\begin{array}{llll} c_{jipl} = c_{plji}, & d_{jipl} = d_{plji}, & a_{ji} = a_{ij}, & b_{ji} = b_{ij}, \\ c_{ji} = c_{ij}, & d_{ji} = d_{ij}, & A_{ij} = A_{ji}, & B_{ij} = B_{ji}, \\ C_{ij} = C_{ji}, & D_{ij} = D_{ji}, & E_{ij} = E_{ji}, & F_{ij} = F_{ji}, \\ s_{ij} = s_{ji}, & r_{ij} = r_{ji}, & t_{ij} = t_{ji}, & v_{ij} = v_{ji}. \end{array}$$

Using the previous equation in the system of equations (2.14), the following constitutive equations are obtained:

$$\sigma_{ji} = c_{jipl}\varepsilon_{pl} + b_{ji}\phi_1 + c_{ji}\phi_2 + d_{ji}\phi_3 - a_{ji}\theta, \tag{2.15}$$

$$\mu_{ji} = d_{ijpl}\varphi_{p,l}, \tag{2.16}$$

$$\begin{aligned}\Omega_{1j} &= A_{ij}\phi_{1,i} + D_{ij}\phi_{2,i} + F_{ij}\phi_{3,i} - r_{ij}T_i,\\ \Omega_{2j} &= D_{ij}\phi_{1,i} + B_{ij}\phi_{2,i} + E_{ij}\phi_{3,i} - t_{ij}T_i,\\ \Omega_{3j} &= F_{ij}\phi_{1,i} + E_{ij}\phi_{2,i} + C_{ij}\phi_{3,i} - v_{ij}T_i,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\Upsilon_1 &= -b_{ji}\varepsilon_{ji} - \alpha_1\phi_1 - \alpha_4\phi_2 - \alpha_6\phi_3 + \ell_1\theta,\\ \Upsilon_2 &= -c_{ji}\varepsilon_{ji} - \alpha_4\phi_1 - \alpha_2\phi_2 - \alpha_5\phi_3 + \ell_2\theta,\\ \Upsilon_3 &= -d_{ji}\varepsilon_{ji} - \alpha_6\phi_1 - \alpha_5\phi_2 - \alpha_3\phi_3 + \ell_3\theta,\end{aligned} \tag{2.18}$$

$$\rho S = a_{ji}\varepsilon_{ji} + \ell_i\phi_i + \frac{a\theta}{T_0}, \tag{2.19}$$

$$\rho\varepsilon_i = -r_{ij}\phi_{1,j} - t_{ij}\phi_{2,j} - v_{ij}\phi_{3,j} - s_{ij}T_j. \tag{2.20}$$

The linear expressions for q_i, Q_i and Q_{ij} are

$$q_i = -[k_{ij}\theta_{,j} + \kappa_{ij}T_j], \tag{2.21}$$

$$Q_i = (K_{ij} - k_{ij})\theta_{,j} + (-\kappa_{ij} + L_{ij})T_j, \tag{2.22}$$

$$Q_{ij} = m_{ijpl}T_{l,p}\ , \tag{2.23}$$

where the consecutive coefficients $k_{ij}, \kappa_{ij}, K_{ij}, L_{ij}$ and m_{ijpl} satisfy the inequality

$$k_{ij}\theta_{,i}\theta_{,j} + (\kappa_{ji} + T_0K_{ij})\theta_{,j}T_i + T_0L_{ij}T_iT_j + T_0m_{jipl}T_{l,p}T_{i,j} \geq 0.$$

The linearized form of inequality (2.10) is

$$\rho T_0\dot{S} = -q_{i,i}. \tag{2.24}$$

In view of equations (15)-(23), equations (2.1)-(2.3), (2.9) and (2.24) become

$$\begin{aligned}&c_{jipl}\varepsilon_{pl,j} + b_{ji}\phi_{1,j} + c_{ji}\phi_{2,j} + d_{ji}\phi_{3,j} - a_{ji}\theta_{,j} + \rho\tilde{F}_i = \rho\ddot{u}_i,\\ &\varepsilon_{ijq}(c_{jqpl}\varepsilon_{pl} + b_{jq}\phi_1 + c_{jq}\phi_2 + d_{jq}\phi_3 - a_{jq}\theta) + d_{ijpl}\varphi_{p,lj} + \rho\tilde{G}_i = \rho J_{ij}\ddot{\varphi}_j,\\ &-b_{ji}\varepsilon_{ji}+A_{ij}\phi_{1,ij}+D_{ij}\phi_{2,ij}+F_{ij}\phi_{3,ij}-\alpha_1\phi_1-\alpha_4\phi_2-\alpha_6\phi_3+\ell_1\theta+\rho\Lambda_1-r_{ij}T_{i,j} = \rho k_1\ddot{\phi}_1,\\ &-c_{ji}\varepsilon_{ji}+D_{ij}\phi_{1,ij}+B_{ij}\phi_{2,ij}+E_{ij}\phi_{3,ij}-\alpha_4\phi_1-\alpha_2\phi_2-\alpha_5\phi_3+\ell_2\theta+\rho\Lambda_2-t_{ij}T_{i,j} = \rho k_2\ddot{\phi}_2,\\ &-d_{ji}\varepsilon_{ji}+F_{ij}\phi_{1,ij}+E_{ij}\phi_{2,ij}+C_{ij}\phi_{3,ij}-\alpha_6\phi_1-\alpha_5\phi_2-\alpha_3\phi_3+\ell_3\theta+\rho\Lambda_3-v_{ij}T_{i,j} = \rho k_3\ddot{\phi}_3.\\ &T_0\left[a_{ji}\dot{\varepsilon}_{ij} + \ell_i\dot{\phi}_i\right] + a\dot{\theta} = k_{ij}\theta_{,ij} + \kappa_{ij}T_{j,i},\\ &m_{jipl}T_{l,pj} - s_{ij}\dot{T}_j - r_{ij}\dot{\phi}_{1,j} - t_{ij}\dot{\phi}_{2,j} - v_{ij}\dot{\phi}_{3,j} = K_{ij}\theta_{,j} + L_{ij}T_j\ .\end{aligned} \tag{2.25}$$

If we take

$$\begin{gathered}c_{jipl} = \lambda\delta_{ji}\delta_{pl} + (\mu + K^*)\delta_{jp}\delta_{il} + \mu\delta_{jl}\delta_{ip}, J_{ij} = J\delta_{ij}, s_{ij} = \chi\delta_{ij},\\ d_{jipl} = \alpha\delta_{ji}\delta_{pl} + \gamma\delta_{jp}\delta_{il} + \beta\delta_{jl}\delta_{ip}, a_{ij} = \vartheta\delta_{ij},\\ b_{ij} = \Re_1\delta_{ij},\ c_{ij} = \Re_2\delta_{ij},\ d_{ij} = \Re_3\delta_{ij},\ A_{ij} = A_1\delta_{ij},\ B_{ij} = A_2\delta_{ij},\\ C_{ij} = A_3\delta_{ij},\ D_{ij} = A_4\delta_{ij},\ E_{ij} = A_5\delta_{ij},\ F_{ij} = A_6\delta_{ij},\\ r_{ij} = r_1\delta_{ij},\ t_{ij} = r_2\delta_{ij},\ v_{ij} = r_3\delta_{ij},\ m_{ijpl} = \kappa_4\delta_{ij}\delta_{pl} + \kappa_6\delta_{ip}\delta_{jl} + \kappa_5\delta_{il}\delta_{jp},\end{gathered}$$

$$k_{ij} = k\delta_{ij}, \;\; \kappa_{ij} = \kappa_1\delta_{ij}, \;\; L_{ij} = \kappa_2\delta_{ij}, \;\; K_{ij} = \kappa_3\delta_{ij},$$

where δ_{ij} is Kronecker's delta and $\lambda, \mu, K^*, \alpha, \beta, \gamma, \vartheta, J, \chi, \kappa_1, ..., \kappa_6, A_1, ..., A_6$, $\Re_1, \Re_2, \Re_3, k, r_1, r_2, r_3$ are material constants in equations (2.25), the governing equations for homogeneous isotropic generalized micropolar thermoelasticity with triple porosity and microtemperatures in absence of body forces and couples are obtained as

$$(\mu + K^*)\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + K^* \operatorname{curl} \boldsymbol{\varphi} + \Re_i\nabla\phi_i - \vartheta\nabla\theta = \rho\ddot{\mathbf{u}},$$

$$K^* \operatorname{curl} \mathbf{u} + (\gamma\Delta - 2K^*)\boldsymbol{\varphi} + (\alpha + \beta)\nabla \operatorname{div} \boldsymbol{\varphi} = \rho J\ddot{\boldsymbol{\varphi}},$$

$$-\Re_1 \operatorname{div} \mathbf{u} + (A_1\Delta - \alpha_1)\phi_1 + (A_4\Delta - \alpha_4)\phi_2 + (A_6\Delta - \alpha_6)\phi_3 + \ell_1\theta - r_1 \operatorname{div} \mathbf{w} = \rho k_1\ddot{\phi}_1,$$

$$-\Re_2 \operatorname{div} \mathbf{u} + (A_4\Delta - \alpha_4)\phi_1 + (A_2\Delta - \alpha_2)\phi_2 + (A_5\Delta - \alpha_5)\phi_3 + \ell_2\theta - r_2 \operatorname{div} \mathbf{w} = \rho k_2\ddot{\phi}_2,$$

$$-\Re_3 \operatorname{div} \mathbf{u} + (A_6\Delta - \alpha_6)\phi_1 + (A_5\Delta - \alpha_5)\phi_2 + (A_3\Delta - \alpha_3)\phi_3 + \ell_3\theta - r_3 \operatorname{div} \mathbf{w} = \rho k_3\ddot{\phi}_3,$$

$$T_0[\vartheta \operatorname{div} \dot{\mathbf{u}} + \ell_i\dot{\phi}_i] + a\dot{\theta} = k\Delta\theta + \kappa_1 \operatorname{div} \mathbf{w},$$

$$\kappa_6\Delta\,\mathbf{w} + (\kappa_4 + \kappa_5)\nabla \operatorname{div} \mathbf{w} - \chi\dot{\mathbf{w}} - r_i\nabla\dot{\phi}_i = \kappa_3\nabla\theta + \kappa_2\,\mathbf{w}, \tag{2.26}$$

where $\mathbf{w} = (T_1, T_2, T_3)$ is the microrotation vector and Δ, ∇ are the Laplacian and Nabla operators, respectively.

3. Steady Oscillations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3:

$$|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \qquad \mathbf{D}_{\mathbf{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$$

The displacement vector, microrotation vector, volume fraction fields, temperature change, and microtemperature vector functions are assumed as:

$$\Big[\mathbf{u}(\mathbf{x}, t), \boldsymbol{\varphi}(\mathbf{x}, t), \boldsymbol{\phi}(\mathbf{x}, t), \theta(\mathbf{x}, t), \mathbf{w}(\mathbf{x}, t)\Big] = \operatorname{Re}\Big[(\mathbf{u}^*, \boldsymbol{\varphi}^*, \boldsymbol{\phi}^*, \theta^*, \mathbf{w}^*)e^{-\iota\omega t}\Big], \tag{3.1}$$

where ω is oscillation frequency.

Using equation (3.1) in the system of equations (2.26) and omitting the asterisk (*) for simplicity, the system of equations for the steady oscillations is obtained as

$$(\mu + K^*)\Delta\mathbf{u} + [(\lambda + \mu)\nabla \operatorname{div} + \rho\omega^2]\mathbf{u} + K^* \operatorname{curl} \boldsymbol{\varphi} + \Re_i\nabla\phi_i - \vartheta\nabla\theta = \mathbf{0},$$

$$K^* \operatorname{curl} \mathbf{u} + \gamma\Delta\boldsymbol{\varphi} + [(\alpha + \beta)\nabla \operatorname{div} - 2K^* + \rho J\omega^2]\boldsymbol{\varphi} = \mathbf{0},$$

$$-\Re_1 \operatorname{div} \mathbf{u} + (A_1\Delta - \beta_1)\phi_1 + (A_4\Delta - \alpha_4)\phi_2 + (A_6\Delta - \alpha_6)\phi_3 + \ell_1\theta - r_1 \operatorname{div} \mathbf{w} = 0,$$

$$-\Re_2 \operatorname{div} \mathbf{u} + (A_4\Delta - \alpha_4)\phi_1 + (A_2\Delta - \beta_2)\phi_2 + (A_5\Delta - \alpha_5)\phi_3 + \ell_2\theta - r_2 \operatorname{div} \mathbf{w} = 0,$$

$$-\Re_3 \operatorname{div} \mathbf{u} + (A_6\Delta - \alpha_6)\phi_1 + (A_5\Delta - \alpha_5)\phi_2 + (A_3\Delta - \beta_3)\phi_3 + \ell_3\theta - r_3 \operatorname{div} \mathbf{w} = 0,$$

$$\iota\omega T_0[\vartheta \operatorname{div} \mathbf{u} + \ell_i\phi_i] + \big[k\Delta + \iota\omega a\big]\theta + \kappa_1 \operatorname{div} \mathbf{w} = 0,$$

$$\iota\omega r_i\nabla\phi_i - \kappa_3\nabla\theta + \big[\kappa_6\Delta + (\kappa_4 + \kappa_5)\nabla\operatorname{div} - \kappa_2 + \iota\omega\chi\big]\mathbf{w} = \mathbf{0}, \tag{3.2}$$

where $\beta_i = \alpha_i - \rho\omega^2 k_i$, $i = 1, 2, 3$.

If we replace ω by $-\iota\tau$, where τ is a complex number and $Re(\tau) > 0$ in the equations (3.2), we obtain the system of equations for the pseudo-oscillations as:

$$(\mu + K^*)\Delta\mathbf{u} + [(\lambda+\mu)\nabla\,\mathrm{div} - \rho\tau^2]\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} + \Re_i\nabla\phi_i - \vartheta\nabla\theta = \mathbf{0},$$
$$K^*\,\mathrm{curl}\,\mathbf{u} + \gamma\Delta\boldsymbol{\varphi} + [(\alpha+\beta)\,\nabla\,\mathrm{div} - 2K^* - \rho J\tau^2]\boldsymbol{\varphi} = \mathbf{0},$$
$$-\Re_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \tilde{\beta}_1)\phi_1 + (A_4\,\Delta - \alpha_4)\phi_2 + (A_6\,\Delta - \alpha_6)\phi_3 + \ell_1\theta - r_1\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\Re_2\,\mathrm{div}\,\mathbf{u} + (A_4\,\Delta - \alpha_4)\phi_1 + (A_2\,\Delta - \tilde{\beta}_2)\phi_2 + (A_5\,\Delta - \alpha_5)\phi_3 + \ell_2\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\Re_3\,\mathrm{div}\,\mathbf{u} + (A_6\,\Delta - \alpha_6)\phi_1 + (A_5\,\Delta - \alpha_5)\phi_2 + (A_3\,\Delta - \tilde{\beta}_3)\phi_3 + \ell_3\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,$$
$$\tau T_0[\vartheta\,\mathrm{div}\,\mathbf{u} + \ell_i\phi_i] + \big[k\Delta + \tau a\big]\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,$$
$$\tau r_i\nabla\phi_i - \kappa_3\nabla\theta + \big[\kappa_6\Delta + (\kappa_4+\kappa_5)\nabla\mathrm{div} - \kappa_2 + \tau\chi\big]\mathbf{w} = \mathbf{0}, \tag{3.3}$$

where

$$\tilde{\beta}_i = \alpha_i + \rho\tau^2 k_i,\ i = 1,2,3.$$

If we put $\rho = 0$, i.e. neglecting inertial effect in equations (3.2), we obtain the system of equations of quasi-static oscillations as:

$$[(\mu + K^*)\Delta + (\lambda+\mu)\nabla\,\mathrm{div}]\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} + \Re_i\nabla\phi_i - \vartheta\nabla\theta = \mathbf{0},$$
$$K^*\,\mathrm{curl}\,\mathbf{u} + \gamma\Delta\boldsymbol{\varphi} + [(\alpha+\beta)\,\nabla\,\mathrm{div} - 2K^*]\boldsymbol{\varphi} = \mathbf{0},$$
$$-\Re_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \alpha_1)\phi_1 + (A_4\,\Delta - \alpha_4)\phi_2 + (A_6\,\Delta - \alpha_6)\phi_3 + \ell_1\theta - r_1\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\Re_2\,\mathrm{div}\,\mathbf{u} + (A_4\,\Delta - \alpha_4)\phi_1 + (A_2\,\Delta - \alpha_2)\phi_2 + (A_5\,\Delta - \alpha_5)\phi_3 + \ell_2\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\Re_3\,\mathrm{div}\,\mathbf{u} + (A_6\,\Delta - \alpha_6)\phi_1 + (A_5\,\Delta - \alpha_5)\phi_2 + (A_3\,\Delta - \alpha_3)\phi_3 + \ell_3\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,$$
$$\iota\omega T_0[\vartheta\,\mathrm{div}\,\mathbf{u} + \ell_i\phi_i] + \big[k\Delta + \iota\omega a\big]\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,$$
$$\iota\omega r_i\nabla\phi_i - \kappa_3\nabla\theta + \big[\kappa_6\Delta + (\kappa_4+\kappa_5)\nabla\mathrm{div} - \kappa_2 + \iota\omega\chi\big]\mathbf{w} = \mathbf{0}. \tag{3.4}$$

If we put $\omega = 0$ in equations (3.2), we obtain the system of equations in the equilibrium theory of micropolar thermoelasticity with triple porosity and microtemperatures as:

$$[(\mu + K^*)\Delta + (\lambda+\mu)\nabla\,\mathrm{div}]\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} + \Re_i\nabla\phi_i - \vartheta\nabla\theta = \mathbf{0},$$
$$K^*\,\mathrm{curl}\,\mathbf{u} + \gamma\Delta\boldsymbol{\varphi} + [(\alpha+\beta)\,\nabla\,\mathrm{div} - 2K^*]\boldsymbol{\varphi} = \mathbf{0},$$
$$-\Re_1\,\mathrm{div}\,\mathbf{u} + (A_1\,\Delta - \alpha_1)\phi_1 + (A_4\,\Delta - \alpha_4)\phi_2 + (A_6\,\Delta - \alpha_6)\phi_3 + \ell_1\theta - r_1\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\Re_2\,\mathrm{div}\,\mathbf{u} + (A_4\,\Delta - \alpha_4)\phi_1 + (A_2\,\Delta - \alpha_2)\phi_2 + (A_5\,\Delta - \alpha_5)\phi_3 + \ell_2\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\Re_3\,\mathrm{div}\,\mathbf{u} + (A_6\,\Delta - \alpha_6)\phi_1 + (A_5\,\Delta - \alpha_5)\phi_2 + (A_3\,\Delta - \alpha_3)\phi_3 + \ell_3\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,$$
$$k\Delta\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,$$
$$-\kappa_3\nabla\theta + \big[\kappa_6\Delta + (\kappa_4+\kappa_5)\nabla\mathrm{div} - \kappa_2\big]\mathbf{w} = \mathbf{0}. \tag{3.5}$$

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{F}^{(i)}(\mathbf{D_x}) = \Big(F^{(i)}_{gh}(\mathbf{D_x})\Big)_{13\times 13}, \tag{3.6}$$

where $F^{(i)}_{gh}(\mathbf{D_x})$; $g, h = 1, ..., 13$ are given in Appendix A.

Here $i = 1, 2, 3, 4$ corresponds to static, pseudo-, quasi-static oscillations and the equilibrium theory of micropolar thermoelasticity with triple porosity and microtemperatures, respectively. The matrices $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 2, 3, 4$ can be obtained from matrix $\mathbf{F}^{(1)}(\mathbf{D_x})$ by taking $\omega = -\iota\tau$, $\rho = 0$ and $\omega = 0$, respectively:

$$\tilde{\mathbf{F}}(\mathbf{D_x}) = \left(\tilde{F}_{gh}(\mathbf{D_x})\right)_{13\times 13},$$

where $\tilde{F}_{gh}(\mathbf{D_x})$, $g, h = 1, ..., 13$ are given in Appendix A.

The system of equations (3.2)-(3.5) can be represented as

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0},\ \ i = 1, 2, 3, 4,$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\phi}, \theta, \mathbf{w})$ is a thirteen-component vector function on E^3. The matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of operator $\mathbf{F}^{(i)}(\mathbf{D_x})$.

Definition 1: The operator $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 1, 2, 3, 4$ is said to be elliptic if $|\tilde{\mathbf{F}}(\mathbf{m})| \neq 0$, where $\mathbf{m} = (m_1, m_2, m_3) \neq \mathbf{0}$.

Since

$$|\tilde{\mathbf{F}}(\mathbf{m})| = \tilde{\mu}^2\gamma^2\tilde{\lambda}\tilde{\alpha}\varrho k\kappa_6^2\kappa_7|\mathbf{m}|^{26},$$

$$\tilde{\lambda} = \lambda + 2\mu + K^*,$$

$$\tilde{\alpha} = \alpha + \beta + \gamma,$$

$$\kappa_7 = \kappa_4 + \kappa_5 + \kappa_6,$$

$$\varrho = \begin{vmatrix} A_1 & A_4 & A_6 \\ A_4 & A_2 & A_5 \\ A_6 & A_5 & A_3 \end{vmatrix},$$

it follows that the operator $\mathbf{F}^{(i)}(\mathbf{D_x})$ is an elliptic differential operator if

$$\tilde{\mu}\gamma\tilde{\lambda}\tilde{\alpha}\varrho k\kappa_6\kappa_7 \neq 0. \tag{3.7}$$

Definition 2: The fundamental solutions of the system of equations (3.2)-(3.5) (fundamental matrices of operators $\mathbf{F}^{(i)}$) are the matrices

$$\mathbf{G}^{(i)}(\mathbf{x}) = \left(G_{gh}^{(i)}(\mathbf{x})\right)_{13\times 13}$$

satisfying conditions

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{G}^{(i)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}),\ \ i = 1, 2, 3, 4, \tag{3.8}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I} = (\delta_{gh})_{13\times 13}$ is the unit matrix and $\mathbf{x} \in \mathrm{E}^3$.

4. Construction of $\mathbf{G}^{(1)}(\mathbf{x})$ in terms of elementary functions

Let us consider the system of non-homogeneous equations

$$\tilde{\mu}\Delta\mathbf{u} + [(\lambda + \mu)\nabla\,\mathrm{div} + \rho\omega^2]\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} - \Re_i\nabla\phi_i + \iota\omega\vartheta T_0\nabla\theta = \mathbf{H}, \tag{4.1}$$

$$K^*\,\mathrm{curl}\,\mathbf{u} + [\gamma\Delta + \tilde{K} + (\alpha + \beta)\,\nabla\,\mathrm{div}]\boldsymbol{\varphi} = \mathbf{V}, \tag{4.2}$$

$$\Re_1 \operatorname{div} \mathbf{u}+(A_1\,\Delta-\beta_1)\phi_1+(A_4\,\Delta-\alpha_4)\phi_2+(A_6\,\Delta-\alpha_6)\phi_3+\iota\omega T_0\ell_1\theta+\iota\omega r_1 \operatorname{div}\mathbf{w} = L_1, \tag{4.3}$$

$$\Re_2 \operatorname{div} \mathbf{u}+(A_4\,\Delta-\alpha_4)\phi_1+(A_2\,\Delta-\beta_2)\phi_2+(A_5\,\Delta-\alpha_5)\phi_3+\iota\omega T_0\ell_2\theta+\iota\omega r_2 \operatorname{div}\mathbf{w} = L_2, \tag{4.4}$$

$$\Re_3 \operatorname{div} \mathbf{u}+(A_6\,\Delta-\alpha_6)\phi_1+(A_5\,\Delta-\alpha_5)\phi_2+(A_3\,\Delta-\beta_3)\phi_3+\iota\omega T_0\ell_3\theta+\iota\omega r_3 \operatorname{div}\mathbf{w} = L_3, \tag{4.5}$$

$$-\vartheta \operatorname{div}\mathbf{u} + \ell_i\phi_i + (k\Delta + \iota\omega a)\theta - \kappa_3 \operatorname{div}\mathbf{w} = Z, \tag{4.6}$$

$$-r_i\nabla\phi_i + \kappa_1\nabla\theta + [\kappa_6\Delta + (\kappa_4+\kappa_5)\nabla\operatorname{div} + \kappa_8]\mathbf{w} = \mathbf{X}, \tag{4.7}$$

where $\mathbf{H}, \mathbf{V}, \mathbf{X}$ are three-component vector functions on E^3; L_i and Z are scalar functions on E^3.

The system of equations (4.1)-(4.7) may also be written in the form

$$\mathbf{F}^{(1)^{tr}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{4.8}$$

where $\mathbf{F}^{(1)^{tr}}$ is the transpose of matrix $\mathbf{F}^{(1)}, \mathbf{Q} = (\mathbf{H}, \mathbf{V}, L_i, Z, \mathbf{X})$ and $\mathbf{x} \in \mathrm{E}^3$.

Applying operator div to the equations (4.1), (4.2) and (4.7), we obtain

$$[\tilde{\lambda}\Delta + \rho\omega^2] \operatorname{div}\mathbf{u} - \Re_i\Delta\phi_i + \iota\omega\vartheta T_0\Delta\theta = \operatorname{div}\mathbf{H}, \tag{4.9}$$

$$[\tilde{\alpha}\Delta + \tilde{K}] \operatorname{div}\boldsymbol{\varphi} = \operatorname{div}\mathbf{V}, \tag{4.10}$$

$$-r_i\Delta\phi_i + \kappa_1\Delta\theta + [\kappa_7\Delta + \kappa_8]\operatorname{div}\mathbf{w} = \operatorname{div}\mathbf{X}. \tag{4.11}$$

Equations (4.3)-(4.6), (4.9) and (4.11) may be expressed in the form

$$\mathbf{N}^{(1)}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}}, \tag{4.12}$$

where $\mathbf{N}^{(1)}, \mathbf{S}$ and $\tilde{\mathbf{Q}}$ are given in Appendix B.

Equations (4.3)-(4.6), (4.9) and (4.11) may also be written in the determinant form as

$$\Gamma^{(1)}(\Delta)\mathbf{S} = \boldsymbol{\Psi}, \tag{4.13}$$

where $\Gamma^{(1)}$ and $\boldsymbol{\Psi}$ are defined in Appendix B.

On expanding $\Gamma^{(1)}(\Delta)$, we can see that

$$\Gamma^{(1)}(\Delta) = \prod_{i=1}^{6}(\Delta + \lambda_i^2),$$

where $\lambda_i^2,\ i = 1, ..., 6$ are the roots of the equation $\Gamma^{(1)}(-m) = 0$ (with respect to m).

Applying operators $\gamma\Delta + \tilde{K}$ and K^* curl to equations (4.1) and (4.2), respectively, we obtain

$$(\gamma\Delta + \tilde{K})\left[\tilde{\mu}\Delta + (\lambda+\mu)\,\nabla\operatorname{div} + \rho\omega^2\right]\mathbf{u} + (\gamma\Delta + \tilde{K})K^*\operatorname{curl}\boldsymbol{\varphi}$$

$$= (\gamma\Delta + \tilde{K})\left[\mathbf{H} + \Re_i\ \nabla\,\phi_i - \iota\omega\vartheta T_0\,\nabla\,\theta\right], \tag{4.14}$$

and

$$(\gamma\Delta + \tilde{K})K^* \operatorname{curl} \boldsymbol{\varphi} = -{K^*}^2 \operatorname{curl}\operatorname{curl} \mathbf{u} + K^* \operatorname{curl} \mathbf{V}. \tag{4.15}$$

Now

$$\operatorname{curl}\operatorname{curl} \mathbf{u} = \nabla \operatorname{div} \mathbf{u} - \Delta \mathbf{u}. \tag{4.16}$$

Using equations (4.15) and (4.16) in equation (4.14) and applying operator $\Gamma^{(1)}(\Delta)$ to the resulting equation with the help of equation (4.13), we get

$$\Gamma^{(1)}(\Delta)\Lambda^{(1)}(\Delta)\mathbf{u} = \boldsymbol{\Psi}', \tag{4.17}$$

where $\Lambda^{(1)}(\Delta)$ and $\boldsymbol{\Psi}'$ are given in Appendix B.

It can be seen that

$$\Lambda^{(1)}(\Delta) = (\Delta + \lambda_7^2)(\Delta + \lambda_8^2),$$

where λ_7^2, λ_8^2 are the roots of the equation $\Lambda^{(1)}(-m) = 0$ (with respect to m).

It follows from equation (4.10) that

$$(\Delta + \lambda_9^2) \operatorname{div} \boldsymbol{\varphi} = \frac{1}{\tilde{\alpha}} \operatorname{div} \mathbf{V} = \Psi_7, \lambda_9^2 = \frac{\tilde{K}}{\tilde{\alpha}}. \tag{4.18}$$

Applying operators K^* curl and $\tilde{\mu}\Delta + \rho\omega^2$ to the equations (4.1) and (4.2), respectively, we acquire

$$[\tilde{\mu}\Delta + \rho\omega^2]K^* \operatorname{curl} \mathbf{u} = K^* \operatorname{curl} \mathbf{H} - {K^*}^2 \operatorname{curl}\operatorname{curl} \boldsymbol{\varphi}, \tag{4.19}$$

and

$$\begin{aligned}(\tilde{\mu}\Delta + \rho\omega^2)(\gamma\Delta + \tilde{K})\boldsymbol{\varphi} + (\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2) \nabla \operatorname{div} \boldsymbol{\varphi} \\ + K^*(\tilde{\mu}\Delta + \rho\omega^2) \operatorname{curl} \mathbf{u} = (\tilde{\mu}\Delta + \rho\omega^2)\mathbf{V}.\end{aligned} \tag{4.20}$$

Now

$$\operatorname{curl}\operatorname{curl} \boldsymbol{\varphi} = \nabla \operatorname{div} \boldsymbol{\varphi} - \Delta \boldsymbol{\varphi}. \tag{4.21}$$

Using equations (4.19) and (4.21) in equation (4.20) and applying the operator $\Delta + \lambda_9^2$ to the resulting equation with the help of equation (4.18), we get

$$\Lambda^{(1)}(\Delta)(\Delta + \lambda_9^2)\boldsymbol{\varphi} = \boldsymbol{\Psi}'', \tag{4.22}$$

where $\boldsymbol{\Psi}''$ is defined in Appendix B.

Applying the operator $\Gamma^{(1)}(\Delta)$ to the equation (4.7), we obtain

$$\Gamma^{(1)}(\Delta)(\Delta + \lambda_{10}^2)\mathbf{w} = \boldsymbol{\Psi}''', \tag{4.23}$$

where λ_{10}^2 and $\boldsymbol{\Psi}'''$ are given in Appendix B.

From equations (4.13), (4.17), (4.22) and (4.23), we obtain

$$\boldsymbol{\Theta}^{(1)}(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\boldsymbol{\Psi}}(\mathbf{x}), \tag{4.24}$$

where $\hat{\boldsymbol{\Psi}}$ and $\boldsymbol{\Theta}^{(1)}(\Delta)$ are given in Appendix C.

The expressions for $\boldsymbol{\Psi}'$, $\boldsymbol{\Psi}''$, $\boldsymbol{\Psi}'''$ and $\Psi_p, p = 2, ..., 5$ can be rewritten in the form

$$\boldsymbol{\Psi}' = \left[\frac{1}{\tilde{N}}(\gamma\Delta + \tilde{K})\Gamma^{(1)}(\Delta)\mathbf{J} + w_{11}^{(1)}(\Delta) \nabla \operatorname{div}\right]\mathbf{H} + w_{21}^{(1)}(\Delta) \operatorname{curl} \mathbf{V}$$

$$+\sum_{i=2}^{5} w^{(1)}_{i+1;1}(\Delta)\,\nabla\, w_i + w^{(1)}_{71}(\Delta)\ \nabla\,\mathrm{div}\,\mathbf{X},$$

$$\mathbf{\Psi}'' = w^{(1)}_{12}(\Delta)\,\mathrm{curl}\,\mathbf{H} + \left[\frac{1}{\tilde{N}}(\Delta+\lambda_9^2)(\tilde{\mu}\Delta+\rho\omega^2)\mathbf{J} + w^{(1)}_{22}(\Delta)\,\nabla\,\mathrm{div}\right]\mathbf{V},$$

$$\mathbf{\Psi}''' = w^{(1)}_{17}(\Delta)\nabla\,\mathrm{div}\,\mathbf{H} + \sum_{i=2}^{5} w^{(1)}_{i+1;7}(\Delta)\nabla\, w_i + \left[\frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\mathbf{J} + w^{(1)}_{77}(\Delta)\nabla\,\mathrm{div}\right]\mathbf{X},$$

$$\Psi_p = w^{(1)}_{1;p+1}(\Delta)\,\mathrm{div}\,\mathbf{H} + \sum_{i=2}^{5} w^{(1)}_{i+1;p+1}(\Delta) w_i + w^{(1)}_{7;p+1}(\Delta)\,\mathrm{div}\,\mathbf{X},\ p=2,...,5, \quad (4.25)$$

where $\boldsymbol{J}=(\delta_{ij})_{3\times 3}$ and $w^{(1)}_{ij}(\Delta)$, $i,j=1,.....,7$ are defined in Appendix C.

From equations (4.25), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{tr}(\mathbf{D_x})\mathbf{Q}(\mathbf{x}), \quad (4.26)$$

where $\mathbf{R}(\mathbf{D_x})$ is also given in Appendix C.

From equations (4.8), (4.24) and (4.26), we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x}) = \mathbf{\Theta}^{(1)}(\Delta). \quad (4.27)$$

We assume that

$$\lambda_p^2 \neq \lambda_l^2 \neq 0;\ \ p,l=1,.....,10;\ \ p\neq l.$$

Let

$$\mathbf{Y}^{(1)}(\mathbf{x}) = \left(Y^{(1)}_{ij}(\mathbf{x})\right)_{13\times 13},\ Y^{(1)}_{pp}(\mathbf{x}) = \sum_{g=1}^{8} r^{(1)}_{1g}\varsigma_g(\mathbf{x}),\ Y^{(1)}_{p+3;p+3}(\mathbf{x}) = \sum_{g=7}^{9} r^{(1)}_{2g}\varsigma_g(\mathbf{x}),$$

$$Y^{(1)}_{ll}(\mathbf{x}) = \sum_{g=1}^{6} r^{(1)}_{3g}\varsigma_g(\mathbf{x}), Y^{(1)}_{p+10;p+10}(\mathbf{x}) = \sum_{g=1}^{6,10} r^{(1)}_{4g}\varsigma_g(\mathbf{x}), Y^{(1)}_{ij}(\mathbf{x}) = 0,$$

$$p=1,2,3;\ \ l=7,...,10;\ \ i,j=1,.....,13;\ \ i\neq j$$

where

$$\varsigma_g(\mathbf{x}) = -\frac{e^{\iota\lambda_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}, r^{(1)}_{1l} = \prod_{i=1,i\neq l}^{8}(\lambda_i^2-\lambda_l^2)^{-1}, r^{(1)}_{2h} = \prod_{i=7,i\neq h}^{9}(\lambda_i^2-\lambda_h^2)^{-1},$$

$$r^{(1)}_{3q} = \prod_{i=1,i\neq q}^{6}(\lambda_i^2-\lambda_q^2)^{-1}, r^{(1)}_{4p} = \prod_{i=1,i\neq p}^{6,10}(\lambda_i^2-\lambda_p^2)^{-1},$$

$$g=1,....,10;\ \ l=1,....,8;\ \ h=7,8,9;\ \ q=1,....,6;\ \ p=1,....,6,10. \quad (4.28)$$

Lemma 1: The matrix $\mathbf{Y}^{(1)}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}^{(1)}(\Delta)$, i.e.

$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}). \quad (4.29)$$

Proof: To prove the lemma, it is sufficient to prove that

$$\Gamma^{(1)}(\Delta)\Lambda^{(1)}(\Delta)Y^{(1)}_{11}(\mathbf{x}) = \delta(\mathbf{x}), \quad (4.30)$$

$$\Lambda^{(1)}(\Delta)(\Delta+\lambda_9^2)Y_{44}^{(1)}(\mathbf{x})=\delta(\mathbf{x}), \tag{4.31}$$

$$\Gamma^{(1)}(\Delta)Y_{77}^{(1)}(\mathbf{x})=\delta(\mathbf{x}), \tag{4.32}$$

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_{10}^2)Y_{11;11}^{(1)}(\mathbf{x})=\delta(\mathbf{x}). \tag{4.33}$$

Consider

$$\sum_{i=1}^{8} r_{1i}^{(1)}=\frac{\sum_{j=1}^{8}(-1)^j z_j}{z_9}, \tag{4.34}$$

where z_i, $i=1,....,9$ are given in Appendix D.

By simplifying the right-hand side of the above relation, we obtain

$$\sum_{i=1}^{8} r_{1i}^{(1)}=0. \tag{4.35}$$

Similarly, we find that

$$\sum_{i=2}^{8} r_{1i}^{(1)}(\lambda_1^2-\lambda_i^2)=0,\sum_{i=3}^{8} r_{1i}^{(1)}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\right]=0,$$

$$\sum_{i=4}^{8} r_{1i}^{(1)}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\right]=0,\sum_{i=5}^{8} r_{1i}^{(1)}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_i^2)\right]=0,$$

$$\sum_{i=6}^{8} r_{1i}^{(1)}\left[\prod_{j=1}^{5}(\lambda_j^2-\lambda_i^2)\right]=0,\sum_{i=7}^{8} r_{1i}^{(1)}\left[\prod_{j=1}^{6}(\lambda_j^2-\lambda_i^2)\right]=0,\prod_{j=1}^{7} r_{18}^{(1)}(\lambda_j^2-\lambda_8^2)=1. \tag{4.36}$$

Also,

$$(\Delta+\lambda_p^2)\varsigma_g(\mathbf{x})=\delta(\mathbf{x})+(\lambda_p^2-\lambda_g^2)\varsigma_g(\mathbf{x})\ p,g=1,...,10. \tag{4.37}$$

Now consider

$$\Gamma^{(1)}(\Delta)\Lambda^{(1)}(\Delta)Y_{11}^{(1)}(\mathbf{x})=\prod_{i=1}^{8}(\Delta+\lambda_i^2)\sum_{g=1}^{8} r_{1g}^{(1)}\varsigma_g(\mathbf{x})$$

$$=\prod_{i=2}^{8}(\Delta+\lambda_i^2)\sum_{g=1}^{8} r_{1g}^{(1)}\Big[\delta(\mathbf{x})+(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]$$

$$=\prod_{i=2}^{8}(\Delta+\lambda_i^2)\Big[\delta(\mathbf{x})\sum_{g=1}^{8} r_{1g}^{(1)}+\sum_{g=2}^{8} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big].$$

Using equations (4.35)-(4.37) in the above relation, we obtain

$$\Gamma^{(1)}(\Delta)\Lambda^{(1)}(\Delta)Y_{11}^{(1)}(\mathbf{x})=\prod_{i=2}^{8}(\Delta+\lambda_i^2)\left[\sum_{g=2}^{8} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]$$

$$=\prod_{i=3}^{8}(\Delta+\lambda_i^2)\left[\sum_{g=2}^{8} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\Big[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\right]$$

$$= \prod_{i=3}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=4}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=4}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=5}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=5}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=6}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=6}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=6}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=7}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=6}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_6^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=7}^{8}(\Delta+\lambda_i^2)\Big[\sum_{g=7}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{6}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= (\Delta+\lambda_8^2)\Big[\sum_{g=7}^{8} r_{1g}^{(1)}\Big[\prod_{j=1}^{6}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_7^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= (\Delta+\lambda_8^2)\varsigma_8(\mathbf{x}) = \delta(\mathbf{x}).$$

Equations (4.31)-(4.33) can be proved in a similar way.

We introduce the matrix

$$\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{R}(\mathbf{D_x})\mathbf{Y}^{(1)}(\mathbf{x}). \tag{4.38}$$

From equations (4.27), (4.29) and (4.38), we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x})\mathbf{Y}^{(1)}(\mathbf{x}) = \mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution to equation (3.8) for $i = 1$.

Theorem 1: If condition (3.7) is satisfied, then the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ defined by the equation (4.38) is the fundamental solution of the system of equations (3.2) and the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is represented in the following form:

$$\mathbf{G}^{(1)}(\mathbf{x}) = \Big(H_{pq}(\mathbf{x})\Big)_{13\times 13} = \begin{pmatrix} \mathbf{H}^{(1)} & \mathbf{H}^{(2)} & \mathbf{H}^{(5)} & \mathbf{H}^{(10)} \\ \mathbf{H}^{(3)} & \mathbf{H}^{(4)} & \mathbf{H}^{(6)} & \mathbf{H}^{(11)} \\ \mathbf{H}^{(7)} & \mathbf{H}^{(8)} & \mathbf{H}^{(9)} & \mathbf{H}^{(15)} \\ \mathbf{H}^{(12)} & \mathbf{H}^{(13)} & \mathbf{H}^{(14)} & \mathbf{H}^{(16)} \end{pmatrix}_{13\times 13}, \tag{4.39}$$

where $\mathbf{H}^{(p)}(\mathbf{x})$, $p = 1, ..., 16$ are given in Appendix D.

5. Construction of matrices $\mathbf{G}^{(i)}(\mathbf{x}),\ i = 2, 3, 4$

5.1. **Pseudo-oscillations.** We introduce the matrix

$$\mathbf{G}^{(2)}(\mathbf{x}) = \mathbf{T}(\mathbf{D_x})\mathbf{Y}^{(2)}(\mathbf{x}), \tag{5.1}$$

where, the matrices $\mathbf{T}(\mathbf{D_x})$ and $\mathbf{Y}^{(2)}(\mathbf{x})$ can be obtained from matrices $\mathbf{R}(\mathbf{D_x})$ and $\mathbf{Y}^{(1)}(\mathbf{x})$ respectively by taking $\omega = -\iota\tau$ and repeating the above procedure after equation (3.8).

Theorem 2: If condition (3.7) is satisfied, then the matrix $\mathbf{G}^{(2)}(\mathbf{x})$ defined by the equation (5.1) is the fundamental solution of the system of equations (3.3).

5.2. **Quasi-static oscillations.** In this case, the matrix $\mathbf{N}^{(3)}(\Delta)$, operators $\Gamma^{(3)}(\Delta), \Lambda^{(3)}(\Delta)$ and matrix operators $\boldsymbol{\Theta}^{(3)}(\Delta)$, $\mathbf{M}(\mathbf{D_x})$, $\mathbf{Y}^{(3)}(\mathbf{x})$ and $\mathbf{G}^{(3)}(\mathbf{x})$ are obtained as

(*i*)

$$\hat{\mathbf{N}}^{(3)}(\Delta) = \Big(\hat{N}^{(3)}_{gh}(\Delta)\Big)_{6\times 6} =$$

$$\begin{pmatrix} \tilde{\lambda} & -\Re_1 & -\Re_2 & -\Re_3 & \iota\omega\vartheta T_0 & 0 \\ \Re_1 & A_1\Delta-\alpha_1 & A_4\Delta-\alpha_4 & A_6\Delta-\alpha_6 & \iota\omega\ell_1 T_0 & \iota\omega r_1 \\ \Re_2 & A_4\Delta-\alpha_4 & A_2\Delta-\alpha_2 & A_5\Delta-\alpha_5 & \iota\omega\ell_2 T_0 & \iota\omega r_2 \\ \Re_3 & A_6\Delta-\alpha_6 & A_5\Delta-\alpha_5 & A_3\Delta-\alpha_3 & \iota\omega\ell_3 T_0 & \iota\omega r_3 \\ -\vartheta & \ell_1 & \ell_2 & \ell_3 & k\Delta+\iota\omega a & -\kappa_3 \\ 0 & -r_1\Delta & -r_2\Delta & -r_3\Delta & \kappa_1\Delta & \kappa_7\Delta+\kappa_8 \end{pmatrix}_{6\times 6}$$

$$\mathbf{N}^{(3)}(\Delta) = \Big(N^{(3)}_{gh}(\Delta)\Big)_{6\times 6} = \Delta\Big(\hat{N}^{(3)}_{gh}(\Delta)\Big)_{6\times 6},$$

(*ii*)

$$\Gamma^{(3)}(\Delta) = \Delta\prod_{i=1}^{5}(\Delta+\tilde{\mu}_i^2), \Lambda^{(3)}(\Delta) = \Delta(\Delta+\tilde{\mu}_6^2) =$$

$$\frac{1}{\tilde{N}}\begin{vmatrix} \tilde{\mu}\Delta & K^*\Delta \\ -K^* & \gamma\Delta-2K^* \end{vmatrix}, \tilde{\mu}_7^2 = -\frac{2K^*}{\tilde{\alpha}},$$

where $\tilde{\mu}_i^2,\ i = 1, ..., 5$ are the roots of the equation $|\hat{\mathbf{N}}^{(3)}(-m)| = 0$ (with respect to m).

(*iii*)

$$\boldsymbol{\Theta}^{(3)}(\Delta) = \Big(\Theta^{(3)}_{gh}(\Delta)\Big)_{13\times 13},$$

$$\Theta_{pp}^{(3)}(\Delta) = \Gamma^{(3)}(\Delta)\Lambda^{(3)}(\Delta) = \Delta^2\prod_{i=1}^{6}(\Delta+\tilde{\mu}_i^2),$$

$$\Theta_{p+3;p+3}^{(3)}(\Delta) = \Lambda^{(3)}(\Delta)(\Delta+\tilde{\mu}_7^2) = \Delta\prod_{i=6}^{7}(\Delta+\tilde{\mu}_i^2),$$

$$\Theta_{ll}^{(3)}(\Delta) = \Gamma^{(3)}(\Delta) = \Delta\prod_{i=1}^{5}(\Delta+\tilde{\mu}_i^2),$$

$$\Theta_{p+10;p+10}^{(3)}(\Delta) = \Gamma^{(3)}(\Delta)(\Delta+\tilde{\mu}_8^2) = \Delta\prod_{i=1}^{5,8}(\Delta+\tilde{\mu}_i^2),\ \tilde{\mu}_8^2 = \lambda_{10}^2,$$

$$\Theta_{gh}^{(3)}(\Delta) = 0;\ p = 1,2,3;\ \ g,h = 1,.....,13;\ \ l = 7,...,10;\ \ g \neq h$$

$$(iv)\ \ w_{11}^{(3)}(\Delta) = -\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[(\lambda+\mu)(\gamma\Delta-2K^*)-K^{*^2}\Bigg]\tilde{N}_{11}^{(3)}(\Delta)+(\gamma\Delta-2K^*)\times \Bigg[-\Re_i\tilde{N}_{1;i+1}^{(3)}(\Delta)+\iota\omega\vartheta T_0\tilde{N}_{15}^{(3)}(\Delta)\Bigg]\Bigg\},$$

$$w_{p+1;1}^{(3)}(\Delta) = -\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[(\lambda+\mu)(\gamma\Delta-2K^*)-K^{*^2}\Bigg]\tilde{N}_{p1}^{(3)}(\Delta)+(\gamma\Delta-2K^*)\times \Bigg[-\Re_i\tilde{N}_{p;i+1}^{(3)}(\Delta)+\iota\omega\vartheta T_0\tilde{N}_{p5}^{(3)}(\Delta)\Bigg]\Bigg\},$$

$$w_{21}^{(3)}(\Delta) = -\frac{K^*\Gamma^{(3)}(\Delta)}{\tilde{N}}, w_{12}^{(3)}(\Delta) = -\frac{K^*(\Delta+\tilde{\mu}_7^2)}{\tilde{N}},$$

$$w_{22}^{(3)}(\Delta) = -\frac{(\alpha+\beta)\tilde{\mu}\Delta-K^{*^2}}{\tilde{N}\tilde{\alpha}},$$

$$w_{1;l+1}^{(3)}(\Delta) = \frac{\tilde{N}_{1l}^{(3)}(\Delta)}{\tilde{M}}, w_{p+1;l+1}^{(3)}(\Delta) = \frac{\tilde{N}_{pl}^{(3)}(\Delta)}{\tilde{M}}, w_{p+1;2}^{(3)}(\Delta) = w_{2;p+1}^{(3)}(\Delta) = 0,$$

$$w_{17}^{(3)}(\Delta) = -\frac{1}{\tilde{M}\kappa_6}\Bigg[(\kappa_4+\kappa_5)\tilde{N}_{16}^{(3)}(\Delta)-r_i\tilde{N}_{1;i+1}^{(3)}(\Delta)+\kappa_1\tilde{N}_{15}^{(3)}(\Delta)\Bigg],$$

$$w_{p+1;7}^{(3)}(\Delta) = -\frac{1}{\tilde{M}\kappa_6}\Bigg[(\kappa_4+\kappa_5)\tilde{N}_{p6}^{(3)}(\Delta)-r_i\tilde{N}_{p;i+1}^{(3)}(\Delta)+\kappa_1\tilde{N}_{p5}^{(3)}(\Delta)\Bigg]$$

$$p = 2,....,6;\ \ l = 2,....,5$$

where $\tilde{N}_{ij}^{(3)}$ is the cofactor of the element $N_{ij}^{(3)}$ of the matrix $\mathbf{N}^{(3)}$.

$$(v)\mathbf{M}(\mathbf{D_x}) = \Big(M_{gh}(\mathbf{D_x})\Big)_{13\times13} = \begin{pmatrix} \mathbf{M}^{(1)} & \mathbf{M}^{(2)} & \mathbf{M}^{(5)} & \mathbf{M}^{(10)} \\ \mathbf{M}^{(3)} & \mathbf{M}^{(4)} & \mathbf{M}^{(6)} & \mathbf{M}^{(11)} \\ \mathbf{M}^{(7)} & \mathbf{M}^{(8)} & \mathbf{M}^{(9)} & \mathbf{M}^{(15)} \\ \mathbf{M}^{(12)} & \mathbf{M}^{(13)} & \mathbf{M}^{(14)} & \mathbf{M}^{(16)} \end{pmatrix}_{13\times13},$$

$$\mathbf{M}^{(i)}(\mathbf{D_x}) = \Big(M_{gh}^{(i)}(\mathbf{D_x})\Big)_{3\times3}, \mathbf{M}^{(j)}(\mathbf{D_x}) = \Big(M_{gh}^{(j)}(\mathbf{D_x})\Big)_{3\times4},$$

$$\mathbf{M}^{(l)}(\mathbf{D_x}) = \Big(M^{(l)}_{gh}(\mathbf{D_x})\Big)_{4\times 3}, \mathbf{M}^{(9)}(\mathbf{D_x}) = \Big(M^{(9)}_{gh}(\mathbf{D_x})\Big)_{4\times 4},$$

$$M^{(1)}_{gh}(\mathbf{D_x}) = \frac{1}{\tilde{N}}(\gamma\Delta - 2K^*)\Gamma^{(3)}(\Delta)\delta_{gh} + w^{(3)}_{11}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$M^{(2)}_{gh}(\mathbf{D_x}) = w^{(3)}_{12}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p}, M^{(3)}_{gh}(\mathbf{D_x}) = w^{(3)}_{21}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p},$$

$$M^{(4)}_{gh}(\mathbf{D_x}) = \frac{1}{\tilde{N}}(\Delta + \tilde{\mu}_7^2)\tilde{\mu}\Delta\,\delta_{gh} + w^{(3)}_{22}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$M^{(5)}_{gh}(\mathbf{D_x}) = w^{(3)}_{1;h+2}(\Delta)\frac{\partial}{\partial x_g}, M^{(q)}_{gh}(\mathbf{D_x}) = 0,$$

$$M^{(7)}_{gh}(\mathbf{D_x}) = w^{(3)}_{g+2;1}(\Delta)\frac{\partial}{\partial x_h}, M^{(9)}_{gh}(\mathbf{D_x}) = w^{(3)}_{g+2;h+2}(\Delta),$$

$$M^{(10)}_{gh}(\mathbf{D_x}) = w^{(3)}_{17}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}, M^{(12)}_{gh}(\mathbf{D_x}) = w^{(3)}_{71}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$M^{(14)}_{gh}(\mathbf{D_x}) = w^{(3)}_{7;h+2}(\Delta)\frac{\partial}{\partial x_g}, M^{(15)}_{gh}(\mathbf{D_x}) = w^{(3)}_{g+2;7}(\Delta)\frac{\partial}{\partial x_h},$$

$$M^{(16)}_{gh}(\mathbf{D_x}) = \frac{1}{\kappa_6}\Gamma^{(3)}(\Delta)\delta_{gh} + w^{(3)}_{77}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}$$

$$i = 1,2,3,4,10,11,12,13,16;\;\; j = 5,6,14;\;\; l = 7,8,15;\;\; q = 6,8,11,13.$$

(*vi*) $$\mathbf{Y}^{(3)}(\mathbf{x}) = \Big(Y^{(3)}_{ij}(\mathbf{x})\Big)_{13\times 13},\;\; Y^{(3)}_{pp}(\mathbf{x}) = r^{(3)}_{11}\varsigma_1^*(\mathbf{x}) + r^{(3)}_{12}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{6} r^{(3)}_{1;g+2}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{p+3;p+3}(\mathbf{x}) = r^{(3)}_{21}\varsigma_1^*(\mathbf{x}) + \sum_{g=6}^{7} r^{(3)}_{2;g-4}\tilde{\varsigma}_g(\mathbf{x}), Y^{(3)}_{ll}(\mathbf{x}) = r^{(3)}_{31}\varsigma_1^*(\mathbf{x}) + \sum_{g=1}^{5} r^{(3)}_{3;g+1}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{p+10;p+10}(\mathbf{x}) = r^{(3)}_{41}\varsigma_1^*(\mathbf{x}) + \sum_{g=1}^{5,8} r^{(3)}_{4;g+1}\tilde{\varsigma}_g(\mathbf{x}), Y^{(3)}_{ij}(\mathbf{x}) = 0$$

$$p = 1,2,3;\;\; l = 7,...,10;\;\; i,j = 1,.....,13;\;\; i \neq j,$$

where

$$\varsigma_1^*(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \varsigma_2^*(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \tilde{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\tilde{\mu}_g|\mathbf{x}|}}{4\pi|\mathbf{x}|};\quad g = 1,.....,8,$$

$$r^{(3)}_{11} = -\sum_{p=1}^{6}\Bigg(\prod_{j=1,j\neq p}^{6}\tilde{\mu}_j^2\Bigg)\prod_{i=1}^{6}\tilde{\mu}_i^{-4}, r^{(3)}_{12} = \prod_{i=1}^{6}\tilde{\mu}_i^{-2}, r^{(3)}_{1;l+2} = \tilde{\mu}_l^{-4}\prod_{i=1,i\neq l}^{6}(\tilde{\mu}_i^2 - \tilde{\mu}_l^2)^{-1},$$

$$r^{(3)}_{21} = \prod_{i=6}^{7}\tilde{\mu}_i^{-2}, r^{(3)}_{2;q-4} = -\tilde{\mu}_q^{-2}\prod_{i=6,i\neq q}^{7}(\tilde{\mu}_i^2 - \tilde{\mu}_q^2)^{-1}, r^{(3)}_{31} = \prod_{i=1}^{5}\tilde{\mu}_i^{-2},$$

$$r^{(3)}_{3;p+1} = -\tilde{\mu}_p^{-2}\prod_{i=1,i\neq p}^{5}(\tilde{\mu}_i^2 - \tilde{\mu}_p^2)^{-1},$$

$$r_{41}^{(3)} = \prod_{i=1}^{5,8} \tilde{\mu}_i^{-2}, r_{4;y+1}^{(3)} = -\tilde{\mu}_y^{-2} \prod_{i=1, i\neq y}^{5,8} (\tilde{\mu}_i^2 - \tilde{\mu}_y^2)^{-1},$$

$$Y_{ij}^{(3)}(\mathbf{x}) = 0;\;\; l = 1, ..., 6;\;\; q = 6, 7;\;\; p = 1, ..., 5;\;\; y = 1, ..., 5, 8.$$

By introducing the matrix

$$\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{M}(\mathbf{D_x})\mathbf{Y}^{(3)}(\mathbf{x}), \tag{5.2}$$

we obtain

$$\mathbf{F}^{(3)}(\mathbf{D_x})\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{F}^{(3)}(\mathbf{D_x})\mathbf{M}(\mathbf{D_x})\mathbf{Y}^{(3)}(\mathbf{x}) = \mathbf{\Theta}^{(3)}(\Delta)\mathbf{Y}^{(3)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(3)}(\mathbf{x})$ is a fundamental solution to equation (3.8) for $i = 3$.

Theorem 3: If condition (3.7) is satisfied, then the matrix $\mathbf{G}^{(3)}(\mathbf{x})$ defined by equation (5.2) is the fundamental solution of the system of equations (3.4).

5.3. **Equilibrium Theory.** In this case, the matrix $\mathbf{N}^{(4)}(\Delta)$, operators $\Gamma^{(4)}(\Delta), \Lambda^{(4)}(\Delta)$ and matrix operators $\mathbf{\Theta}^{(4)}(\Delta)$, $\mathbf{Z}(\mathbf{D_x})$, $\mathbf{Y}^{(4)}(\mathbf{x})$ and $\mathbf{G}^{(4)}(\mathbf{x})$ are obtained as

$$(i)\hat{\mathbf{N}}^{(4)}(\Delta) = \left(\hat{N}_{gh}^{(4)}(\Delta)\right)_{4\times 4} = \begin{pmatrix} \tilde{\lambda} & -\Re_1 & -\Re_2 & -\Re_3 \\ \Re_1 & A_1\Delta - \alpha_1 & A_4\Delta - \alpha_4 & A_6\Delta - \alpha_6 \\ \Re_2 & A_4\Delta - \alpha_4 & A_2\Delta - \alpha_2 & A_5\Delta - \alpha_5 \\ \Re_3 & A_6\Delta - \alpha_6 & A_5\Delta - \alpha_5 & A_3\Delta - \alpha_3 \end{pmatrix}_{4\times 4},$$

$$\mathbf{N}^{(4)}(\Delta) = \left(N_{gh}^{(4)}(\Delta)\right)_{4\times 4} = \Delta\left(\hat{N}_{gh}^{(4)}(\Delta)\right)_{4\times 4},$$

$$(ii)\Gamma^{(4)}(\Delta) = \Delta \prod_{i=1}^{3}(\Delta + \omega_i^2), \Lambda^{(4)}(\Delta) = \Delta(\Delta + \omega_4^2), \omega_4^2 = \tilde{\mu}_6^2, \omega_5^2 = \tilde{\mu}_7^2, \omega_6^2 = \frac{\kappa_1\kappa_3 - \kappa_2 k}{k\kappa_7}, \omega_7^2 = -\frac{\kappa_2}{\kappa_6},$$

where ω_i^2, $i = 1, 2, 3$ are the roots of the equation $|\hat{\mathbf{N}}^{(4)}(-m)| = 0$ (with respect to m).

$$(iii) \qquad \mathbf{\Theta}^{(4)}(\Delta) = \left(\Theta_{gh}^{(4)}(\Delta)\right)_{13\times 13},$$

$$\Theta_{pp}^{(4)}(\Delta) = \Gamma^{(4)}(\Delta)\Lambda^{(4)}(\Delta) = \Delta^2 \prod_{i=1}^{4}(\Delta + \omega_i^2),$$

$$\Theta_{p+3;p+3}^{(4)}(\Delta) = \Lambda^{(4)}(\Delta)(\Delta + \omega_5^2) = \Delta \prod_{i=4}^{5}(\Delta + \omega_i^2),$$

$$\Theta_{p+6;p+6}^{(4)}(\Delta) = \Gamma^{(4)}(\Delta) = \Delta \prod_{i=1}^{3}(\Delta + \omega_i^2),$$

$$\Theta^{(4)}_{10;10}(\Delta)=\Gamma^{(4)}(\Delta)\Delta(\Delta+\omega_6^2)=\Delta^2\prod_{i=1}^{3,6}(\Delta+\omega_i^2),$$

$$\Theta^{(4)}_{p+10;p+10}(\Delta)=\Gamma^{(4)}(\Delta)\Delta(\Delta+\omega_6^2)(\Delta+\omega_7^2)=\Delta^2\prod_{i=1}^{3,6,7}(\Delta+\omega_i^2),$$

$$\Theta^{(4)}_{gh}(\Delta)=0;\;\; p=1,2,3;\;\; g,h=1,.....,13;\;\; g\neq h.$$

$$(iv)\;\; w^{(4)}_{11}(\Delta)=-\frac{1}{\tilde{\lambda}\tilde{N}\varrho}\left\{\left[(\lambda+\mu)(\gamma\Delta-2K^*)-K^{*^2}\right]\tilde{N}^{(4)}_{11}(\Delta)-(\gamma\Delta-2K^*)\Re_i\tilde{N}^{(4)}_{1;i+1}(\Delta)\right\},$$

$$w^{(4)}_{l+1;1}(\Delta)=-\frac{1}{\tilde{\lambda}\tilde{N}\varrho}\left\{\left[(\lambda+\mu)(\gamma\Delta-2K^*)-K^{*^2}\right]\tilde{N}^{(4)}_{l1}(\Delta)-(\gamma\Delta-2K^*)\Re_i\tilde{N}^{(4)}_{l;i+1}(\Delta)\right\},$$

$$w^{(4)}_{21}(\Delta)=-\frac{K^*\Gamma^{(4)}(\Delta)}{\tilde{N}},w^{(4)}_{12}(\Delta)=-\frac{K^*(\Delta+\omega_5^2)}{\tilde{N}},w^{(4)}_{22}(\Delta)=w^{(3)}_{22}(\Delta),$$

$$w^{(4)}_{1;l+1}(\Delta)=\frac{\tilde{N}^{(4)}_{1l}(\Delta)}{\varrho\tilde{\lambda}},\;\; w^{(4)}_{p+1;l+1}(\Delta)=\frac{\tilde{N}^{(4)}_{pl}(\Delta)}{\varrho\tilde{\lambda}},$$

$$w^{(4)}_{16}(\Delta)=\frac{1}{k\kappa_7\tilde{\lambda}\varrho}\left[\kappa_3\Delta r_i\tilde{N}^{(4)}_{1;i+1}(\Delta)+(\kappa_7\Delta-\kappa_2)\left[\vartheta\tilde{N}^{(4)}_{11}(\Delta)-\ell_i\tilde{N}^{(4)}_{1;i+1}(\Delta)\right]\right],$$

$$w^{(4)}_{l+1;6}(\Delta)=\frac{1}{k\kappa_7\tilde{\lambda}\varrho}\left[\kappa_3\Delta r_i\tilde{N}^{(4)}_{l;i+1}(\Delta)+(\kappa_7\Delta-\kappa_2)\left[\vartheta\tilde{N}^{(4)}_{l1}(\Delta)-\ell_i\tilde{N}^{(4)}_{l;i+1}(\Delta)\right]\right],$$

$$w^{(4)}_{17}(\Delta)=\frac{(\Delta+\omega_7^2)}{k\kappa_7\tilde{\lambda}\varrho}\left[k\Delta r_i\tilde{N}^{(4)}_{1;i+1}(\Delta)-\kappa_1\left[\vartheta\tilde{N}^{(4)}_{11}(\Delta)-\ell_i\tilde{N}^{(4)}_{1;i+1}(\Delta)\right]\right],$$

$$w^{(4)}_{l+1;7}(\Delta)=\frac{(\Delta+\omega_7^2)}{k\kappa_7\tilde{\lambda}\varrho}\left[k\Delta r_i\tilde{N}^{(4)}_{l;i+1}(\Delta)-\kappa_1\left[\vartheta\tilde{N}^{(4)}_{l1}(\Delta)-\ell_i\tilde{N}^{(4)}_{l;i+1}(\Delta)\right]\right],$$

$$w^{(4)}_{67}(\Delta)=-\frac{\kappa_1\Gamma^{(4)}(\Delta)(\Delta+\omega_7^2)}{k\kappa_7},\;\; w^{(4)}_{76}(\Delta)=\frac{\kappa_3\Gamma^{(4)}(\Delta)}{k\kappa_7},$$

$$w^{(4)}_{66}(\Delta)=\frac{(\kappa_7\Delta-\kappa_2)\Gamma^{(4)}(\Delta)}{k\kappa_7},w^{(4)}_{77}(\Delta)=-\frac{\Gamma^{(4)}(\Delta)[\kappa_1\kappa_3+\Delta k(\kappa_4+\kappa_5)]}{k\kappa_6\kappa_7},$$

$$w^{(4)}_{q+1;2}(\Delta)=w^{(4)}_{2;q+1}(\Delta)=w^{(4)}_{6j}(\Delta)=w^{(4)}_{7j}(\Delta)=0,$$

$$p,l=2,3,4;\;\; j=1,3,4,5;\;\; q=2,....,6$$

where $\tilde{N}^{(4)}_{ij}$ is the cofactor of the element $N^{(4)}_{ij}$ of the matrix $\mathbf{N}^{(4)}$.

$$(v)\mathbf{Z}(\mathbf{D_x})=\Big(Z_{gh}(\mathbf{D_x})\Big)_{13\times13}=\begin{pmatrix}\mathbf{Z}^{(1)} & \mathbf{Z}^{(2)} & \mathbf{Z}^{(5)} & \mathbf{Z}^{(10)}\\ \mathbf{Z}^{(3)} & \mathbf{Z}^{(4)} & \mathbf{Z}^{(6)} & \mathbf{Z}^{(11)}\\ \mathbf{Z}^{(7)} & \mathbf{Z}^{(8)} & \mathbf{Z}^{(9)} & \mathbf{Z}^{(15)}\\ \mathbf{Z}^{(12)} & \mathbf{Z}^{(13)} & \mathbf{Z}^{(14)} & \mathbf{Z}^{(16)}\end{pmatrix}_{13\times13},$$

$$\mathbf{Z}^{(i)}(\mathbf{D_x})=\Big(Z^{(i)}_{gh}(\mathbf{D_x})\Big)_{3\times3},\mathbf{Z}^{(j)}(\mathbf{D_x})=\Big(Z^{(j)}_{gh}(\mathbf{D_x})\Big)_{3\times4},$$

$$\mathbf{Z}^{(l)}(\mathbf{D_x})=\Big(Z^{(l)}_{gh}(\mathbf{D_x})\Big)_{4\times3},\mathbf{Z}^{(9)}(\mathbf{D_x})=\Big(Z^{(9)}_{gh}(\mathbf{D_x})\Big)_{4\times4},$$

$$Z^{(1)}_{gh}(\mathbf{D_x}) = \frac{1}{\tilde{N}}(\gamma\Delta - 2K^*)\Gamma^{(4)}(\Delta)\delta_{gh} + w^{(4)}_{11}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$Z^{(2)}_{gh}(\mathbf{D_x}) = w^{(4)}_{12}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p}, Z^{(3)}_{gh}(\mathbf{D_x}) = w^{(4)}_{21}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p},$$

$$Z^{(4)}_{gh}(\mathbf{D_x}) = \frac{1}{\tilde{N}}(\Delta + \omega_5^2)\tilde{\mu}\Delta\,\delta_{gh} + w^{(4)}_{22}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$Z^{(5)}_{gh}(\mathbf{D_x}) = w^{(4)}_{1;h+2}(\Delta)\frac{\partial}{\partial x_g}, Z^{(q)}_{gh}(\mathbf{D_x}) = 0,$$

$$Z^{(7)}_{gh}(\mathbf{D_x}) = w^{(4)}_{g+2;1}(\Delta)\frac{\partial}{\partial x_h}, Z^{(9)}_{gh}(\mathbf{D_x}) = w^{(4)}_{g+2;h+2}(\Delta),$$

$$Z^{(10)}_{gh}(\mathbf{D_x}) = w^{(4)}_{17}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$Z^{(14)}_{gh}(\mathbf{D_x}) = w^{(4)}_{7;h+2}(\Delta)\frac{\partial}{\partial x_g}, Z^{(15)}_{gh}(\mathbf{D_x}) = w^{(4)}_{g+2;7}(\Delta)\frac{\partial}{\partial x_h},$$

$$Z^{(16)}_{gh}(\mathbf{D_x}) = \frac{1}{\kappa_6}\Delta(\Delta + \omega_6^2)\Gamma^{(4)}(\Delta)\delta_{gh} + w^{(4)}_{77}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}$$

$$i = 1,2,3,4,10,11,12,13,16;\;\; j = 5,6,14;\;\; l = 7,8,15;\;\; q = 6,8,11,12,13.$$

$$(vi) \quad \mathbf{Y}^{(4)}(\mathbf{x}) = \Big(Y^{(4)}_{ij}(\mathbf{x})\Big)_{13\times 13},\;\; Y^{(4)}_{pp}(\mathbf{x}) = r^{(4)}_{11}\varsigma^*_1(\mathbf{x}) + r^{(4)}_{12}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{4} r^{(4)}_{1;g+2}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{p+3;p+3}(\mathbf{x}) = r^{(4)}_{21}\varsigma^*_1(\mathbf{x}) + \sum_{g=4}^{5} r^{(4)}_{2;g-2}\hat{\varsigma}_g(\mathbf{x}), Y^{(4)}_{p+6;p+6}(\mathbf{x}) = r^{(4)}_{31}\varsigma^*_1(\mathbf{x}) + \sum_{g=1}^{3} r^{(4)}_{3;g+1}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{10;10}(\mathbf{x}) = r^{(4)}_{41}\varsigma^*_1(\mathbf{x}) + r^{(4)}_{42}\varsigma^*_2(\mathbf{x})\sum_{g=1}^{3,6} r^{(4)}_{4;g+2}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{p+10;p+10}(\mathbf{x}) = r^{(4)}_{51}\varsigma^*_1(\mathbf{x}) + r^{(4)}_{52}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{3,6,7} r^{(4)}_{5;g+2}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{ij}(\mathbf{x}) = 0;\;\; p = 1,2,3;\;\; i,j = 1,.....,13;\;\; i \neq j,$$

where

$$\hat{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\omega_g|\mathbf{x}|}}{4\pi|\mathbf{x}|};\quad g = 1,.....,7,$$

$$r^{(4)}_{11} = -\sum_{p=1}^{4}\Big(\prod_{j=1,j\neq p}^{4}\omega_j^2\Big)\prod_{i=1}^{4}\omega_i^{-4},$$

$$r^{(4)}_{12} = \prod_{i=1}^{4}\omega_i^{-2}, r^{(4)}_{1;l+2} = \omega_l^{-4}\prod_{i=1,i\neq l}^{4}(\omega_i^2 - \omega_l^2)^{-1},$$

$$r^{(4)}_{21} = \prod_{i=4}^{5}\omega_i^{-2}, r^{(4)}_{2;q-2} = -\omega_q^{-2}\prod_{i=4,i\neq q}^{5}(\omega_i^2 - \omega_q^2)^{-1}, r^{(4)}_{31} = \prod_{i=1}^{3}\omega_i^{-2},$$

$$r^{(4)}_{3;e+1} = -\omega_e^{-2} \prod_{i=1,i\neq e}^{3} (\omega_i^2 - \omega_e^2)^{-1},$$

$$r^{(4)}_{41} = -\sum_{p=1}^{3,6} \Bigg(\prod_{j=1,j\neq p}^{3,6} \omega_j^2 \Bigg) \prod_{i=1}^{3,6} \omega_i^{-4}, r^{(4)}_{42} = \prod_{i=1}^{3,6} \omega_i^{-2},$$

$$r^{(4)}_{4;z+2} = \omega_z^{-4} \prod_{i=1,i\neq z}^{3,6} (\omega_i^2 - \omega_z^2)^{-1}, r^{(4)}_{51} = -\sum_{p=1}^{3,6,7} \Bigg(\prod_{j=1,j\neq p}^{3,6,7} \omega_j^2 \Bigg) \prod_{i=1}^{3,6,7} \omega_i^{-4},$$

$$r^{(4)}_{52} = \prod_{i=1}^{3,6,7} \omega_i^{-2}, r^{(4)}_{5;y+2} = \omega_y^{-4} \prod_{i=1,i\neq y}^{3,6,7} (\omega_i^2 - \omega_y^2)^{-1},$$

$$l = 1, ..., 4; \quad q = 4, 5; \quad e = 1, 2, 3; \quad z = 1, 2, 3, 6; \quad y = 1, 2, 3, 6, 7.$$

If we introduce the matrix

$$\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}). \tag{5.3}$$

then, we obtain

$$\mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{Z}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}) = \mathbf{\Theta}^{(4)}(\Delta)\mathbf{Y}^{(4)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(4)}(\mathbf{x})$ is a solution to equation (3.8) for $i = 4$.

Theorem 4: If condition (3.7) is satisfied, then the matrix $\mathbf{G}^{(4)}(\mathbf{x})$ defined by equation (5.3) is the fundamental solution of the system of equations (3.5).

6. Basic Properties of $\mathbf{G}^{(1)}(\mathbf{x})$

Theorem 5: Each column of the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution of the system of equations (3.2) at every point $\mathbf{x} \in \mathrm{E}^3$ except the origin.

Theorem 6: If condition (3.7) is satisfied, then the fundamental solution of the system of equations $\tilde{\mathbf{F}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ is the matrix

$$\mathbf{W}(\mathbf{x}) = \Big(W_{gh}(\mathbf{x}) \Big)_{13\times 13} = \begin{pmatrix} \mathbf{W}^{(1)} & \mathbf{W}^{(2)} & \mathbf{W}^{(5)} & \mathbf{W}^{(10)} \\ \mathbf{W}^{(3)} & \mathbf{W}^{(4)} & \mathbf{W}^{(6)} & \mathbf{W}^{(11)} \\ \mathbf{W}^{(7)} & \mathbf{W}^{(8)} & \mathbf{W}^{(9)} & \mathbf{W}^{(15)} \\ \mathbf{W}^{(12)} & \mathbf{W}^{(13)} & \mathbf{W}^{(14)} & \mathbf{W}^{(16)} \end{pmatrix}_{13\times 13}, \tag{6.1}$$

where $\mathbf{W}^{(p)}(\mathbf{x})$, $p = 1, ..., 16$ are defined in Appendix E.

Lemma 2: If condition (3.7) is satisfied, then

$$\Delta w^{(1)}_{11}(\Delta) = \frac{1}{\tilde{M}} \Lambda^{(1)}(\Delta) \tilde{N}^{(1)}_{11}(\Delta) - \frac{1}{\tilde{N}} \Gamma^{(1)}(\Delta)(\gamma\Delta + \tilde{K}), \tag{6.2}$$

$$\Delta w^{(1)}_{22}(\Delta) = \frac{1}{\tilde{\alpha}} \left[\Lambda^{(1)}(\Delta) - \frac{1}{\tilde{N}} (\tilde{\alpha}\Delta + \tilde{K})(\tilde{\mu}\Delta + \rho\omega^2) \right], \tag{6.3}$$

$$\Delta w^{(1)}_{77}(\Delta) = \frac{1}{\tilde{M}} (\Delta + \lambda^2_{10}) \tilde{N}^{(1)}_{66}(\Delta) - \frac{1}{\kappa_6} \Gamma^{(1)}(\Delta). \tag{6.4}$$

Proof: Consider

$$w_{11}^{(1)}(\Delta) = -\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[(\lambda+\mu)(\gamma\Delta+\tilde{K}) - K^{*^2}\Bigg]\tilde{N}_{11}^{(1)}(\Delta) + (\gamma\Delta+\tilde{K})\times \\ \Bigg[-\Re_i\tilde{N}_{1;i+1}^{(1)}(\Delta) + \iota\omega\vartheta T_0\tilde{N}_{15}^{(1)}(\Delta)\Bigg]\Bigg\}.$$

Now

$$\Gamma^{(1)}(\Delta) = \frac{1}{\tilde{M}}\,|\mathbf{N}^{(1)}(\Delta)| = \\ \frac{1}{\tilde{M}}\Bigg\{[\tilde{\lambda}\Delta+\rho\omega^2]\tilde{N}_{11}^{(1)} - \Re_i\Delta\tilde{N}_{1;i+1}^{(1)} + \iota\omega\vartheta T_0\Delta\tilde{N}_{15}^{(1)}(\Delta)\Bigg\}.$$

Therefore,

$$\begin{aligned}\Delta w_{11}^{(1)}(\Delta) &= -\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[(\lambda+\mu)(\gamma\Delta+\tilde{K}) - K^{*^2}\Bigg]\Delta\tilde{N}_{11}^{(1)}(\Delta) + (\gamma\Delta+\tilde{K})\times \\ &\qquad \Bigg[-\Re_i\Delta\tilde{N}_{1;i+1}^{(1)}(\Delta) + \iota\omega\vartheta T_0\Delta\tilde{N}_{15}^{(1)}(\Delta)\Bigg]\Bigg\} \\ &= -\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[\tilde{M}\Gamma^{(1)}(\Delta) - (\tilde{\lambda}\Delta+\rho\omega^2)\tilde{N}_{11}^{(1)}\Bigg](\gamma\Delta+\tilde{K}) \\ &\qquad +\Bigg[(\lambda+\mu)(\gamma\Delta+\tilde{K}) - K^{*^2}\Bigg]\Delta\tilde{N}_{11}^{(1)}\Bigg\} \\ &= \frac{1}{\tilde{M}}\Lambda^{(1)}(\Delta)\tilde{N}_{11}^{(1)}(\Delta) - \frac{1}{\tilde{N}}\Gamma^{(1)}(\Delta)(\gamma\Delta+\tilde{K}).\end{aligned}$$

Also taking the R.H.S. of equation (6.3),

$$\begin{aligned}&\frac{1}{\tilde{N}\tilde{\alpha}}\Bigg\{\Bigg[(\tilde{\mu}\Delta+\rho\omega^2)(\gamma\Delta+\tilde{K}) + K^{*^2}\Delta\Bigg] - (\tilde{\alpha}\Delta+\tilde{K})(\tilde{\mu}\Delta+\rho\omega^2)\Bigg\} \\ &= -\frac{1}{\tilde{N}\tilde{\alpha}}\Delta\Bigg[(\alpha+\beta)(\tilde{\mu}\Delta+\rho\omega^2) - K^{*^2}\Bigg] = \Delta w_{22}^{(1)}(\Delta).\end{aligned}$$

Equation (6.4) can be proved in similar way.

Theorem 7: If condition (3.7) is satisfied and $\mathbf{x}\in \mathrm{E}^3 - \{\mathbf{0}\}$, then

$$\mathbf{H}^{(1)}(\mathbf{x}) = \nabla\,\mathrm{div}\sum_{j=1}^{6} x_{1j}\varsigma_j(\mathbf{x}) - \mathrm{curl}\,\mathrm{curl}\sum_{e=6}^{8} x_{1e}\varsigma_e(\mathbf{x}),$$

$$\mathbf{H}^{(2)}(\mathbf{x}) = \mathbf{H}^{(3)}(\mathbf{x}) = \mathrm{curl}\sum_{e=7}^{8} x_{2e}\varsigma_e(\mathbf{x}),$$

$$\mathbf{H}^{(4)}(\mathbf{x}) = \nabla\,\mathrm{div}\,[x_{49}\varsigma_9(\mathbf{x})] - \mathrm{curl}\,\mathrm{curl}\sum_{e=7}^{8} x_{4e}\varsigma_e(\mathbf{x}),$$

$$H_{iq}^{(5)}(\mathbf{x}) = \frac{\partial}{\partial x_i}\sum_{j=1}^{6} x_{5qj}\varsigma_j(\mathbf{x}), \mathbf{H}^{(p)}(\mathbf{x}) = \mathbf{0},$$

$$H_{qi}^{(7)}(\mathbf{x}) = \frac{\partial}{\partial x_i}\sum_{j=1}^{6} x_{7qj}\varsigma_j(\mathbf{x}), H_{ql}^{(9)}(\mathbf{x}) = \sum_{j=1}^{6} x_{9qlj}\varsigma_j(\mathbf{x}),$$

$$\mathbf{H}^{(10)}(\mathbf{x}) = \nabla\,\mathrm{div}\sum_{j=1}^{6} x_{10j}\varsigma_j(\mathbf{x}), \mathbf{H}^{(12)}(\mathbf{x}) = \nabla\,\mathrm{div}\sum_{j=1}^{6} x_{12j}\varsigma_j(\mathbf{x}),$$

$$H_{iq}^{(14)}(\mathbf{x}) = \frac{\partial}{\partial x_i}\sum_{j=1}^{6} x_{14qj}\varsigma_j(\mathbf{x}), H_{qi}^{(15)}(\mathbf{x}) = \frac{\partial}{\partial x_i}\sum_{j=1}^{6} x_{15qj}\varsigma_j(\mathbf{x}),$$

$$\mathbf{H}^{(16)}(\mathbf{x}) = \nabla\,\mathrm{div}\sum_{j=1}^{6} x_{16j}\varsigma_j(\mathbf{x}) + \mathrm{curl}\,\mathrm{curl}\,[x_{167}\varsigma_{10}(\mathbf{x})]$$

$$i = 1,2,3;\;\; q,l = 1,....,4;\;\; p = 6,8,11,13$$

where

$$x_{1j} = -\frac{r_{3j}^{(1)}}{\tilde{M}\lambda_j^2}\tilde{N}_{11}^{(1)}(-\lambda_j^2), x_{1e} = \frac{(-1)^{e+1}(\gamma\lambda_e^2 - \tilde{K})}{\tilde{N}\lambda_e^2(\lambda_8^2 - \lambda_7^2)}, x_{2e} = \frac{(-1)^e K^*}{\tilde{N}(\lambda_8^2 - \lambda_7^2)},$$

$$x_{4e} = \frac{(-1)^{e+1}(\tilde{\mu}\lambda_e^2 - \rho\omega^2)}{\tilde{N}\lambda_e^2(\lambda_8^2 - \lambda_7^2)}, x_{49} = -\frac{1}{\tilde{\alpha}\lambda_9^2}, x_{5qj} = \frac{r_{3j}^{(1)}}{\tilde{M}}\tilde{N}_{1;q+1}^{(1)}(-\lambda_j^2),$$

$$x_{7qj} = -\frac{r_{3j}^{(1)}}{\tilde{M}\lambda_j^2}\tilde{N}_{q+1;1}^{(1)}(-\lambda_j^2), x_{9qlj} = \frac{r_{3j}^{(1)}}{\tilde{M}}\tilde{N}_{q+1;l+1}^{(1)}(-\lambda_j^2),$$

$$x_{10j} = \frac{r_{3j}^{(1)}}{\tilde{M}}\tilde{N}_{16}^{(1)}(-\lambda_j^2), x_{12j} = -\frac{r_{3j}^{(1)}}{\tilde{M}\lambda_j^2}\tilde{N}_{61}^{(1)}(-\lambda_j^2),$$

$$x_{14qj} = \frac{r_{3j}^{(1)}}{\tilde{M}}\tilde{N}_{6;q+1}^{(1)}(-\lambda_j^2), x_{15qj} = -\frac{r_{3j}^{(1)}}{\tilde{M}\lambda_j^2}\tilde{N}_{q+1;6}^{(1)}(-\lambda_j^2),$$

$$x_{16j} = -\frac{r_{3j}^{(1)}}{\tilde{M}\lambda_j^2}\tilde{N}_{66}^{(1)}(-\lambda_j^2), \qquad x_{167} = \frac{1}{\kappa_6\lambda_{10}^2} \tag{6.5}$$

$$j = 1,...,6;\;\; q,l = 1,...,4;\;\; e = 7,8.$$

Proof: It follows from equation (4.37) that

$$\Delta\varsigma_j(\mathbf{x}) = -\lambda_j^2\varsigma_j(\mathbf{x});\;\; j = 1,....,10.$$

Thus, we have

$$-\frac{1}{\lambda_j^2}(\nabla\,\mathrm{div} - \mathrm{curl}\,\mathrm{curl})\varsigma_j(\mathbf{x}) = \mathbf{J}\,\varsigma_j(\mathbf{x}),\;\; \mathbf{x} \neq \mathbf{0}.$$

Consider

$$\mathbf{H}^{(1)}(\mathbf{x}) = \mathbf{R}^{(1)}(\mathbf{D_x})Y_{11}^{(1)}(\mathbf{x})$$

$$= \left\{\frac{1}{\tilde{N}}(\gamma\Delta + \tilde{K})\mathbf{J}\,\Gamma^{(1)}(\Delta) + w_{11}^{(1)}(\Delta)\,\nabla\,\mathrm{div}\right\}\sum_{j=1}^{8} r_{1j}^{(1)}\varsigma_j(\mathbf{x})$$

$$= \sum_{j=1}^{8} r_{1j}^{(1)} \Big\{ \Big[-\frac{1}{\tilde{N}\lambda_j^2}(-\gamma\lambda_j^2 + \tilde{K})\Gamma^{(1)}(-\lambda_j^2) + w_{11}^{(1)}(-\lambda_j^2) \Big] \nabla \operatorname{div} + \frac{1}{\tilde{N}\lambda_j^2}(-\gamma\lambda_j^2 + \tilde{K})\Gamma^{(1)}(-\lambda_j^2) \operatorname{curl}\operatorname{curl} \Big\} \varsigma_j(\mathbf{x}). \tag{6.6}$$

From equation (6.2), we have

$$w_{11}^{(1)}(-\lambda_j^2) = -\frac{1}{\tilde{M}\lambda_j^2}\Lambda^{(1)}(-\lambda_j^2)\tilde{N}_{11}^{(1)}(-\lambda_j^2) + \frac{1}{\tilde{N}\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)(-\gamma\lambda_j^2 + \tilde{K}).$$

Using the above equation in equation (6.6), we get

$$\mathbf{H}^{(1)}(\mathbf{x}) = \sum_{j=1}^{8} r_{1j}^{(1)} \Big\{ \Big[-\frac{1}{\tilde{M}\lambda_j^2}\Lambda^{(1)}(-\lambda_j^2)\tilde{N}_{11}^{(1)}(-\lambda_j^2) \Big] \nabla \operatorname{div} + \frac{1}{\tilde{N}\lambda_j^2}(-\gamma\lambda_j^2 + \tilde{K})\Gamma^{(1)}(-\lambda_j^2) \operatorname{curl}\operatorname{curl} \Big\} \varsigma_j(\mathbf{x}). \tag{6.7}$$

Now,

$$\Gamma^{(1)}(-\lambda_j^2) r_{1j}^{(1)} = 0; \quad j = 1, ..., 6$$

$$\Gamma^{(1)}(-\lambda_j^2) r_{1j}^{(1)} = \frac{(-1)^{j+1}}{\lambda_8^2 - \lambda_7^2}; \quad j = 7, 8$$

and

$$\begin{aligned} \Lambda^{(1)}(-\lambda_j^2) r_{1j}^{(1)} &= r_{3j}^{(1)}; \quad j = 1, ..., 6 \\ \Lambda^{(1)}(-\lambda_j^2) r_{1j}^{(1)} &= 0; \quad j = 7, 8. \end{aligned} \tag{6.8}$$

By virtue of equation (6.8), equation (6.7) becomes

$$\begin{aligned} \mathbf{H}^{(1)}(\mathbf{x}) &= \nabla \operatorname{div} \sum_{j=1}^{6} \Big[-\frac{1}{\tilde{M}\lambda_j^2} r_{3j}^{(1)} \tilde{N}_{11}^{(1)}(-\lambda_j^2) \Big] \varsigma_j(\mathbf{x}) + \\ &\quad \operatorname{curl}\operatorname{curl} \sum_{e=7}^{8} \frac{(-1)^{e+1}(-\gamma\lambda_e^2 + \tilde{K})}{\tilde{N}\lambda_e^2(\lambda_8^2 - \lambda_7^2)} \varsigma_e(\mathbf{x}) \\ &= \nabla \operatorname{div} \sum_{j=1}^{6} x_{1j}\varsigma_j(\mathbf{x}) - \operatorname{curl}\operatorname{curl} \sum_{e=7}^{8} x_{1e}\varsigma_e(\mathbf{x}). \end{aligned}$$

The remaining formulae of the above theorem can be proved in a similar way.

Lemma 3: If condition (3.7) is satisfied, then

$$\sum_{j=1}^{6} r_{3j}^{(1)} = \sum_{j=1}^{6} r_{3j}^{(1)}\lambda_j^2 = \sum_{j=1}^{6} r_{3j}^{(1)}\lambda_j^4 = \sum_{j=1}^{6} r_{3j}^{(1)}\lambda_j^6 = \sum_{j=1}^{6} r_{3j}^{(1)}\lambda_j^8 = 0,$$

$$\sum_{j=1}^{6} r_{3j}^{(1)}\lambda_j^{10} = -1, \sum_{j=1}^{6} \frac{r_{3j}^{(1)}}{\lambda_j^2} = \prod_{i=1}^{6} \lambda_i^{-2} = \frac{\tilde{M}}{\rho\omega^2 \tilde{N}_{11}^{(1)}(0)} = \frac{\tilde{M}}{\kappa_8 \tilde{N}_{66}^{(1)}(0)}, \tag{6.9}$$

and

$$\sum_{j=1}^{6} x_{1j} = -(\rho\omega^2)^{-1}, \sum_{j=1}^{6} x_{1j}\lambda_j^2 = -\tilde{\lambda}^{-1}, \sum_{e=7}^{8} x_{1e}\lambda_e^2 = -\tilde{\mu}^{-1},$$

$$\sum_{j=1}^{6} x_{16j} = -\kappa_8^{-1}, \sum_{j=1}^{6} x_{16j}\lambda_j^2 = -\kappa_7^{-1}, \sum_{j=1}^{6} x_{911j} = \frac{A_2A_3 - A_5^2}{\varrho},$$

$$\sum_{j=1}^{6} x_{922j} = \frac{A_1A_3 - A_6^2}{\varrho}, \sum_{j=1}^{6} x_{933j} = \frac{A_1A_2 - A_4^2}{\varrho}, \sum_{j=1}^{6} x_{944j} = k^{-1}. \tag{6.10}$$

Proof: Consider

$$\tilde{N}_{11}^{(1)}(-\lambda_j^2) = -k\kappa_7\varrho\lambda_j^{10} + B_1\lambda_j^8 + B_2\lambda_j^6 + B_3\lambda_j^4 + B_4\lambda_j^2 + \tilde{N}_{11}^{(1)}(0), \tag{6.11}$$

where B_p, $p = 1, ..., 4$ are coefficients, independent of λ_j and skipped due to lengthy calculations.

Using equation (4.28), relations (6.9) can be proved by direct calculations.

From equations (6.9) and (6.11), we get

$$\sum_{j=1}^{6} \frac{r_{3j}^{(1)}}{\lambda_j^2}\tilde{N}_{11}^{(1)}(-\lambda_j^2) = \sum_{j=1}^{6} r_{3j}^{(1)}[-k\kappa_7\varrho\lambda_j^8 + B_1\lambda_j^6 + B_2\lambda_j^4 + B_3\lambda_j^2 + B_4 + \tilde{N}_{11}^{(1)}(0)\lambda_j^{-2}]$$

$$= \tilde{N}_{11}^{(1)}(0)\sum_{j=1}^{6} \frac{r_{3j}^{(1)}}{\lambda_j^2} = \frac{\tilde{M}}{\rho\omega^2},$$

and

$$\sum_{j=1}^{6} r_{3j}^{(1)}\tilde{N}_{11}^{(1)}(-\lambda_j^2) = \sum_{j=1}^{6} r_{3j}^{(1)}\big[-k\kappa_7\varrho\lambda_j^{10} + B_1\lambda_j^8 + B_2\lambda_j^6 + B_3\lambda_j^4 + B_4\lambda_j^2 + \tilde{N}_{11}^{(1)}(0)\big] = k\kappa_7\varrho.$$

Therefore, from equation (6.5), we have

$$\sum_{j=1}^{6} x_{1j} = -\sum_{j=1}^{6} \frac{r_{3j}^{(1)}}{\tilde{M}\lambda_j^2}\tilde{N}_{11}^{(1)}(-\lambda_j^2) = -(\rho\omega^2)^{-1},$$

$$\sum_{j=1}^{6} x_{1j}\lambda_j^2 = -\sum_{j=1}^{6} \frac{r_{3j}^{(1)}}{\tilde{M}}\tilde{N}_{11}^{(1)}(-\lambda_j^2) = -\frac{k\kappa_7\varrho}{\tilde{M}} = -\tilde{\lambda}^{-1}.$$

Also, we obtain

$$\sum_{e=7}^{8} x_{1e}\lambda_e^2 = -\frac{\gamma}{\tilde{N}} = -\tilde{\mu}^{-1}.$$

Similarly, we can prove the remaining results of equation (6.10).

Theorem 8: The relations

$$G^{(1)}_{pq}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|);\ \ p,q = 1,....,13. \tag{6.12}$$

hold in the neighborhood of the origin.

Proof: Consider

$$\mathbf{H}^{(1)}(\mathbf{x}) - \mathbf{W}^{(1)}(\mathbf{x}) = \nabla \operatorname{div} \tilde{Y}_{11}(\mathbf{x}) - \operatorname{curl}\operatorname{curl} \tilde{Y}_{22}(\mathbf{x}), \tag{6.13}$$

where

$$\tilde{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{6} x_{1j}\varsigma_j(\mathbf{x}) - \frac{\varsigma_2^*(\mathbf{x})}{\tilde{\lambda}},$$

$$\tilde{Y}_{22}(\mathbf{x}) = \sum_{e=7}^{8} x_{1e}\varsigma_e(\mathbf{x}) - \frac{\varsigma_2^*(\mathbf{x})}{\tilde{\mu}}. \tag{6.14}$$

From equation (6.14), we have

$$\tilde{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{6} \frac{-x_{1j}}{4\pi} \sum_{g=0}^{\infty} \frac{\iota^g \lambda_j^g}{g!} |\mathbf{x}|^{g-1} + \frac{|\mathbf{x}|}{8\pi\tilde{\lambda}}$$

$$= -\frac{1}{8\pi}\left[2\sum_{j=1}^{6} x_{1j} \sum_{g=0}^{\infty} \frac{\iota^g \lambda_j^g}{g!} |\mathbf{x}|^{g-1} - \frac{|\mathbf{x}|}{\tilde{\lambda}}\right]$$

$$= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{j=1}^{6} x_{1j} - |\mathbf{x}|\left(\sum_{j=1}^{6} x_{1j}\lambda_j^2 + \frac{1}{\tilde{\lambda}}\right)\right] - \frac{\iota}{4\pi}\sum_{j=1}^{6} x_{1j}\lambda_j + \tilde{Y}_{33}(\mathbf{x}). \tag{6.15}$$

Similarly,

$$\tilde{Y}_{22}(\mathbf{x}) = -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{e=7}^{8} x_{1e} - |\mathbf{x}|\left(\sum_{e=7}^{8} x_{1e}\lambda_e^2 + \frac{1}{\tilde{\mu}}\right)\right] - \frac{\iota}{4\pi}\sum_{e=7}^{8} x_{1e}\lambda_e + \tilde{Y}_{44}(\mathbf{x}), \tag{6.16}$$

where

$$\tilde{Y}_{33}(\mathbf{x}) = -\frac{1}{4\pi}\sum_{j=1}^{6} x_{1j} \sum_{g=3}^{\infty} \frac{\iota^g \lambda_j^g}{g!} |\mathbf{x}|^{g-1},$$

$$\tilde{Y}_{44}(\mathbf{x}) = -\frac{1}{4\pi}\sum_{e=7}^{8} x_{1e} \sum_{g=3}^{\infty} \frac{\iota^g \lambda_e^g}{g!} |\mathbf{x}|^{g-1}. \tag{6.17}$$

Clearly

$$\tilde{Y}_{qq}(\mathbf{x}) = O(|\mathbf{x}|^2),\ \frac{\partial}{\partial x_e}\tilde{Y}_{qq}(\mathbf{x}) = O(|\mathbf{x}|),$$

$$\frac{\partial^2}{\partial x_e \partial x_i}\tilde{Y}_{qq}(\mathbf{x}) = \ \text{constant} + O(|\mathbf{x}|);\ \ e,i = 1,2,3;\ \ q = 3,4. \tag{6.18}$$

Consider

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{x_i}{|\mathbf{x}|^3},\ \ \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = \left[\frac{3x_i^2}{|\mathbf{x}|^5} - \frac{1}{|\mathbf{x}|^3}\right]$$

Hence,

$$\Delta\frac{1}{|\mathbf{x}|}=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right)=0.$$

Therefore,

$$(\nabla\,\mathrm{div}-\mathrm{curl}\,\mathrm{curl})\frac{1}{|\mathbf{x}|}=\mathbf{J}\Delta\frac{1}{|\mathbf{x}|}=\mathbf{0}. \tag{6.19}$$

Equation (6.13) with the aid of equations (6.10) and (6.15)-(6.19) becomes

$$\mathbf{H}^{(1)}(\mathbf{x})-\mathbf{W}^{(1)}(\mathbf{x})=\nabla\,\mathrm{div}\,\tilde{Y}_{33}(\mathbf{x})-\mathrm{curl}\,\mathrm{curl}\,\tilde{Y}_{44}(\mathbf{x})=\mathrm{constant}+O(|\mathbf{x}|).$$

Similarly other formulae of equation (6.12) can be proved.

Therefore, matrix $\mathbf{W}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{G}^{(1)}(\mathbf{x})$ in the neighborhood of the origin.

7. Conclusions:

This paper focuses on deriving the linear theory of micropolar thermoelasticity with triple porosity and microtemperatures, without relying on Darcy's law. The main achievements and conclusions of the study can be summarized as follows:

1. The governing equations for micropolar thermoelasticity with triple porosity and microtemperatures are reduced in an isotropic medium. This reduction simplifies the analysis and allows for a more accessible understanding of the material behavior.
2. The fundamental solution, denoted as $\mathbf{G}^{(1)}(\mathbf{x})$, is derived for the system of equations (3.2) describing steady oscillations. This solution, expressed in terms of elementary functions, provides insights into the material's response under steady-state conditions.
3. Additionally, the fundamental solutions $\mathbf{G}^{(i)}(\mathbf{x})$, $i = 2, 3, 4$ for the system of equations (3.3)-(3.5) in the cases of pseudo-static oscillations, quasi-static oscillations, and equilibrium are also obtained.
4. The derived fundamental solution $\mathbf{G}^{(1)}(\mathbf{x})$ enables the investigation of three-dimensional boundary value problems in the theory of triple porosity micropolar thermoelastic solids with microtemperatures using the potential method. This method provides a valuable approach for analyzing and solving boundary value problems within the linear theory of micropolar thermoelasticity with triple porosity and microtemperatures.

In summary, this paper's achievements in deriving the fundamental solutions for the linear theory of micropolar thermoelasticity with triple porosity and microtemperatures contribute to the understanding and analysis of these materials' behavior. The results obtained provide a foundation for further research and practical applications in the field of micropolar thermoelasticity with triple porosity and microtemperatures.

Appendix A. Coefficients for the differential operator (3.6)

$$F^{(1)}_{pq}(\mathbf{D_x}) = [\tilde{\mu}\Delta + \rho\omega^2]\delta_{pq} + (\lambda+\mu)\frac{\partial^2}{\partial x_p \partial x_q}, F^{(1)}_{p;q+3}(\mathbf{D_x}) = F^{(1)}_{p+3;q}(\mathbf{D_x}) =$$

$$K^* \sum_{j=1}^{3} \varepsilon_{pjq}\frac{\partial}{\partial x_j}, F^{(1)}_{p;q+6}(\mathbf{D_x}) = -F^{(1)}_{q+6;p}(\mathbf{D_x}) = \Re_q \frac{\partial}{\partial x_p},$$

$$F^{(1)}_{p;10}(\mathbf{D_x}) = -\vartheta\frac{\partial}{\partial x_p}, F^{(1)}_{p;q+10}(\mathbf{D_x}) = 0,$$

$$F^{(1)}_{p+3;q+3}(\mathbf{D_x}) = (\gamma\Delta + \tilde{K})\delta_{pq} + (\alpha+\beta)\frac{\partial^2}{\partial x_p \partial x_q}, F^{(1)}_{p+3;l}(\mathbf{D_x}) = F^{(1)}_{l;p+3}(\mathbf{D_x}) = 0,$$

$$F^{(1)}_{p+6;p+6}(\mathbf{D_x}) = A_p\Delta - \beta_p, F^{(1)}_{78}(\mathbf{D_x}) = F^{(1)}_{87}(\mathbf{D_x}) = A_4\Delta - \alpha_4,$$

$$F^{(1)}_{79}(\mathbf{D_x}) = F^{(1)}_{97}(\mathbf{D_x}) = A_6\Delta - \alpha_6, F^{(1)}_{89}(\mathbf{D_x}) = F^{(1)}_{98}(\mathbf{D_x}) = A_5\Delta - \alpha_5,$$

$$F^{(1)}_{p+6;10}(\mathbf{D_x}) = \ell_p, F^{(1)}_{p+6;q+10}(\mathbf{D_x}) = -r_p\frac{\partial}{\partial x_q}, F^{(1)}_{10;q}(\mathbf{D_x}) = \iota\omega\vartheta T_0\frac{\partial}{\partial x_q},$$

$$F^{(1)}_{10;q+6}(\mathbf{D_x}) = \iota\omega\ell_q T_0, F^{(1)}_{10;10}(\mathbf{D_x}) = k\Delta + \iota\omega a,$$

$$F^{(1)}_{10;q+10}(\mathbf{D_x}) = \kappa_1\frac{\partial}{\partial x_q}, F^{(1)}_{p+10;q}(\mathbf{D_x}) = 0, F^{(1)}_{p+10;q+6}(\mathbf{D_x}) = \iota\omega r_q\frac{\partial}{\partial x_p},$$

$$F^{(1)}_{p+10;10}(\mathbf{D_x}) = -\kappa_3\frac{\partial}{\partial x_p}, F^{(1)}_{p+10;q+10}(\mathbf{D_x}) = (\kappa_6\Delta + \kappa_8)\delta_{pq} + (\kappa_4+\kappa_5)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$\tilde{\mu} = \mu + K^*, \qquad \tilde{K} = -2K^* + \rho J\omega^2, \qquad \kappa_8 = -\kappa_2 + \iota\omega\chi$$

$$p,q = 1,2,3;\ \ l = 7, ..., 13.$$

$$\tilde{F}_{pq}(\mathbf{D_x}) = \tilde{\mu}\Delta\delta_{pq} + (\lambda+\mu)\frac{\partial^2}{\partial x_p \partial x_q}, \tilde{F}_{p+3;q+3}(\mathbf{D_x}) = \gamma\Delta\delta_{pq} + (\alpha+\beta)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$\tilde{F}_{p+6;p+6}(\mathbf{D_x}) = A_p\Delta, \tilde{F}_{78}(\mathbf{D_x}) = \tilde{F}_{87}(\mathbf{D_x}) = A_4\Delta, \tilde{F}_{79}(\mathbf{D_x}) = \tilde{F}_{97}(\mathbf{D_x}) = A_6\Delta,$$

$$\tilde{F}_{89}(\mathbf{D_x}) = \tilde{F}_{98}(\mathbf{D_x}) = A_5\Delta, \tilde{F}_{10;10}(\mathbf{D_x}) = k\Delta,$$

$$\tilde{F}_{p+10;q+10}(\mathbf{D_x}) = \kappa_6\Delta\delta_{pq} + (\kappa_4+\kappa_5)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$\tilde{F}_{pi}(\mathbf{D_x}) = \tilde{F}_{ip}(\mathbf{D_x}) = \tilde{F}_{p+3;j}(\mathbf{D_x}) = \tilde{F}_{j;p+3}(\mathbf{D_x}) = 0,$$

$$\tilde{F}_{p+6;l}(\mathbf{D_x}) = \tilde{F}_{l;p+6}(\mathbf{D_x}) = \tilde{F}_{10;q+10}(\mathbf{D_x}) = \tilde{F}_{q+10;10}(\mathbf{D_x}) = 0,$$

$$p,q = 1,2,3;\ \ j = 7, ..., 13;\ \ i = 4, ..., 13;\ \ l = 10, 11, 12, 13.$$

Appendix B. Some quantities in equations (4.12), (4.13), (4.17), (4.22) and (4.23)

$\mathbf{S} = (\operatorname{div}\mathbf{u}, \boldsymbol{\phi}, \theta, \operatorname{div}\mathbf{w})$, $\tilde{\mathbf{Q}} = (w_1, ..., w_6) = (\operatorname{div}\mathbf{H}, L_i, Z, \operatorname{div}\mathbf{X})$ and

$$\mathbf{N}^{(1)}(\Delta) = \left(N_{gh}^{(1)}(\Delta)\right)_{6\times 6} =$$

$$= \begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 & -\Re_1\Delta & -\Re_2\Delta & -\Re_3\Delta & \iota\omega\vartheta T_0\Delta & 0 \\ \Re_1 & A_1\Delta - \beta_1 & A_4\Delta - \alpha_4 & A_6\Delta - \alpha_6 & \iota\omega\ell_1 T_0 & \iota\omega r_1 \\ \Re_2 & A_4\Delta - \alpha_4 & A_2\Delta - \beta_2 & A_5\Delta - \alpha_5 & \iota\omega\ell_2 T_0 & \iota\omega r_2 \\ \Re_3 & A_6\Delta - \alpha_6 & A_5\Delta - \alpha_5 & A_3\Delta - \beta_3 & \iota\omega\ell_3 T_0 & \iota\omega r_3 \\ -\vartheta & \ell_1 & \ell_2 & \ell_3 & k\Delta + \iota\omega a & -\kappa_3 \\ 0 & -r_1\Delta & -r_2\Delta & -r_3\Delta & \kappa_1\Delta & \kappa_7\Delta + \kappa_8 \end{pmatrix}_{6\times 6},$$

$$\boldsymbol{\Psi} = (\Psi_1, ..., \Psi_6), \Psi_p = \frac{1}{\tilde{M}}\sum_{i=1}^{6}\tilde{N}_{ip}^{(1)}(\Delta)w_i,$$

$$\Gamma^{(1)}(\Delta) = \frac{1}{\tilde{M}}\,|\mathbf{N}^{(1)}(\Delta)|,\ \ \tilde{M} = \tilde{\lambda}k\kappa_7\varrho \ \ p = 1, ..., 6$$

and $\tilde{N}_{ip}^{(1)}$ is the cofactor of the element $N_{ip}^{(1)}$ of the matrix $\mathbf{N}^{(1)}$.

$$\Lambda^{(1)}(\Delta) = \frac{1}{\tilde{N}}\begin{vmatrix} \tilde{\mu}\Delta + \rho\omega^2 & K^*\Delta \\ -K^* & \gamma\Delta + \tilde{K} \end{vmatrix}, \tilde{N} = \gamma\tilde{\mu},$$

$$\boldsymbol{\Psi}' = \frac{1}{\tilde{N}}\Big\{-\Big[(\gamma\Delta + \tilde{K})(\lambda + \mu) - {K^*}^2\Big]\nabla\,\Psi_1 + (\gamma\Delta + \tilde{K})\times$$

$$\Big[\Gamma^{(1)}(\Delta)\mathbf{H} + \Re_i\,\nabla\,\Psi_{i+1} - \iota\omega\vartheta T_0\,\nabla\,\Psi_5\Big] - K^*\Gamma^{(1)}(\Delta)\operatorname{curl}\mathbf{V}\Big\}.$$

$$\boldsymbol{\Psi}'' = \frac{1}{\tilde{N}}\Big\{-\Big[(\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2) - {K^*}^2\Big]\nabla\,\Psi_7 +$$

$$(\tilde{\mu}\Delta + \rho\omega^2)(\Delta + \lambda_9^2)\mathbf{V} - K^*(\Delta + \lambda_9^2)\operatorname{curl}\mathbf{H}\Big\}.$$

$$\lambda_{10}^2 = \frac{\kappa_8}{\kappa_6},\quad \boldsymbol{\Psi}''' = \frac{1}{\kappa_6}\Big\{\Gamma^{(1)}(\Delta)\mathbf{X} - \nabla\Big[(\kappa_4 + \kappa_5)\Psi_6 - r_i\,\Psi_{i+1} + \kappa_1\,\Psi_5\Big]\Big\}.$$

Appendix C. Some quantities in equations (4.24),(4.25) and (4.26)

$$\hat{\boldsymbol{\Psi}} = (\boldsymbol{\Psi}', \boldsymbol{\Psi}'', \Psi_2, ..., \Psi_5, \boldsymbol{\Psi}'''),$$

$$\boldsymbol{\Theta}^{(1)}(\Delta) = \left(\Theta_{gh}^{(1)}(\Delta)\right)_{13\times 13},$$

$$\Theta_{pp}^{(1)}(\Delta) = \Gamma^{(1)}(\Delta)\Lambda^{(1)}(\Delta) = \prod_{i=1}^{8}(\Delta + \lambda_i^2),$$

$$\Theta^{(1)}_{p+3;p+3}(\Delta)=\Lambda^{(1)}(\Delta)(\Delta+\lambda_9^2)=\prod_{i=7}^{9}(\Delta+\lambda_i^2),$$

$$\Theta^{(1)}_{ll}(\Delta)=\Gamma^{(1)}(\Delta)=\prod_{i=1}^{6}(\Delta+\lambda_i^2),$$

$$\Theta^{(1)}_{p+10;p+10}(\Delta)=\Gamma^{(1)}(\Delta)(\Delta+\lambda_{10}^2)=\prod_{i=1}^{6,10}(\Delta+\lambda_i^2),\Theta^{(1)}_{gh}(\Delta)=0,$$

$$p=1,2,3\ g,h=1,.....,13\ l=7,....,10\ g\neq h.$$

$$w^{(1)}_{11}(\Delta)=-\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[(\lambda+\mu)(\gamma\Delta+\tilde{K})-{K^*}^2\Bigg]\tilde{N}^{(1)}_{11}(\Delta)+(\gamma\Delta+\tilde{K})\times$$
$$\Bigg[-\Re_i\tilde{N}^{(1)}_{1;i+1}(\Delta)+\iota\omega\vartheta T_0\tilde{N}^{(1)}_{15}(\Delta)\Bigg]\Bigg\},$$

$$w^{(1)}_{p+1;1}(\Delta)=-\frac{1}{\tilde{M}\tilde{N}}\Bigg\{\Bigg[(\lambda+\mu)(\gamma\Delta+\tilde{K})-{K^*}^2\Bigg]\tilde{N}^{(1)}_{p1}(\Delta)+(\gamma\Delta+\tilde{K})\times$$
$$\Bigg[-\Re_i\tilde{N}^{(1)}_{p;i+1}(\Delta)+\iota\omega\vartheta T_0\tilde{N}^{(1)}_{p5}(\Delta)\Bigg]\Bigg\},$$

$$w^{(1)}_{21}(\Delta)=-\frac{K^*\Gamma^{(1)}(\Delta)}{\tilde{N}},w^{(1)}_{12}(\Delta)=-\frac{K^*(\Delta+\lambda_9^2)}{\tilde{N}},$$

$$w^{(1)}_{22}(\Delta)=-\frac{(\alpha+\beta)(\tilde{\mu}\Delta+\rho\omega^2)-{K^*}^2}{\tilde{N}\tilde{\alpha}},w^{(1)}_{1;l+1}(\Delta)=\frac{\tilde{N}^{(1)}_{1l}(\Delta)}{\tilde{M}},$$

$$w^{(1)}_{p+1;l+1}(\Delta)=\frac{\tilde{N}^{(1)}_{pl}(\Delta)}{\tilde{M}},w^{(1)}_{2;p+1}(\Delta)=w^{(1)}_{p+1;2}(\Delta)=0,$$

$$w^{(1)}_{17}(\Delta)=-\frac{1}{\tilde{M}\kappa_6}\Bigg[(\kappa_4+\kappa_5)\tilde{N}^{(1)}_{16}(\Delta)-r_i\tilde{N}^{(1)}_{1;i+1}(\Delta)+\kappa_1\tilde{N}^{(1)}_{15}(\Delta)\Bigg],$$

$$w^{(1)}_{p+1;7}(\Delta)=-\frac{1}{\tilde{M}\kappa_6}\Bigg[(\kappa_4+\kappa_5)\tilde{N}^{(1)}_{p6}(\Delta)-r_i\tilde{N}^{(1)}_{p;i+1}(\Delta)+\kappa_1\tilde{N}^{(1)}_{p5}(\Delta)\Bigg],$$

$$p=2,....,6;\ \ l=2,....,5.$$

$$\mathbf{R}(\mathbf{D_x})=\Big(R_{gh}(\mathbf{D_x})\Big)_{13\times13}=\begin{pmatrix}\mathbf{R}^{(1)} & \mathbf{R}^{(2)} & \mathbf{R}^{(5)} & \mathbf{R}^{(10)}\\ \mathbf{R}^{(3)} & \mathbf{R}^{(4)} & \mathbf{R}^{(6)} & \mathbf{R}^{(11)}\\ \mathbf{R}^{(7)} & \mathbf{R}^{(8)} & \mathbf{R}^{(9)} & \mathbf{R}^{(15)}\\ \mathbf{R}^{(12)} & \mathbf{R}^{(13)} & \mathbf{R}^{(14)} & \mathbf{R}^{(16)}\end{pmatrix}_{13\times13},$$

$$\mathbf{R}^{(i)}(\mathbf{D_x})=\Big(R^{(i)}_{gh}(\mathbf{D_x})\Big)_{3\times3},\mathbf{R}^{(j)}(\mathbf{D_x})=\Big(R^{(j)}_{gh}(\mathbf{D_x})\Big)_{3\times4},$$

$$\mathbf{R}^{(l)}(\mathbf{D_x})=\Big(R^{(l)}_{gh}(\mathbf{D_x})\Big)_{4\times3},\mathbf{R}^{(9)}(\mathbf{D_x})=\Big(R^{(9)}_{gh}(\mathbf{D_x})\Big)_{4\times4},$$

$$R^{(1)}_{gh}(\mathbf{D_x})=\frac{1}{\tilde{N}}(\gamma\Delta+\tilde{K})\Gamma^{(1)}(\Delta)\delta_{gh}+w^{(1)}_{11}(\Delta)\frac{\partial^2}{\partial x_g\partial x_h},$$

$$R_{gh}^{(2)}(\mathbf{D_x}) = w_{12}^{(1)}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p}, R_{gh}^{(3)}(\mathbf{D_x}) = w_{21}^{(1)}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p},$$

$$R_{gh}^{(4)}(\mathbf{D_x}) = \frac{1}{\tilde{N}}(\Delta+\lambda_9^2)(\tilde{\mu}\Delta+\rho\omega^2)\delta_{gh} + w_{22}^{(1)}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$R_{gh}^{(5)}(\mathbf{D_x}) = w_{1;h+2}^{(1)}(\Delta)\frac{\partial}{\partial x_g}, R_{gh}^{(q)}(\mathbf{D_x}) = 0,$$

$$R_{gh}^{(7)}(\mathbf{D_x}) = w_{g+2;1}^{(1)}(\Delta)\frac{\partial}{\partial x_h}, R_{gh}^{(9)}(\mathbf{D_x}) = w_{g+2;h+2}^{(1)}(\Delta),$$

$$R_{gh}^{(10)}(\mathbf{D_x}) = w_{17}^{(1)}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}, R_{gh}^{(12)}(\mathbf{D_x}) = w_{71}^{(1)}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$R_{gh}^{(14)}(\mathbf{D_x}) = w_{7;h+2}^{(1)}(\Delta)\frac{\partial}{\partial x_g}, R_{gh}^{(15)}(\mathbf{D_x}) = w_{g+2;7}^{(1)}(\Delta)\frac{\partial}{\partial x_h},$$

$$R_{gh}^{(16)}(\mathbf{D_x}) = \frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\delta_{gh} + w_{77}^{(1)}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$i = 1,2,3,4,10,11,12,13,16;\;\; j = 5,6,14;\;\; l = 7,8,15;\;\; q = 6,8,11,13.$$

Appendix D. Some quantities in relations (4.34) and (4.39)

$$z_1 = \prod_{i=3}^{8}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{8}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{8}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{8}(\lambda_5^2-\lambda_p^2)\prod_{q=7}^{8}(\lambda_6^2-\lambda_q^2)(\lambda_7^2-\lambda_8^2),$$

$$z_2 = \prod_{i=3}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{8}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{8}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{8}(\lambda_5^2-\lambda_p^2)\prod_{q=7}^{8}(\lambda_6^2-\lambda_q^2)(\lambda_7^2-\lambda_8^2),$$

$$z_3 = \prod_{i=2,i\neq 3}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{8}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{8}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{8}(\lambda_5^2-\lambda_p^2)\prod_{q=7}^{8}(\lambda_6^2-\lambda_q^2)(\lambda_7^2-\lambda_8^2),$$

$$z_4 = \prod_{i=2,i\neq 4}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 4}^{8}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{8}(\lambda_3^2-\lambda_l^2)\prod_{p=6}^{8}(\lambda_5^2-\lambda_p^2)\prod_{q=7}^{8}(\lambda_6^2-\lambda_q^2)(\lambda_7^2-\lambda_8^2),$$

$$z_5 = \prod_{i=2,i\neq 5}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 5}^{8}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq 5}^{8}(\lambda_3^2-\lambda_l^2)\prod_{p=6}^{8}(\lambda_4^2-\lambda_p^2)\prod_{q=7}^{8}(\lambda_6^2-\lambda_q^2)(\lambda_7^2-\lambda_8^2),$$

$$z_6 = \prod_{i=2,i\neq 6}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 6}^{8}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq 6}^{8}(\lambda_3^2-\lambda_l^2)\prod_{p=5,p\neq 6}^{8}(\lambda_4^2-\lambda_p^2)\prod_{q=7}^{8}(\lambda_5^2-\lambda_q^2)(\lambda_7^2-\lambda_8^2),$$

$$z_7 = \prod_{i=2,i\neq 7}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 7}^{8}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq 7}^{8}(\lambda_3^2-\lambda_l^2)\prod_{p=5,p\neq 7}^{8}(\lambda_4^2-\lambda_p^2)\prod_{q=6,q\neq 7}^{8}(\lambda_5^2-\lambda_q^2)(\lambda_6^2-\lambda_8^2),$$

$$z_8 = \prod_{i=2}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{7}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{7}(\lambda_3^2-\lambda_l^2)\prod_{p=5}^{7}(\lambda_4^2-\lambda_p^2)\prod_{q=6}^{7}(\lambda_5^2-\lambda_q^2)(\lambda_6^2-\lambda_7^2),$$

$$z_9=\prod_{i=2}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{8}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{8}(\lambda_3^2-\lambda_l^2)\prod_{p=5}^{8}(\lambda_4^2-\lambda_p^2)\prod_{q=6}^{8}(\lambda_5^2-\lambda_q^2)\prod_{z=7}^{8}(\lambda_6^2-\lambda_z^2)(\lambda_7^2-\lambda_8^2).$$

$$\mathbf{H}^{(i)}(\mathbf{x})=\Big(H_{gh}^{(i)}(\mathbf{x})\Big)_{3\times 3},\mathbf{H}^{(j)}(\mathbf{x})=\Big(H_{gh}^{(j)}(\mathbf{x})\Big)_{3\times 4},$$

$$\mathbf{H}^{(l)}(\mathbf{x})=\Big(H_{gh}^{(l)}(\mathbf{x})\Big)_{4\times 3},\mathbf{H}^{(9)}(\mathbf{x})=\Big(H_{gh}^{(9)}(\mathbf{x})\Big)_{4\times 4},$$

$$\mathbf{H}^{(z)}(\mathbf{x})=\mathbf{R}^{(z)}(\mathbf{D}_{\mathbf{x}})Y_{11}^{(1)}(\mathbf{x}),\mathbf{H}^{(y)}(\mathbf{x})=\mathbf{R}^{(y)}(\mathbf{D}_{\mathbf{x}})Y_{44}^{(1)}(\mathbf{x}),\mathbf{H}^{(e)}(\mathbf{x})=\mathbf{R}^{(e)}(\mathbf{D}_{\mathbf{x}})Y_{77}^{(1)}(\mathbf{x}),$$

$$\mathbf{H}^{(n)}(\mathbf{x})=\mathbf{R}^{(n)}(\mathbf{D}_{\mathbf{x}})Y_{11;11}^{(1)}(\mathbf{x}),\mathbf{H}^{(p)}(\mathbf{x})=\mathbf{0}$$

$$i=1,2,3,4,10,11,12,13,16;\;\; j=5,6,14;\;\; l=7,8,15;$$
$$z=1,3,7,12;\;\; y=2,4;\;\; e=5,9,14;\;\; n=10,15,16;\;\; p=6,8,11,13.$$

Appendix E. Elemnts of the matrix in equation (6.1)

$$\mathbf{W}^{(i)}(\mathbf{x})=\Big(W_{gh}^{(i)}(\mathbf{x})\Big)_{3\times 3},\mathbf{W}^{(j)}(\mathbf{x})=\Big(W_{gh}^{(j)}(\mathbf{x})\Big)_{3\times 4},\mathbf{W}^{(l)}(\mathbf{x})=\Big(W_{gh}^{(l)}(\mathbf{x})\Big)_{4\times 3},$$

$$\mathbf{W}^{(9)}(\mathbf{x})=\Big(W_{gh}^{(9)}(\mathbf{x})\Big)_{4\times 4},W_{gh}^{(1)}(\mathbf{x})=\left[\frac{1}{\tilde{\tilde{\lambda}}}\nabla\operatorname{div}-\frac{1}{\tilde{\mu}}\operatorname{curl}\operatorname{curl}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{gh}^{(p)}(\mathbf{x})=0,W_{gh}^{(4)}(\mathbf{x})=\left[\frac{1}{\tilde{\alpha}}\nabla\operatorname{div}-\frac{1}{\gamma}\operatorname{curl}\operatorname{curl}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{11}^{(9)}(\mathbf{x})=\frac{A_2A_3-A_5^2}{\varrho}\varsigma_1^*(\mathbf{x}),W_{12}^{(9)}(\mathbf{x})=W_{21}^{(9)}(\mathbf{x})=\frac{A_5A_6-A_4A_3}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{22}^{(9)}(\mathbf{x})=\frac{A_1A_3-A_6^2}{\varrho}\varsigma_1^*(\mathbf{x}),W_{33}^{(9)}(\mathbf{x})=\frac{A_1A_2-A_4^2}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{13}^{(9)}(\mathbf{x})=W_{31}^{(9)}(\mathbf{x})=\frac{A_4A_5-A_2A_6}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{23}^{(9)}(\mathbf{x})=W_{32}^{(9)}(\mathbf{x})=\frac{A_4A_6-A_1A_5}{\varrho}\varsigma_1^*(\mathbf{x}),$$

$$W_{e4}^{(9)}(\mathbf{x})=W_{4e}^{(9)}(\mathbf{x})=0,W_{44}^{(9)}(\mathbf{x})=\frac{1}{k}\varsigma_1^*(\mathbf{x}),$$

$$W_{gh}^{(16)}(\mathbf{x})=\left[\frac{1}{\kappa_7}\nabla\operatorname{div}-\frac{1}{\kappa_6}\operatorname{curl}\operatorname{curl}\right]\varsigma_2^*(\mathbf{x}),$$

$$i=1,...,4,10,...,13,16;\;\; j=5,6,14;\;\; l=7,8,15;$$
$$e=1,2,3;\;\; p=2,3,5,...,8,10,...,15.$$

References

1. Eringen, A. C. *Foundations of Micropolar Thermoelasticity.* International Center for Mechanical Science, Courses and Lectures, No. 23, Spriner, Berlin, 1970.
2. Eringen, A. C. *Microcontinuum Field Theories I: Foundations and Solids.* 1st ed. Springer New York, 1999. doi: 10.1007/978-1-4612-0555-5.
3. Nowacki, W. "Couple Stresses in the Theory of Thermoelasticity I." *Bulletin of the Polish Academy of Sciences: Technical Sciences,* **14**, (1966), pp. 129–138.
4. Nowacki, W. "Couple Stresses in the Theory of Thermoelasticity II." *Bulletin of the Polish Academy of Sciences: Technical Sciences,* **14**, (1966), pp. 263–272.
5. Nowacki, W. "Couple Stresses in the Theory of Thermoelasticity III." *Bulletin of the Polish Academy of Sciences: Technical Sciences,* **14**, (1966), pp. 801–809.
6. Boschi, E. and Iesan, D. "A generalized theory of linear micropolar thermoelasticity." *Meccanica,* **8**, (1973), pp. 154–157, doi: 10.1007/BF02128724.
7. Svanadze, M. "A generalized theory of linear micropolar thermoelasticity." *Meccanica,* **51**, (2016), pp. 1825–1837, doi: 10.1007/s11012-015-0334-6.
8. Straughan, B. "Uniqueness and stability in triple porosity thermoelasticity." *Rendiconti Lincei-Matematica E Applicazioni,* **28**, (2017), pp. 191–208, doi: 10.4171/RLM/758.
9. Kansal, T. "Fundamental solutions in generalized theory of thermoelastic diffusion with triple porosity." *Engineering Transactions,* **71**, (2023), pp. 473–505, url: https://hal.science/hal-03686057v1/document.
10. Svanadze, M. "External boundary value problems in the quasi static theory of elasticity for triple porosity materials." *PAMM,* **16**, (2023), pp. 495–496, doi: 10.1002/pamm.201610236.
11. Svanadze, M. "Boundary value problems in the theory of thermoelasticity for triple porosity materials." *Mechanics of Solids, Structures and Fluids; NDE, Diagnosis, and Prognosis,* **9**, (2016), pp. 1–10, doi: 10.1115/IMECE2016-65046.
12. Svanadze, M. "External boundary value problems in the quasi static theory of triple porosity thermoelasticity." *PAMM,* **17**, (2017), pp. 471–472, doi: 10.1002/pamm.201710205.
13. Svanadze, M. "Potential method in the theory of elasticity for triple porosity materials." *Journal of Elasticity,* **130**, (2018), pp. 1–24, doi: 10.1007/s10659-017-9629-2.
14. Svanadze, M. "Potential method in the linear theory of triple porosity thermoelasticity." *Journal of Mathematical Analysis and Applications,* **461**, (2018), pp. 1585–1605, doi: 10.1016/j.jmaa.2017.12.022.
15. Svanadze, M. "On the linear equilibrium theory of elasticity for materials with triple voids." *The Quarterly Journal of Mechanics and Applied Mathematics,* **71**, (2018), pp. 329–348, doi: 10.1093/qjmam/hby008.
16. Straughan, B. "Mathematical aspects of multi-porosity continua." *Advances in Mechanics and Mathematics,* **38**, (2017), pp. 1–208, doi: 10.1007/978-3-319-70172-1.

17. Grot, R. A. "Thermodynamics of a continuum with microstructure." *International Journal of Engineering Science,* **7**, (1969), pp. 801–814, doi: 10.1016/0020-7225(69)90062-7.
18. Iesan, D. and Quintanilla, R. "On a theory of thermoelasticity with microtemperatures." *Journal of Thermal Stresses,* **23**, (2000), pp. 199–215. doi: 10.1080/014957300280407.
19. Iesan, D. "On a theory of micromorphic elastic solids with microtemperatures." *Journal of Thermal Stresses,* **24**, (2001), pp. 737–752. doi: DOI : 10 . 1080 / 014957301300324882.
20. Iesan, D. "Thermoelasticity of bodies with microstructure and microtemperatures." *International Journal of Solids and Structures,* **44**, (2007), pp. 8648–8662. doi: 10.1016/j.ijsolstr.2007.06.027.
21. Kansal, T. "Linear analysis of micromorphic thermoelastic materials with microtemperatures and triple porosity." *Theoretical and Applied Mechanics,* **51**, (2024), pp. 137–163. doi: 10.1177/1081286518761.
22. Svanadze, M. "Fundamental solutions in the linear theory of thermoelasticity for solids with triple porosity." *Mathematics and Mechanics of Solids,* **24**, (2019), pp. 919–938. doi: 10.1177/1081286518761.
23. Kansal, T. "The theory of thermoelasticity with double Porosity and microtemperatures." *CMST,* **28**, (2022), pp. 87–107. doi: 10.12921/cmst.2022.0000016.