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GREEN FUNCTIONS FOR COUPLED BOUNDARY VALUE PROBLEMS WITH APPLICATIONS TO STEPPED BEAMS MADE OF HETEROGENEOUS MATERIAL

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Abstract. The main objective of the present paper is to clarify the effect of the axial load on the eigenfrequencies of axially loaded and pinned-pinned stepped beams made of heterogeneous material. To this end, we shall consider how the Green functions of the corresponding coupled boundary value problems can be determined. After finding these Green Functions, the vibration problems of the unloaded and loaded stepped beams are reduced to eigenvalue problems governed by homogeneous Fredholm integral equations. These are solved numerically.

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1. Introduction

Beams can be found in many machines or structures as their vital elements. For that reason their mechanical behavior has been the subject of studies for a long time [1–3].

One of the major topics of interest is their vibrations [4–6]. When it comes to stepped beams, the continuous-mass transfer matrix method is extended in [7] to incorporate further effects such as rotatory inertia. The beams can have multiple steps and can carry an arbitrary number of lumped mass elements. De Rosa et al. [8] consider stepped beams assuming the Euler-Bernoulli hypothesis. The beams rest on an elastic foundation, whose stiffness can change at the steps. The frequency equation is solved numerically. Stepped beams with lumped masses made of axially functionally graded material are investigated in [9] using the lumped mass transfer matrix method. In article [10], the applied method is Adomian Decomposition, which proves to be effective for this kind of issue.

The Green function was first used in 1828 [11] for electrostatic issues. Since then, it has gained ground [12, 13]. Several three-point boundary value issues defined by third-order nonlinear differential equations are discussed in [14] with Green functions. The findings of [15] were extended for degenerated ordinary differential equation systems in [16, 17] for beam vibrations. The topic of stepped beam vibrations with a

Green function technique is addressed in [18], although for fixed-fixed support conditions. Paper [18] is devoted to coupled eigenvalue problems for which it presents a definition of the Green function determined for fixed-fixed stepped beams with the aim of clarifying their vibration problems, including the issue of what happens if the beam is subjected to axial forces.

The paper is organized into six sections. Sections 2 and 3 present the definition of the coupled boundary value problems and the definition of the Green functions that belong to them. The properties of these Green functions are detailed in Section 4. The definition plays an important role in the determination of the Green functions since it is constructive and allows calculation of the Green functions. Section 5 considers what form the coupled eigenvalue problems take. The issue of the stepped beams is tackled in Sections 6 and 7, which together with Section 8 constitute the main part of the present paper. They contain the calculation of the Green functions for the unloaded stepped beams and the axially loaded stepped beams as well. As regards their vibration problems, the corresponding eigenvalue problems are reduced to Fredholm integral equations that are solved numerically. Section 8 presents the numerical solutions for the vibration problem when the beam is axially loaded. The last section contains the concluding remarks.

2. Coupled boundary value problems

We shall consider a pair of inhomogeneous ordinary differential equations (ODEs)

$$L_i[y_i(x)] = r_i(x), \qquad i = 1,2 \tag{1a}$$

where the differential operators $L_i[y_i(x)]$ of order 2κ are defined by the relations

$$L_i[y_i(x)] = \sum_{n=0}^{2\kappa} p_{ni}(x)\, y_i^{(n)}(x), \qquad \frac{\mathrm{d}^n(\ldots)}{\mathrm{d}x^n} = (\ldots)^{(n)}, \qquad i = 1,2. \tag{1b}$$

Note that the order of these ODEs are the same.

Let b be an inner point in the interval $x \in [0, \ell = 1]$ for which it holds that $0 < b < \ell$, $b = \ell_1$, $\ell - b = \ell_2$ and $\ell_1 + \ell_2 = \ell = 1$. It is assumed that $\kappa \geq 1$ is a natural number. The functions ($p_{n1}(x)$ and $r_1(x)$) $\{r_{n2}(x)$ and $r_2(x)\}$ are continuous if ($x \in [0, b)$) $\{x \in (b, \ell = 1]\}$ and $p_{2\kappa i}(x) \neq 0$.

It is assumed that ODEs (1) are associated with the following boundary and continuity conditions:

$$U_{0r}[y_1] = \sum_{n=1}^{2\kappa} \alpha_{nr1}\, y_1^{(n-1)}(0) = 0\,, \qquad r = 1,2,\ldots,\kappa \tag{2a}$$

$$\begin{aligned} U_{br}[y_1, y_2] &= U_{br1}[y_1] - U_{br2}[y_2] = \\ &= \sum_{n=1}^{2\kappa} \Big(\beta_{nr1}\, y_1^{(n-1)}(b) - \beta_{nr2}\, y_2^{(n-1)}(b)\Big) = 0\,, \qquad r = 1,2,\ldots,2\kappa \end{aligned} \tag{2b}$$

$$U_{1r}[y_2] = \sum_{n=1}^{2\kappa} \gamma_{nr2}\, y_2^{(n-1)}(\ell) = 0\,, \qquad r = 1,2,\ldots,\kappa \tag{2c}$$

where α_{nrI}, β_{nrI}, β_{nrII}, and γ_{nrII} are real constants.

ODEs (1) with boundary and continuity conditions (2) determine a coupled boundary value problem, since the solutions $y_1(x)$, $y_2(x)$ should satisfy continuity conditions (2b).

Let us denote the linearly independent particular solutions of ODEs (1b) by $y_{mi}(x)$ $(m = 1, 2, \dots, 2\kappa)$. With $y_{mi}(x)$, the general solutions $y_i(x)$ are of the form

$$y_1(x) = \sum_{m=1}^{2\kappa} \mathcal{A}_{m1} y_{m1}(x), \qquad \text{if } x \in [0, b]; \tag{3a}$$

$$y_2(x) = \sum_{\ell=1}^{2\kappa} \mathcal{A}_{m2} y_{m2}(x), \qquad \text{if } x \in [b, \ell = 1]; \tag{3b}$$

where $\mathcal{A}_{m1}$ and $\mathcal{A}_{m2}$ are undetermined integration constants.

The integration constants $\mathcal{A}_{\ell 1}$ and $\mathcal{A}_{\ell 2}$ can be obtained from the boundary and continuity conditions:

$$\sum_{m=1}^{2\kappa} \mathcal{A}_{m1} U_{0r}[y_{m1}] = 0, \qquad r = 1, 2, \dots, \kappa \tag{4a}$$

$$\sum_{m=1}^{2\kappa} \left(\mathcal{A}_{m1} U_{br1}[y_{m1}] - \mathcal{A}_{m2} U_{br2}[y_{m2}]\right) = 0, \qquad r = 1, 2, \dots, 2\kappa \tag{4b}$$

$$\sum_{\ell=1}^{2\kappa} \mathcal{A}_{m2} U_{1r}[y_{m2}] = 0, \qquad r = 1, 2, \dots, \kappa \,. \tag{4c}$$

If we know the Green function $G(x, \xi)$ that belongs to the coupled boundary value problem (1), (2), then we seek the solution in the following form:

$$y(x) = \int_{\xi=0}^{\ell=1} G(x, \xi)\, r(\xi)\, \mathrm{d}\xi \,, \tag{5a}$$

where

$$y(x) = \begin{cases} y_1(x) & \text{if } x \in [0, b) \\ y_2(x) & \text{if } x \in (b, \ell = 1] \end{cases} \quad \text{and} \quad r(\xi) = \begin{cases} r_1(\xi) & \text{if } \xi \in [0, b)\,, \\ r_2(\xi) & \text{if } \xi \in (b, \ell = 1]\,. \end{cases} \tag{5b}$$

3. Green's functions of coupled boundary value problems

Let $G(x, \xi)$ be the Green function that belongs to the coupled boundary value problem (1), (2). It is defined by the following formula and properties [18].
Formula:

$$G(x, \xi) = \begin{cases} G_{11}(x, \xi) & \text{if } x, \xi \in [0, b], \\ G_{21}(x, \xi) & \text{if } x \in [b, \ell] \text{ and } \xi \in [0, b], \\ G_{12}(x, \xi) & \text{if } x \in [0, b] \text{ and } \xi \in [b, \ell], \\ G_{22}(x, \xi) & \text{if } x, \xi \in [b, \ell], \end{cases} \tag{6}$$

Properties:

1. Let ξ be an arbitrarily fixed coordinate in $[0, b]$

(i) The function $G_{11}(x,\xi)$ is a continuous function of x and ξ in the triangles $0 \le x \le \xi \le b$ and $0 \le \xi \le x \le b$ – see Figure 1. In addition it is 2κ times differentiable with respect to x and the derivatives

$$\frac{\partial^n G_{11}(x,\xi)}{\partial x^n} = G_{11}^{(n)}(x,\xi)\,, \qquad n = 1, 2, \ldots, 2\kappa \tag{7a}$$

are also continuous functions of x and ξ in the triangles $0 \le x \le \xi \le b$ and $0 \le \xi \le x \le b$.

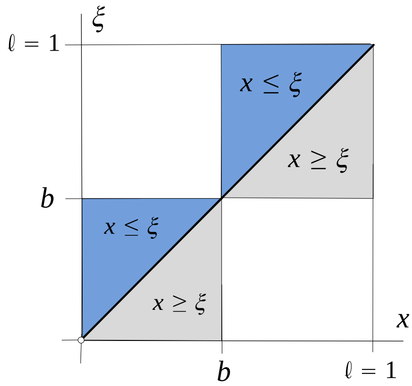

Figure 1. Triangular domains

The function $G_{11}(x,\xi)$ and its derivatives

$$G_{11}^{(n)}(x,\xi) = \frac{\partial^n G_{11}(x,\xi)}{\partial x^n}\,, \qquad n = 1, 2, \ldots, 2\kappa - 2 \tag{7b}$$

should be continuous for $x = \xi$:

$$\lim_{\varepsilon\to 0}\left[G_{11}^{(n)}(\xi+\varepsilon,\xi) - G_{11}^{(n)}(\xi-\varepsilon,\xi)\right] = \\ = \left[G_{11}^{(n)}(\xi+0,\xi) - G_{11}^{(n)}(\xi-0,\xi)\right] = 0 \quad n = 0, 1, 2, \ldots 2\kappa - 2 \tag{7c}$$

the derivative $G_{1I}^{(2\kappa-1)}(x,\xi)$ should, however, have a jump if $x = \xi$:

$$\lim_{\varepsilon\to 0}\left[G_{1I}^{(2\kappa-1)}(\xi+\varepsilon,\xi) - G_{1I}^{(2\kappa-1)}(\xi-\varepsilon,\xi)\right] = \\ = \left[G_{1I}^{(2\kappa-1)}(\xi+0,\xi) - G_{1I}^{(2\kappa-1)}(\xi-0,\xi)\right] = \frac{1}{p_{2\kappa 1}(\xi)}. \tag{7d}$$

(ii) In contrast, the function $G_{21}(x,\xi)$ and its derivatives

$$G_{21}^{(n)}(x,\xi) = \frac{\partial^n G_{21}(x,\xi)}{\partial x^n}\,, \qquad n = 1, 2, \ldots, 2\kappa \tag{7e}$$

are all continuous functions for any x in $[b, \ell]$

2. Let ξ be fixed in $[b,\ell]$.
(i) The function $G_{12}(x,\xi)$ and its derivatives

$$G_{12}^{(n)}(x,\xi)=\frac{\partial^n G_{12}(x,\xi)}{\partial x^n}\,,\qquad n=1,2,\dots,2\kappa \tag{8a}$$

are all continuous functions for any x in $[0,b]$.
(ii) The function $G_{22}(x,\xi)$ is a continuous function of x and ξ in the triangles $b\le x\le \xi\le \ell$ and $b\le \xi\le x\le \ell$ – see again Figure 1. In addition it is 2κ times differentiable with respect to x and the derivatives

$$\frac{\partial^n G_{22}(x,\xi)}{\partial x^n}=G_{22}^{(n)}(x,\xi)\,,\qquad n=1,2,\dots,2\kappa \tag{8b}$$

are also continuous functions of x and ξ in the triangles $b\le x\le \xi\le \ell$ and $b\le \xi\le x\le \ell$.

The function $G_{22}(x,\xi)$ and its derivatives

$$G_{22}^{(n)}(x,\xi)=\frac{\partial^n G_{22}(x,\xi)}{\partial x^n}\,,\qquad n=1,2,\dots,2\kappa-2 \tag{8c}$$

are also continuous for any $x=\xi$ in $[b,\ell]$:

$$\lim_{\varepsilon\to 0}\left[G_{22}^{(n)}(\xi+\varepsilon,\xi)-G_{22}^{(n)}(\xi-\varepsilon,\xi)\right]= \\ =\left[G_{22}^{(n)}(\xi+0,\xi)-G_{22}^{(n)}(\xi-0,\xi)\right]=0\,,\quad n=0,1,2,\dots 2\kappa-2; \tag{8d}$$

the derivative $G_{22}^{(2\kappa-1)}(x,\xi)$ should, however, have a jump if $x=\xi$:

$$\lim_{\varepsilon\to 0}\left[G_{22}^{(2\kappa-1)}(\xi+\varepsilon,\xi)-G_{22}^{(2\kappa-1)}(\xi-\varepsilon,\xi)\right]= \\ =\left[G_{22}^{(2\kappa-1)}(\xi+0,\xi)-G_{22}^{(2\kappa-1)}(\xi-0,\xi)\right]=\frac{1}{p_{2\kappa 2}(\xi)}. \tag{8e}$$

3. Let α be an arbitrary but finite non-zero constant. For a fixed $\xi\in[0,\ell]$ the product $G(x,\xi)\alpha$ as a function of x ($x\ne\xi$) should satisfy the homogeneous differential equations

$$\begin{aligned} L_1\left[G(x,\xi)\alpha\right]=0,\quad &\text{if}\quad x\in[0,b];\\ L_2\left[G(x,\xi)\alpha\right]=0,\quad &\text{if}\quad x\in[b,\ell]. \end{aligned} \tag{9}$$

4. The product $G(x,\xi)\alpha$ as a function of x should satisfy the boundary conditions and the continuity conditions

$$\begin{aligned} &\sum_{n=1}^{2\kappa}\alpha_{nr1}\,G^{(n-1)}(0)=0\,, && r=1,\dots,\kappa\\ &\sum_{n=1}^{2\kappa}\Big(\beta_{nr1}\,G^{(n-1)}(b-0)-\beta_{nr2}\,G^{(n-1)}(b+0)\Big)=0\,, && r=1,\dots,2\kappa\\ &\sum_{n=1}^{2\kappa}\gamma_{nr2}\,G^{(n-1)}(\ell)=0\,. && r=1,\dots,\kappa \end{aligned} \tag{10}$$

The above boundary and continuity conditions should be satisfied by the functions pairs

$$\{G_{11}(x,\xi), G_{21}(x,\xi)\}\,,$$
$$\{G_{12}(x,\xi), G_{22}(x,\xi)\}\,,$$

as well.

Remark 1. It can be proved by following the line of thought of a similar proof presented in [17] that the Green function defined above satisfies not only differential equation (1) but boundary and continuity conditions (2) as well.

Remark 2. The definition of the Green function is a constructive one since it makes possible to calculate the elements of the Green function.

Remark 3. Consider the inhomogeneous coupled boundary value problem defined by differential equations (1) with the boundary and continuity conditions (2). Let us assume that we know the corresponding Green function. Then the solution is given by the integral (5).

4. Properties of the Green function

4.1. **Self-Adjointness.** Assume that the functions

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in [0,b] \\ u_2(x) & \text{if } x \in [b,\ell] \end{cases} \tag{11a}$$

and

$$v(x) = \begin{cases} v_1(x) & \text{if } x \in [0,b] \\ v_2(x) & \text{if } x \in [b,\ell] \end{cases}\,. \tag{11b}$$

satisfy the boundary and continuity conditions (2) and are continuously differentiable 2κ times. Then they are called comparison functions. It is obvious that the solutions $y_1(x)$ and $y_2(x)$ of the coupled boundary value problem (1) and (2) are also comparative functions. Formula

$$(u,v)_L = \int_0^b u_1(x)\, L_1[v_1(x)]\,\mathrm{d}x + \int_b^\ell u_2(x)\, L_2[v_2(x)]\,\mathrm{d}x \tag{12}$$

taken on the set of the comparison functions $u(x)$, $v(x)$ is a product defined on the differential operators L_1 and L_2.

The coupled boundary value problem (1) and (2) is said to be self-adjoint if the product (12) is commutative, i.e., it holds that

$$(u,v)_L = (v,u)_L\,. \tag{13}$$

Condition (13) is called the condition of self-adjointness.

It can be proved (see [18]) that the Green function of coupled and self-adjoint boundary value problems is a symmetric function of ξ and x:

$$G(x,\xi) = G(\xi,x)\;. \tag{14}$$

5. Coupled eigenvalue problems

Consider differential equations

$$K_i\,[y_i] = \lambda\,M_i\,[y_i]\,, \qquad i=1,2 \tag{15a}$$

where $y_1(x)$, $x\in[0,b]$ and $y_2(x)$, $x\in[b,\ell]$; $(0<b<\ell=1)$ are again the unknown functions while λ is an unknown parameter (the eigenvalue sought). Differential operators $K_i\,[y_i]$ and $M_i\,[y_i]$ are defined by the equations

$$\begin{aligned} K_i\,[y_i] &= \sum_{n=0}^{\kappa}(-1)^n\left[f_{ni}(x)y_i^{(n)}(x)\right]^{(n)}\,, \qquad \frac{\mathrm{d}^n(\ldots)}{\mathrm{d}x^n} = (\ldots)^{(n)}\,;\\ M_i\,[y_i] &= \sum_{n=0}^{\mu}(-1)^n\left[g_{ni}(x)y_i^{(n)}(x)\right]^{(n)}\,, \qquad \kappa>\mu\geq 1 \end{aligned} \tag{15b}$$

in which the real function $(f_{ni}(x))$ $[g_{ni}(x)]$ is assumed to be differentiable continuously (κ) $[\mu]$ times and

$$f_{\kappa i}(x)\neq 0 \qquad \text{if } x\in[0,b] \tag{15c}$$

$$g_{\mu i}(x)\neq 0 \qquad \text{if } x\in[b,\ell]\,. \tag{15d}$$

The order of the differential operator on the left side of (15a) – this is 2κ – is greater than 2μ: the latter is the order of the differential operator on the right side.

We shall assume that ODEs (15) are associated with the homogeneous boundary and continuity conditions given by equations (2).

Let $u(x)$ and $v(x)$ $x\in[0,\ell]$ be two comparative functions for the eigenvalue problem (15), (2) – see (11). If we perform successive partial integration we get the following formulae for the products $(u,v)_K$ and $(u,v)_M$:

$$\begin{aligned} (u,v)_K = &\left[\sum_{n=0}^{\kappa}\sum_{r=0}^{n-1}(-1)^{(n+r)}u_1^{(r)}(x)\left[f_{n1}(x)\,v_1^{(n)}(x)\right]^{(n-1-r)}\right]_0^{b-0} + \\ &+\left[\sum_{n=0}^{\kappa}\sum_{r=0}^{n-1}(-1)^{(n+r)}u_2^{(r)}(x)\left[f_{n2}(x)\,v_2^{(n)}(x)\right]^{(n-1-r)}\right]_{b+0}^{\ell} + \\ &+\sum_{n=0}^{\kappa}\int_0^b u_1^{(n)}(x)f_n(x)v_1^{(n)}(x)\,\mathrm{d}x + \sum_{n=0}^{\kappa}\int_b^\ell u_2^{(n)}(x)f_n(x)v_2^{(n)}(x)\,\mathrm{d}x = \\ &= K_0(u,v) + \sum_{n=0}^{\kappa}\int_0^b u_1^{(n)}(x)f_n(x)v_1^{(n)}(x)\,\mathrm{d}x + \sum_{n=0}^{\kappa}\int_b^\ell u_2^{(n)}(x)f_n(x)v_2^{(n)}(x)\,\mathrm{d}x\,, \end{aligned} \tag{16a}$$

and

$$\begin{aligned} (u,v)_M = &\left[\sum_{n=0}^{\mu}\sum_{r=0}^{n-1}(-1)^{(n+r)}u_1^{(r)}(x)\left[g_{n1}(x)\,v_1^{(n)}(x)\right]^{(n-1-r)}\right]_0^{b-0} + \\ &+\left[\sum_{n=0}^{\mu}\sum_{r=0}^{n-1}(-1)^{(n+r)}u_2^{(r)}(x)\left[g_{n2}(x)\,v_2^{(n)}(x)\right]^{(n-1-r)}\right]_{b+0}^{\ell} + \end{aligned}$$

$$+\sum_{n=0}^{\mu}\int_0^b u_1^{(n)}(x)g_{n1}(x)v_1^{(n)}(x)\,\mathrm{d}x+\sum_{n=0}^{\mu}\int_b^{\ell} u_2^{(n)}(x)g_{n2}(x)v_2^{(n)}(x)\,\mathrm{d}x=$$

$$=M_0(u,v)+\sum_{n=0}^{\mu}\int_0^b u_1^{(n)}(x)g_{n1}(x)v_1^{(n)}(x)\,\mathrm{d}x+\sum_{n=0}^{\mu}\int_b^{\ell} u_2^{(n)}(x)g_{n2}(x)v_2^{(n)}(x)\,\mathrm{d}x. \tag{16b}$$

The expressions $K_0(u,v)$ and $M_0(u,v)$ are defined by the right sides of equations (16). They are referred to as boundary and continuity expressions. If

$$K_0(u,v)=K_0(v,u) \qquad \text{and} \qquad M_0(u,v)=M_0(v,u) \tag{17}$$

then the coupled eigenvalue problem determined by equations (15), (2) is obviously self-adjoint. The coupled eigenvalue problem is called simple if

$$M_1[y]=g_{01}(x)y_1(x) \qquad \text{and} \qquad M_2[y]=g_{02}(x)y_2(x). \tag{18}$$

Assume that the eigenvalue problem considered is simple. Assume further that the Green function that belongs to the coupled differential equations

$$K_i\left[y_i(x)\right]=r_i(x), \qquad i=1,2 \tag{19}$$

associated with boundary condition and continuity conditions (2) is known. Then it holds that

$$y(x)=\lambda\int_0^{\ell} G(x,\xi)g_0(\xi)y(\xi)\,\mathrm{d}\xi\,, \tag{20}$$

where

$$y(x)=\begin{cases} y_1(x) & \text{if } \xi\in[0,b)\,,\\ y_2(x) & \text{if } \xi\in(b,\ell]\end{cases} \qquad \text{and} \qquad g_0(x)=\begin{cases} g_{01}(x) & \text{if } \xi\in[0,b)\,,\\ g_{02}(x) & \text{if } \xi\in(b,\ell]\end{cases}$$

is the eigenfunction $y(x)$ that belong to the eigenvalue λ while the structure of $G(x,\xi)$ is given by (6). In this way the coupled eigenvalue problem is reduced to an eigenvalue problem governed by a homogeneous Fredholm integral equation. Assume that the original eigenvalue problem is self-adjoint and positive definite, i.e. it holds, among others, that $g_0(\xi)>0$ ($\xi\in[0,\ell]$). Under these conditions the previous Fredholm integral equation can be rewritten into the form

$$\mathcal{Y}(x)=\lambda\int_0^{\ell}\mathcal{K}(x,\xi)\mathcal{Y}(\xi)\,\mathrm{d}\xi\,, \tag{21}$$

where

$$\mathcal{Y}(x)=\sqrt{g_0(x)}y(x), \qquad \mathcal{K}(x,\xi)=\sqrt{g_0(x)}G(x,\xi)\sqrt{g_0(\xi)} \tag{22}$$

in which $\mathcal{Y}(x)$ is a new unknown function and the kernel $\mathcal{K}(x,\xi)$ is symmetric.

6. Stepped beams

6.1. **Governing equations for heterogeneous stepped beam problems.** Figure 2 shows a pinned-pinned heterogeneous stepped beam (PPStp beam). The axial force N ($N>0$) is compressive in this figure. The transverse coordinates are $\hat{y},\hat{z}$, while the longitudinal is $\hat{x}=\hat{\xi}$. The coordinate plane $\hat{x}\hat{z}$ is a symmetry plane of the beam.

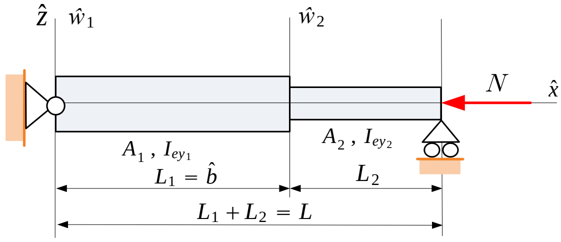

Figure 2. Heterogeneous stepped beam

The cross-sectional areas A_i, $(i = 1, 2)$ are constants. The beam is assumed to have heterogeneous cross sections, which means that the modulus of elasticity E satisfies the condition $E(\hat{y}, \hat{z}) = E(-\hat{y}, \hat{z})$. In this case we speak about cross-sectional heterogeneity. The length of the beam is L, the discontinuity in the cross sections is at $\hat{b}$. We should mention that the E-weighted first moment [19] $Q_{\hat{y}}$ is zero in this coordinate system:

$$Q_{\hat{y}} = \int_A \hat{z}E(\hat{y}, \hat{z})\mathrm{d}A = 0\,. \tag{23}$$

The E-weighted moments of inertia [19] are defined by the equations

$$I_{ey_1} = \int_{A_1} E(\hat{y}, \hat{z})z^2\,\mathrm{d}A, \quad I_{ey_2} = \int_{A_2} E(\hat{y}, \hat{z})z^2\,\mathrm{d}A. \tag{24}$$

The beam is subjected to distributed forces $\hat{f}_{y1}(\hat{x})$, $\hat{x} \in [0, L_1)$, $\hat{f}_{y2}(\hat{x})$, $\hat{x} \in [(L_1, L]$ acting on the center line $\hat{x}$. The vertical displacements on the center line are denoted by $\hat{w}_1$, $\hat{x} \in [0, L_1)$ and $\hat{w}_2$, $\hat{x} \in [0, L_1)$.

In what follows we shall introduce the following dimensionless quantities:

$$\begin{gathered} x = \hat{x}/L, \qquad \xi = \hat{\xi}/L, \qquad w_i = \hat{w}_i \;\; L\ (i = 1, 2), \\ b = \hat{b}/L\,, \qquad \ell = \left.\frac{x}{L}\right|_{x=L} = 1\,., \end{gathered} \tag{25}$$

(a) Equilibrium problems of PPStp beams with cross-sectional heterogeneity are governed by the following differential equations [19]:

$$\begin{gathered} K_i(w_i(x)) = I_{ey_i} w_i^{(4)} = f_{zi}(x)\,, \quad f_{zi} = L^3 \hat{f}_{zi}, \quad x \in \begin{cases} [0, b) & \text{if } i = 1 \\ (b, \ell) & \text{if } i = 2 \end{cases} \\ \frac{\mathrm{d}^k w_i}{\mathrm{d}x^k} = w_i^{(k)}, \;\; (k = 1, \dots, 4) \end{gathered} \tag{26}$$

ODEs $(26)_1$ are associated with the following boundary and continuity conditions:

$$w_1(0) = 0\,, \quad w_2^{(1)}(0) = 0\,; \qquad w_2(\ell) = 0\,, \quad w_2^{(2)}(\ell) = 0\,. \tag{27a}$$

$$w_1(b - 0) = w_2(b + 0) \qquad w_1^{(1)}(b - 0) = w_2^{(1)}(b + 0) \tag{27b}$$

$$I_{ey_1} w_1^{(2)}(b - 0) = I_{ey_2} w_2^{(2)}(b + 0) \qquad I_{ey_1} w_1^{(3)}(b - 0) = I_{ey_2} w_2^{(3)}(b + 0) \tag{27c}$$

ODEs $(26)_1$ with boundary and continuity conditions (27) constitute a coupled boundary value problem.

With the Green function that belongs to the coupled boundary value problem $(26)_1$, (27) solution for the dimensionless deflection $w(x)$ ($w(x) = w_1(x)$ if $x \in [0, b]$; $w(x) = w_2(x)$ if $x \in [b, \ell]$) is given by the following equation:

$$w(x) = \int_0^\ell G(x,\xi) f(\xi) \mathrm{d}\xi, \qquad f(\xi) = \begin{cases} f_{z1}(\xi) & \text{if } \xi \in [0, b], \\ f_{z2}(\xi) & \text{if } \xi \in [b, \ell]. \end{cases} \tag{28}$$

(b) Vibration problems of PPStp beams. As regards the free vibrations of PPStp beams it holds that

$$f(\xi) = \begin{cases} \rho_{a1} A_1 L^4 \omega^2 w_1(x) & \text{if } \xi \in [0, b], \\ \rho_{a2} A_2 L^4 \omega^2 w_2(x) & \text{if } \xi \in [b, \ell]. \end{cases} = \underbrace{\rho_{a1} A_1 L^4 \omega^2}_{\lambda} w(\xi) \begin{cases} 1 & \text{if } \xi \in [0, b], \\ \dfrac{\rho_{a2} A_2}{\rho_{a1} A_1} & \text{if } \xi \in [b, \ell]. \end{cases} \tag{29}$$

in which $w_i(x)$ is the dimensionless amplitude, ρ_{ai} is the average density on A_i, while ω stands for the circular frequency of the vibrations. With these notations the differential equations

$$\begin{aligned} K_1(w_1(x)) &= I_{ey_1} w_i^{(4)} = \underbrace{\rho_{a1} A_1 L^4 \omega^2}_{\lambda} w_1(x)\,, \\ K_2(w_1(x)) &= I_{ey_2} w_2^{(4)} = \lambda \frac{\rho_{a2} A_2}{\rho_{a1} A_1} w_2(x) \end{aligned} \tag{30}$$

are satisfied by $w_i(x)$. Differential equations (30) with the boundary and continuity conditions (27) determine a coupled eigenvalue problem for which λ is the eigenvalue. Recalling (28), we may conclude that this eigenvalue problem is governed by the homogeneous Fredholm integral equation

$$w(x) = \lambda \int_0^\ell G(x,\xi) w(\xi) \left\{ \begin{array}{ll} 1 & \text{if } \xi \in [0, b], \\ \dfrac{\rho_{a2} A_2}{\rho_{a1} A_1} & \text{if } \xi \in [b, \ell]. \end{array} \right\} \mathrm{d}\xi. \tag{31}$$

6.2. **Calculation of the Green function.**

6.2.1. *Particular solutions.* The linearly independent particular solutions of the differential equation $K_i(w_i(x)) = 0$ are very simple functions:

$$w_{11} = w_{12} = 1\,, \quad w_{21} = w_{22} = x\,, \quad w_{31} = w_{32} = x^2\,, \quad w_{41} = w_{42} = x^3\,. \tag{32}$$

6.2.2. *Calculations of the Green function if $\xi \in (0, b)$.* We shall assume that

$$\left.\begin{aligned} G_{11}(x,\xi) &= \sum_{m=1}^{4} (a_{mI}(\xi) + b_{mI}(\xi)) w_{1m}(x), \qquad x < \xi; \\ G_{11}(x,\xi) &= \sum_{m=1}^{4} (a_{mI}(\xi) - b_{mI}(\xi)) w_{1m}(x), \qquad x > \xi; \end{aligned}\right. \quad x \in [0, b] \tag{33a}$$

$$G_{21}(x,\xi) = \sum_{m=1}^{4} c_{mI}(\xi) w_{2m}(x), \qquad x \in [b,\ell] \tag{33b}$$

where the coefficients $a_{m1}(\xi), b_{m1}(\xi)$ and $c_{m1}(\xi)$ are unknown functions. This selection ensures the fulfillment of the following properties of the definition: 1. (ii) and 3. Fulfillment of Property 1. (i) leads to the following equation system:

$$\begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \\ 0 & 1 & 2\xi & 3\xi^2 \\ 0 & 0 & 2 & 6\xi \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2I_{ey_1}} \end{bmatrix} \tag{34}$$

from where

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix} = \frac{1}{12I_{ey_1}} \begin{bmatrix} \xi^3 \\ -3\xi^2 \\ 3\xi \\ -1 \end{bmatrix}. \tag{35}$$

Property 4 of the definition requires that the boundary and continuity conditions should all be satisfied. Therefore equations (27) yield the following equation system: Boundary conditions at $x = 0$:

$$\sum_{m=1}^{4} a_{m1} w_{m1}(0) = -\sum_{m=1}^{4} b_{m1} w_{m1}(0)\,, \tag{36a}$$

$$\sum_{m=1}^{4} a_{m1} w_{m1}^{(2)}(0) = -\sum_{m=1}^{4} b_{m1} w_{m1}^{(2)}(0)\,. \tag{36b}$$

Continuity conditions at $x = b$:

$$\sum_{m=1}^{4} a_{m1} w_{m1}(b) - \sum_{m=1}^{4} c_{mi} w_{m2}(b) = \sum_{m=1}^{4} b_{m1} w_{m1}(b)\,, \tag{36c}$$

$$\sum_{m=1}^{4} a_{m1} w_{m1}^{(1)}(b) - \sum_{m=1}^{4} c_{mi} w_{m2}^{(1)}(b) = \sum_{m=1}^{4} b_{m1} w_{m1}^{(1)}(b)\,, \tag{36d}$$

$$\sum_{m=1}^{4} a_{m1} w_{m1}^{(2)}(b) - \underbrace{\frac{I_{ey_2}}{I_{ey_1}}}_{\alpha} \sum_{m=1}^{4} c_{mi} w_{m2}^{(2)}(b) = \sum_{m=1}^{4} b_{mi} w_{m1}^{(2)}(b)\,, \tag{36e}$$

$$\sum_{m=1}^{4} a_{m1} w_{m1}^{(3)}(b) - \underbrace{\frac{I_{ey_2}}{I_{ey_1}}}_{\alpha} \sum_{m=1}^{4} c_{mi} w_{m2}^{(3)}(b) = \sum_{m=1}^{4} b_{mi} w_{m1}^{(3)}(b)\,. \tag{36f}$$

Boundary conditions at $x = \ell$:

$$\sum_{m=1}^{4} c_{m1} w_{m2}(0) = 0\,, \tag{36g}$$

$$\sum_{m=1}^{4} c_{m1} w_{m2}^{(2)}(0) = 0\,. \tag{36h}$$

After substituting w_{m1}, w_{m2}, and b_{m1}, equation system (36) assumes the following matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & b^3 & -1 & -b & -b^2 & -b^3 \\ 0 & 1 & 0 & 3b^2 & 0 & -1 & -2b & -3b^2 \\ 0 & 0 & 0 & 6b & 0 & 0 & -2\alpha & -6\alpha b \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & -6\alpha \\ 0 & 0 & 0 & 0 & 1 & \ell & \ell^2 & \ell^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \end{bmatrix} = \frac{1}{12 I_{ey_1}} \begin{bmatrix} -\xi^3 \\ -3\xi \\ 2\xi^3 - 3\xi^2 b + 6\xi b^2 - b^3 \\ -3\xi^2 + 12\xi b - 3b^2 \\ 12\xi - 6b \\ -6 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

Making use of the closed form solutions for a_{m1}, b_{m1} and c_{m1} ($m = 1, \ldots, 4$), we get $G_{11}(x,\xi)$ and $G_{21}(x,\xi)$ from equations (33):

$$\begin{aligned} G_{11}(x,\xi) = {} & \frac{1}{12 I_{ey_1}} \left\{ (-\xi^3 \pm \xi^3) + \right. \\ & + \left[\frac{\xi}{\alpha \ell^2} \left(4\,(\ell - b)^3 + \alpha \left(12\ell b\,(\ell - b) + 2\xi^2 \ell + 4b^3 - 3\xi \ell^2 \right) \right) \pm \left(-3\xi^2 \right) \right] x + \\ & \left. + \left(-3\xi \pm 3\xi \right) x^2 + \left(-\frac{1}{\ell}\,(\ell - 2\xi) \pm (-1) \right) x^3 \right\}, \end{aligned} \tag{38a}$$

$$G_{21}(x,\xi) = \frac{2\xi\,(\ell - x)}{12 I_{ey_1} \alpha \ell^2} \left(2x\ell^2 - 3\ell b^2 - x^2 \ell + 2b^3 + \alpha \left(3\ell b^2 - \xi^2 \ell - 2b^3 \right) \right). \tag{38b}$$

6.2.3. *Calculation of the Green function if $\xi \in (b, \ell)$:* In this case it is assumed that

$$G_{12}(x,\xi) = \sum_{m=1}^{4} c_{m2}(\xi) w_{m1}(x), \qquad x \in [0, b]; \tag{39a}$$

$$\left.\begin{aligned} G_{22}(x,\xi) &= \sum_{m=1}^{4} (a_{m2}(\xi) + b_{m2}(\xi)) w_{m2}(x), \qquad x \le \xi \\ G_{22}(x,\xi) &= \sum_{m=1}^{4} (a_{m2}(\xi) - b_{m2}(\xi)) w_{m2}(x), \qquad x \ge \xi \end{aligned}\right\} \quad x \in [b, \ell]; \tag{39b}$$

in which the coefficients $a_{m2}(\xi), b_{m2}(\xi)$, and $c_{m2}(\xi)$ are again the unknowns.

Recalling the calculation steps that resulted in solution (35), we obtain that

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \end{bmatrix} = \frac{1}{12I_{ey_2}} \begin{bmatrix} \xi^3 \\ -3\xi^2 \\ 3\xi \\ -1 \end{bmatrix}. \tag{40}$$

The boundary conditions at $x = 0$, $x = \ell$ and the continuity conditions at $x = b$ – the calculations are based on equations (36) but the details are omitted – lead to the following equation system:

$$\begin{bmatrix} 1 & b & b^2 & b^3 & 0 & -b & 0 & -b^3 \\ 0 & 1 & 2b & 3b^2 & 0 & -1 & 0 & -3b^2 \\ 0 & 0 & 2\alpha & 6\alpha b & 0 & 0 & 0 & -6b \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & \ell & \ell^2 & \ell^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6\ell & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} = \frac{1}{12I_{ey_2}} \begin{bmatrix} -\xi^3 + 3b\xi^2 - 3b^2\xi + b^3 \\ 3\xi^2 - 6b\xi + 3b^2 \\ \alpha\,(6b - 6\xi) \\ \alpha \\ 0 \\ 0 \\ \xi^3 - 3\ell\xi^2 + 3\ell^2\xi - \ell^3 \\ 6\xi - 6\ell \end{bmatrix} \tag{41}$$

Utilizing the closed form solutions for a_{m1}, b_{m1}, and c_{m1}, the following formulae are obtained for $G_{11}(x,\xi)$ and $G_{21}(x,\xi)$ from equations (39):

$$G_{12}(x,\xi) = \frac{2x\,(\ell - \xi)}{12I_{ey_2}\ell^2}\left(2\xi\ell^2 - 3\ell b^2 - \xi^2\ell + 2b^3 + \alpha\left(3\ell b^2 - 2b^3 - x^2\ell\right)\right), \tag{42a}$$

$$\begin{aligned} G_{22}(x,\xi) &= \frac{1}{12I_{ey_2}\ell}\left(4b^3\xi - \ell\xi^3 - 4\ell b^3 + 4\alpha b^3\,(\ell - \xi)\right) \pm \frac{\xi^3}{12I_{ey_2}} + \\ &+ \left(\frac{1}{12I_{ey_2}\ell^2}\left(4\ell^3\xi + 2\ell\xi^3 - 3\ell^2\xi^2 - 4b^3\xi + 4\ell b^3 + 4b^3\alpha\xi - 4\ell b^3\alpha\right) \pm \frac{-3\xi^2}{12I_{ey_2}}\right) x + \\ &+ \left(\frac{-3\xi}{12I_{ey_2}} \pm \frac{3\xi}{12I_{ey_2}}\right) x^2 + \left(-\frac{1}{12I_{ey_2}\ell}\,(\ell - 2\xi) \pm \frac{-1}{12I_{ey_2}}\right) x^3 \end{aligned} \tag{42b}$$

Remark 4. Recalling and applying then formula (16) to differential equations (26), we may conclude that $K_0(u,v) = 0$ in (16). This means that the coupled boundary value problem defined by (16) and (27) is self-adjoint. Consequently the Green function should be symmetric, i.e., it holds that

$$G(x,\xi) = G(\xi,x).$$

It is clear from a comparison of (38b) and (42a) that $G_{12}(x,\xi) = G_{21}(\xi,x)$. It can also be checked by paper-and-pencil calculations that $G_{11}(x,\xi) = G_{11}(\xi,x)$ and $G_{22}(x,\xi) = G_{22}(\xi,x)$.

Remark 5. The unit of the Green function is $1/\mathrm{N\,mm}^2$.

Remark 6. Let us introduce the dimensionless distributed load

$$\mathfrak{f}_{zi} = \frac{f_{zi}}{I_{ey_i}} = \frac{L^3 \hat{f}_{zi}}{I_{ey_i}} \tag{43}$$

and multiply equations $(26)_1$ by $1/I_{ey_i}$. The result is

$$w_i^{(4)} = \mathfrak{f}_{zi}(x)\,. \tag{44}$$

Note that differential equations (44) with the boundary and continuity conditions (27) determine now a three-point boundary value problem – therefore the coupling has been removed. The dimensionless Green function for this three-point boundary value problem is given by the equation

$$\mathcal{G}(x,\xi) = \begin{cases} \mathcal{G}_{11}(x,\xi) = I_{ey_1} G_{11}(x,\xi) & \text{if } x,\xi \in [0,b], \\ \mathcal{G}_{21}(x,\xi) = I_{ey_1} G_{21}(x,\xi) & \text{if } x \in [b,\ell] \text{ and } \xi \in [0,b], \\ \mathcal{G}_{12}(x,\xi) = I_{ey_2} G_{12}(x,\xi) & \text{if } x \in [0,b] \text{ and } \xi \in [b,\ell], \\ \mathcal{G}_{22}(x,\xi) = I_{ey_2} G_{22}(x,\xi) & \text{if } x,\xi \in [b,\ell]. \end{cases} \tag{45}$$

It is worthy of mention that $\mathcal{G}(x,\xi)$ depends on I_{ey_1} and I_{ey_2} via α only, The presence of this parameter reflects the fact that the beam considered is stepped. The solution for the equilibrium problem is then

$$w(x) = \int_0^\ell \mathcal{G}(x,\xi)\mathfrak{f}(\xi)\mathrm{d}\xi, \qquad \mathfrak{f}(\xi) = \begin{cases} \mathfrak{f}_{z1}(\xi) & \text{if } \xi \in [0,b], \\ \mathfrak{f}_{z2}(\xi) & \text{if } \xi \in [b,\ell]. \end{cases}$$

Though the three-point boundary value problem (44), (27) is not self-adjoint, the following symmetry conditions are obviously satisfied:

$$\begin{cases} \mathcal{G}_{11}(x,\xi) = \mathcal{G}_{11}(\xi,x) & \text{if } x,\xi \in [0,b], \\ \frac{\mathcal{G}_{21}(x,\xi)}{I_{ey1}} = \frac{\mathcal{G}_{12}(\xi,x)}{I_{ey2}} & \text{if } x \in [b,\ell] \text{ and } \xi \in [0,b], \\ \mathcal{G}_{22}(x,\xi) = \mathcal{G}_{22}(\xi,x) & \text{if } x,\xi \in [b,\ell]. \end{cases} \tag{46}$$

If we write $\hat{b}$, L, $\hat{x}$ and $\hat{\xi}$ for b, ℓ, x and ξ in formulate (45) we obtain the Green function for the case when we use a selected length unit. Then the unit of the Green function is the cube of the length unit selected.

For the purpose of displaying the behavior of the Green function, Figure 3 depicts then graph when $L = 100$ mm, $\hat{b} = 50$ mm, $\hat{\xi} = 75$ mm and $\alpha = 0.52200625$.

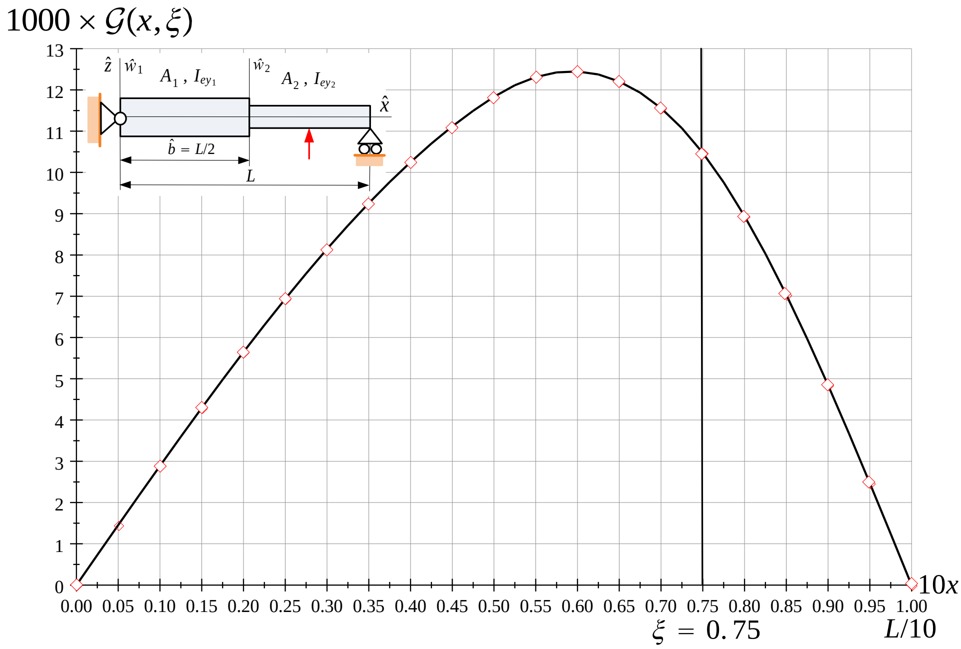

Figure 3. The Green function of a PPStp beam

Remark 7. With (45) the eigenvalue problem (31) for λ can be rewritten into the following form

$$w(x) = \chi \int_0^{\ell} \mathcal{G}(x,\xi) w(\xi) \left\{ \begin{array}{ll} 1 & \text{if } \xi \in [0,b], \\ \kappa & \text{if } \xi \in [b,\ell]. \end{array} \right\} \mathrm{d}\xi \,, \tag{47}$$

where

$$\chi = \frac{\lambda}{I_{ey1}} = \frac{\rho_{a1} A_1 L^4}{I_{ey1}} \omega^2, \qquad \text{and} \qquad \kappa = \frac{\rho_{a2} A_2 I_{ey_1}}{\rho_{a1} A_1 I_{ey_2}} \tag{48}$$

is the new eigenvalue.

6.3. **Example 1.** Consider the stepped beam shown in Figure 4. We shall assume that $\nu = 0.95, 0.90, 0.85, 0.80, 0.75$ if $\hat{x} \in (\hat{b}, L]$. It is also assumed that $D_1 = 50\,\text{mm}$, while $E_1 = E_2 = E_{steel} = 2.0 \times 10^5\,\text{N/mm}^2$. The length L of the beam is 800 mm, the location of the middle support is given by the parameter $\hat{b}$. The surface densities have the following values: $\rho_1 = \rho_2 = \rho_{\text{steel}} = 7850 \quad \text{kg}/10^9\text{mm}^3$. Under the previous conditions Table 1 shows the characteristic data for the various cross sections.

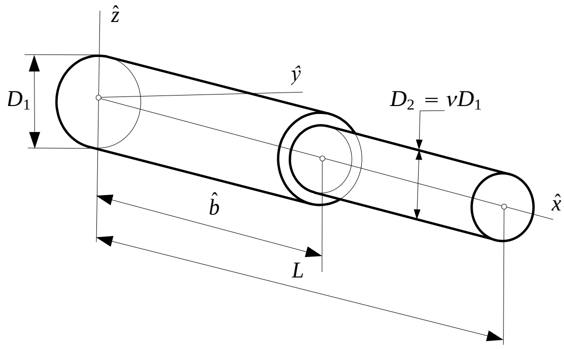

Figure 4. Stepped beam with circular cross section

Table 1. Data for the cross sections

ν	$\rho_a = \rho_1 = \rho_2$ kg/mm^3	$I_{ey1} \times 10^{-13}$ kg mm$^3/s^2$	$I_{ey2} \times 10^{-13}$ kg mm$^3/s^2$	α	κ
0.95			4.997747756	0.81450625	1.052631579
0.90			4.025779180	0.65610000	1.234586718
0.85	7.850×10^{-6}	6.135923152	3.202990235	0.52200625	1.384099617
0.80			2.513274123	0.40960000	1.562500000
0.75			1.473235149	0.31640625	1.777777778

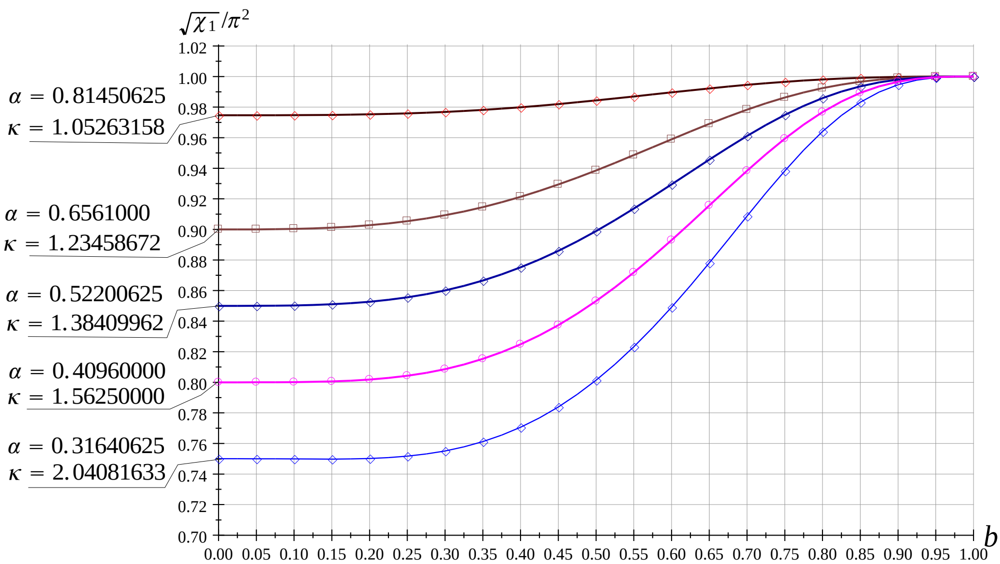

Figure 5. The first eigenvalue as a function of b; α and κ are parameters

The eigenvalue problem (47)–(48) is solved numerically by using a solution algorithm based on the boundary element method and published in [17].

Figure 5 shows the computational results for $\sqrt{\chi_1}/4.73004^2$ as a function of the dimensionless parameter b. Each curve in Figure 5 corresponds to a different value of the parameter α.

Assume that $b = 0.5$. If $\nu = 0.8$ we have $\alpha = 0.4096$ and $\kappa = 1.5625$. It follows from equation (48) that

$$\omega_1 = \frac{1}{L^2}\sqrt{\frac{I_{ey_1}}{\rho_{a1}A_1}\chi_1} = \frac{1}{800^2} \times \left(\sqrt{\frac{6.135\,923\,152 \times 10^{13}}{7.850 \times 10^{-6} \times 25^2 \times \pi}}\right) \times 8.420\,160\,122 = \\ = 830.100\,277\,1 \ \text{r/sec.} \quad (49)$$

If there is no step in the beam

$$\sqrt{\chi_1} = \pi^2 \times 1.0 = 9.869\,604\,4019.$$

and

$$\omega_1 = \frac{1}{800^2} \times \left(\sqrt{\frac{6.135\,923\,152 \times 10^{13}}{7.850 \times 10^{-6} \times 25^2 \times \pi}}\right) \times 9.869\,604\,4019 = 972.993\,533\,4 \ \text{r/sec.}$$

These results are compared with finite element calculations using Ansys. For mesh generation, a total of 360 uniform hexahedral elements (SOLID185) were used to discretize the geometry. A good agreement has been found:

Table 2. Comparison to FEM results

Eigenfrequency (Hz)	Our solution	Ansys solution	Relative error
Stepped beam	$\frac{830.100\,277\,1}{2\pi} = 132.115$	131.18	0.707%
Unifrom beam	$\frac{972.\,993\,53}{2\pi} = 154.\,865$	154.15	0.462%

When calculating the relative error our solution was the denominator.

7. Axially loaded stepped beams

7.1. **Governing equations.** We shall consider three different problems for axially loaded heterogeneous beams.

(a) Equilibrium problems. If a PPStp beam with cross-sectional heterogeneity is axially loaded, equilibrium problems are governed by the ODEs

$$\begin{aligned} K_{1a}(w_1(x)) &= I_{ey_1}w_1^{(4)} \pm N_1L^2w_1^{(2)} = f_{z1}(x), \ x \in [0, b]; \\ K_{2a}(w_2(x)) &= I_{ey_2}w_2^{(4)} \pm N_2L^2w_2^{(2)} = f_{z2}(\hat{x}), \ x \in [b, l], \end{aligned} \quad (50)$$

where N_1 and N_2 ($N_1 > 0$, $N_2 > 0$) are the axial forces acting on the beam. Their signs are (positive) [negative] if the considered axial force is (compressive) [tensile]. ODEs (50) are associated with boundary and continuity conditions (27). Note that boundary value problem (50), (27) is again a coupled boundary value problem.

In the sequel we shall assume that $N_1 = N_2 = N$.

If we know the Green functions $G = G_c(x,\xi)$ (N is compressive) and $G = G_t(x,\xi)$ (N is tensile) solution for the dimensionless deflection $w(x)$ ($w(x) = w_1(x)$ if $x \in [0,b]$; $w(x) = w_2(x)$ if $x \in [b,\ell]$) is given by integral (28) in which ($G(x,\xi) = G_c(x,\xi)$ if the axial force is compressive) $[G(x,\xi) = G_t(x,\xi)$ if the axial force is tensile$]$.

Remark 8. It can be checked with ease that the coupled boundary value problem (50), (27) is self-adjoint.

(b) Stability problems. If $f_{z1}(x) = f_{z2}(x) = 0$, $N_1 = N_2 = N$ and the sign of N is positive we get

$$\begin{aligned} K_{1as}(w_1(x)) &= w_1^{(4)} + \mathcal{N}_1 w_1^{(2)} = 0, \quad \mathcal{N}_1 = \frac{NL^2}{I_{ey_1}}, \quad x \in [0,b]; \\ K_{2as}(w_2(x)) &= w_2^{(4)} + \mathcal{N}_2 w_2^{(2)} = 0, \quad \mathcal{N}_2 = \frac{NL^2}{I_{ey_2}}, \quad x \in [b,l]. \end{aligned} \tag{51}$$

ODEs (51) with boundary and continuity conditions (27) constitute a coupled eigenvalue problem for which N is the eigenvalue sought. This problem is solved in Appendix A.

(c) Vibration problems. If the axially loaded beams vibrate, the amplitudes should fulfill ODEs

$$\begin{aligned} K_{1av}(w_1(x)) &= w_1^{(4)} \pm \mathcal{N}_1 w_1^{(2)} = \chi w_1, \quad \chi = \frac{\lambda}{I_{ey_1}} = \frac{\rho_{a1} A_1 L^4 \omega^2}{I_{ey_1}}, \quad x \in [0,b]; \\ K_{2av}(w_2(x)) &= w_2^{(4)} \pm \mathcal{N}_2 w_2^{(2)} = \chi\kappa w_2, \quad \chi\kappa = \frac{\rho_{a2} A_2 L^4 \omega^2}{I_{ey_2}}, \quad x \in [b,l], \end{aligned} \tag{52}$$

which are associated with boundary and continuity conditions (27). ODEs (52) with boundary and continuity conditions (27) constitute a coupled eigenvalue problem with χ as the eigenvalue.

7.2. **Calculation of the Green function for compressive axial force.** Let us introduce the quantities

$$p_1 = \sqrt{\mathcal{N}_1}, \qquad p_2 = \sqrt{\mathcal{N}_2} = p_1\sqrt{\alpha}. \tag{53}$$

With (53), solutions to the dimensionless displacements in equations (50) – the signs of $N_1 L^2 w_1^{(2)}$ and $N_2 L^2 w_2^{(2)}$ are positive – are given by

$$w_1 = \sum_{\ell=1}^{4} a_{\ell 1} w_{\ell 1}(x) = a_{11} + a_{21}x + a_{31}\cos p_1 x + a_{41}\sin p_1 x, \qquad p_1 = \sqrt{\mathcal{N}_1}; \tag{54a}$$

$$w_2 = \sum_{\ell=1}^{4} a_{\ell 2} w_{\ell 2}(x) = a_{12} + a_{22}x + a_{32}\cos p_2 x + a_{42}\sin p_2 x, \qquad p_2 = \sqrt{\mathcal{N}_2}. \tag{54b}$$

The structure of the Green function is presented by equation (6).
If $\xi \in (0,b)$ we shall assume that

$$\left.\begin{aligned} G_{c11}(x,\xi) &= \sum_{m=1}^{4}(a_{m1}(\xi)+b_{m1}(\xi))w_{1m}(x), \qquad x<\xi \\ G_{c11}(x,\xi) &= \sum_{m=1}^{4}(a_{m1}(\xi)-b_{m1}(\xi))w_{1m}(x), \qquad x>\xi \end{aligned}\right\} \quad x\in[0,b] \tag{55a}$$

$$G_{c21}(x,\xi)=\sum_{m=1}^{4}c_{m1}(\xi)w_{2m}(x). \qquad x\in[b,\ell] \tag{55b}$$

Here, the coefficients $a_{m1}(\xi), b_{m1}(\xi)$, and $c_{m1}(\xi)$ are the unknown functions. If we follow the calculation steps detailed in Subsection 6.2.2 we get the equation systems:

$$\begin{bmatrix} 1 & \xi & \cos p_1\xi & \sin p_1\xi \\ 0 & 1 & -p_1\sin p_1\xi & p_1\cos p_1\xi \\ 0 & 0 & -p_1^2\cos p_1\xi & -p_1^2\sin p_1\xi \\ 0 & 0 & p_1^3\sin p_1\xi & -p_1^3\cos p_1\xi \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{2I_{ey_1}} \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}=\frac{1}{2I_{ey_1}}\begin{bmatrix} \frac{\xi}{p_1^2} \\ -\frac{1}{p_1^2} \\ -\frac{\sin p_1\xi}{p_1^3} \\ \frac{\cos p_1\xi}{p_1^3} \end{bmatrix} \tag{56}$$

and

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & b & \cos p_1 b & \sin p_1 b & -1 & -b & -\cos p_2 b & -\sin p_2 b \\ 0 & 1 & -p_1\sin p_1 b & p_1\cos p_1 b & 0 & -1 & p_2\sin p_2 b & -p_2\cos p_2 b \\ 0 & 0 & -p_1^2\cos p_1 b & -p_1^2\sin p_1 b & 0 & 0 & \alpha p_2^2\cos p_2 b & \alpha p_2^2\sin p_2 b \\ 0 & 0 & p_1^3\sin p_1 b & -p_1^3\cos p_1 b & 0 & 0 & -\alpha p_2^3\sin p_2 b & \alpha p_2^3\cos p_2 b \\ 0 & 0 & 0 & 0 & 1 & \ell & \cos p_2\ell & \sin p_2\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos p_2\ell & \sin p_2\ell \end{bmatrix}\begin{bmatrix} a_{1I} \\ a_{2I} \\ a_{3I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix}=$$

$$=\frac{1}{2I_{ey_1}p_1^3}\begin{bmatrix} -p_1\xi+\sin p_1\xi \\ \sin p_1\xi \\ p_1\xi-p_1 b+\sin p_1(b-\xi) \\ -p_1+p_1\cos p_1(b-\xi) \\ -p_1^2\sin p_1(b-\xi) \\ -p_1^3\cos p_1(\xi-b) \\ 0 \\ 0 \end{bmatrix} \tag{57}$$

If $\xi \in (b,\ell)$ it is assumed that

$$\left.\begin{aligned} G_{c22}(x,\xi) &= \sum_{m=1}^{4}(a_{m2}(\xi)+b_{m2}(\xi))w_{2m}(x), \qquad x<\xi \\ G_{c22}(x,\xi) &= \sum_{m=1}^{4}(a_{m2}(\xi)-b_{m2I}(\xi))w_{2m}(x), \qquad x>\xi \end{aligned}\right\} \quad x\in[b,\ell] \tag{58a}$$

$$G_{c12}(x,\xi)=\sum_{m=1}^{4}c_{m2}(\xi)w_{1m}(x), \qquad x\in[0,b] \tag{58b}$$

where the coefficients $a_{m2}(\xi), b_{m2}(\xi)$ and $c_{m2}(\xi)$ are the unknowns. By repeating the calculation steps presented in Subsection 6.2.3 the following equation system can be obtained for these unknown coefficients:

$$\begin{bmatrix} 1 & \xi & \cos p_2\xi & \sin p_2\xi \\ 0 & 1 & -p_2\sin p_2\xi & p_2\cos p_2\xi \\ 0 & 0 & -p_2^2\cos p_2\xi & -p_2^2\sin p_2\xi \\ 0 & 0 & p_2^3\sin p_2\xi & -p_2^3\cos p_2\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{2I_{ey_1}} \end{bmatrix}, \quad \begin{bmatrix} b_{1II} \\ b_{2II} \\ b_{3II} \\ b_{4II} \end{bmatrix} = \frac{1}{2I_{ey_1}} \begin{bmatrix} \frac{\xi}{p_2^2} \\ -\frac{1}{p_2^2} \\ -\frac{\sin p_2\xi}{p_2^3} \\ \frac{\cos p_2\xi}{p_2^3} \end{bmatrix} \tag{59}$$

and

$$\begin{bmatrix} 1 & b & \cos p_2 b & \sin p_2 b & 0 & -b & 0 & -\sin p_1 b \\ 0 & 1 & -p_2\sin p_2 b & p_2\cos p_2 b & 0 & -1 & 0 & -p_1\cos p_1 b \\ 0 & 0 & -\alpha p_2^2\cos p_2 b & -\alpha p_2^2\sin p_2 b & 0 & 0 & 0 & p_1^2\sin p_1 b \\ 0 & 0 & \alpha p_2^3\sin p_2 b & -\alpha p_2^3\cos p_2 b & 0 & 0 & 0 & p_1^3\cos p_1 b \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & \ell & \cos p_2\ell & \sin p_2\ell & 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos p_2\ell & -\sin p_2\ell & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4II} \\ c_{1II} \\ c_{2II} \\ c_{3II} \\ c_{4II} \end{bmatrix} =$$

$$= -\frac{1}{2I_{ey_2}p_2^3} \begin{bmatrix} p_2\xi - p_2 b + \sin p_2(b-\xi) \\ p_2\cos p_2(b-\xi) - p_2 \\ -\alpha p_2^2\sin p_2(b-\xi) \\ -\alpha p_2^3\cos p_2(b-\xi) \\ 0 \\ 0 \\ -p_2\xi + p_2\ell - \sin p_2(\ell-\xi) \\ \sin p_2(\ell-\xi) \end{bmatrix} \tag{60}$$

The closed form solutions for $a_{11}(\xi), \ldots, c_{42}(\xi)$ obtained by solving equations (57) and (60) are very long formulae and for this reason they are not presented here.

Remark 9. The Green function $G_c(x\xi)$ is symmetric, i.e., it holds that $G_c(x,\xi) = G_c(\xi,\xi)$. Fulfillment of the symmetry condition is checked by numerical computations since the paper-and-pencil calculations for checking the symmetry condition are very time consuming.

Remark 10. The dimensionless Green functions $\mathcal{G}_c(x,\xi)$ can be calculated by using equation (45). $\mathcal{G}_c(x,\xi)$ fulfills symmetry conditions (46).

Remark 11. Assume that $b = 0.5$, $\xi = 0.75$ and $\alpha = 0.52200625$. Then the critical value of the dimensionless compressive force p_2 is equal to 3.55896485 – see Figure 8. Under these conditions, Figure 6 depicts the Green function $\mathcal{G}(x,\xi)$.

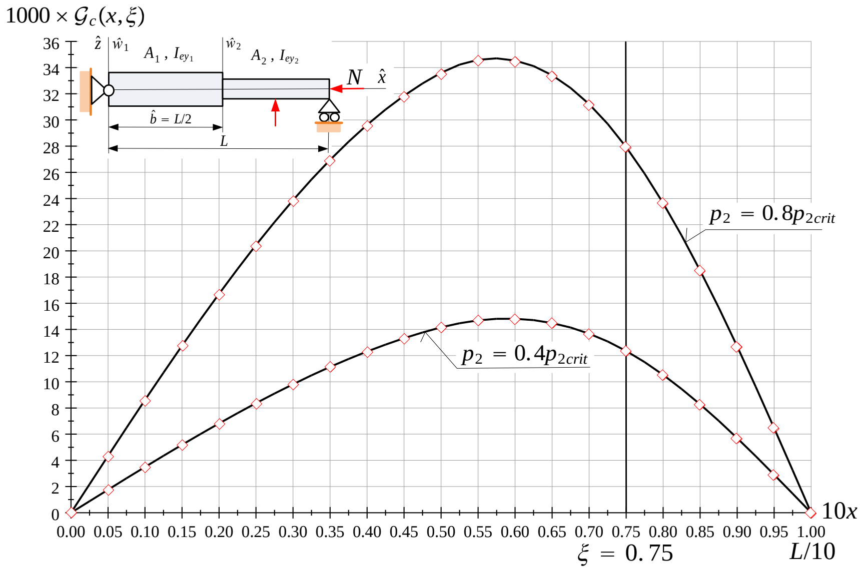

Figure 6. The Green function of a PPStp beam subjected to a compressive force

7.3. **Calculation of the Green function for tensile axial force.** Recalling equations (50), solutions to the dimensionless displacements – the signs of $N_1L^2w_1^{(2)}$ and $N_2L^2w_2^{(2)}$ is negative – are given by

$$w_1 = \sum_{\ell=1}^{4} a_{\ell 1}w_{\ell 1}(x) = a_{11} + a_{21}x + a_{31}\cosh p_1x + a_{41}\sinh p_1x, \tag{61a}$$

$$w_2 = \sum_{\ell=1}^{4} a_{\ell 2}w_{\ell 2}(x) = a_{12} + a_{22}x + a_{32}\cosh p_2x + a_{42}\sinh p_2x. \tag{61b}$$

The structure of the Green function is the same as earlier - see equations (6). If $\xi \in (0, b)$ it is assumed that

$$\left.\begin{aligned} G_{t11}(x,\xi) &= \sum_{m=1}^{4}(a_{m1}(\xi)+b_{m1}(\xi))w_{1m}(x), \qquad x<\xi \\ G_{t11}(x,\xi) &= \sum_{m=1}^{4}(a_{m1}(\xi)-b_{m1}(\xi))w_{1m}(x), \qquad x>\xi \end{aligned}\right\} \quad x\in[0,b]; \tag{62a}$$

$$G_{t21}(x,\xi) = \sum_{m=1}^{4} c_{m1}(\xi)w_{2m}(x), \qquad x \in [b,\ell] \tag{62b}$$

where the coefficients $a_{m1}(\xi), b_{m1}(\xi)$ and $c_{m1}(\xi)$ are again the unknown functions. Application of the calculation steps detailed in Subsection 6.2.2 yields

$$\begin{bmatrix} 1 & \xi & \cosh p_1\xi & \sinh p_1\xi \\ 0 & 1 & p_1\sinh p_1\xi & p_1\cosh p_1\xi \\ 0 & 0 & p_1^2\cosh p_1\xi & p_1^2\sinh p_1\xi \\ 0 & 0 & p_1^3\sinh p_1\xi & p_1^3\cosh p_1\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \frac{1}{2I_{ey_1}} \begin{bmatrix} -\frac{\xi}{p_1^2} \\ \frac{1}{p_1^2} \\ \frac{\sinh p_1\xi}{p_1^3} \\ -\frac{\cosh p_1\xi}{p_1^3} \end{bmatrix} \tag{63}$$

and

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & b & \cosh p_1b & \sinh p_1b & -1 & -b & -\cosh p_2b & -\sinh p_2b \\ 0 & 1 & p_1\sinh p_1b & p_1\cosh p_1b & 0 & -1 & -p_2\sinh p_2b & -p_2\cosh p_2b \\ 0 & 0 & p_1^2\cosh p_1b & p_1^2\cosh p_1b & 0 & 0 & -\alpha p_2^2\cosh p_2b & -\alpha p_2^2\sinh p_2b \\ 0 & 0 & p_1^3\sinh p_1b & p_1^3\cosh p_1b & 0 & 0 & -\alpha p_2^3\sinh p_2b & -\alpha p_2^3\cosh p_2b \\ 0 & 0 & 0 & 0 & 1 & \ell & \cosh p_2\ell & \sinh p_2\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & \cosh p_2\ell & \sinh p_2\ell \end{bmatrix} \begin{bmatrix} a_{1I} \\ a_{2I} \\ a_{3I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix} =$$

$$= \frac{1}{2I_{ey_1}p_1^3} \begin{bmatrix} p_1\xi - \sinh p_1\xi \\ -\sinh p_1\xi \\ -p_1\xi + p_1b - \sinh p_1(b-\xi) \\ p_1 - p_1\cosh p_1(b-\xi) \\ -p_1^2\sinh p_1(b-\xi) \\ -p_1^3\cosh p_1(b-\xi) \\ 0 \\ 0 \end{bmatrix} \tag{64}$$

If $\xi \in (b,\ell)$ we shall assume that:

$$\begin{aligned} G_{t22}(x,\xi) &= \sum_{m=1}^{4} (a_{m2}(\xi) + b_{m2}(\xi))w_{2m}(x), \qquad x < \xi \\ G_{t22}(x,\xi) &= \sum_{m=1}^{4} (a_{m2}(\xi) - b_{m2I}(\xi))w_{2m}(x), \qquad x > \xi \end{aligned} \qquad x \in [b,\ell] \tag{65a}$$

$$G_{t12}(x,\xi) = \sum_{m=1}^{4} c_{m2}(\xi)w_{1m}(x). \qquad x \in [0,b] \tag{65b}$$

Recalling the the calculation steps presented in Subsection 6.2.3 we obtain the following equation systems

$$\begin{bmatrix} 1 & \xi & \cosh p_2\xi & \sinh p_2\xi \\ 0 & 1 & p_2\sinh p_2\xi & p_2\cosh p_2\xi \\ 0 & 0 & p_2^2\cosh p_2\xi & p_2^2\sinh p_2\xi \\ 0 & 0 & p_2^3\sinh p_2\xi & p_2^3\cosh p_2\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2I_{ey_2}} \end{bmatrix}, \qquad \begin{bmatrix} b_{1II} \\ b_{2II} \\ b_{3II} \\ b_{4II} \end{bmatrix} = \frac{1}{2I_{ey_2}} \begin{bmatrix} -\frac{\xi}{p_2^2} \\ \frac{1}{p_2^2} \\ \frac{\sinh p_2\xi}{p_2^3} \\ -\frac{\cosh p_2\xi}{p_2^3} \end{bmatrix} \tag{66}$$

and

$$\begin{bmatrix} 1 & b & \cosh p_2 b & \sinh p_2 b & 0 & -b & 0 & -\sinh p_1 b \\ 0 & 1 & p_2 \sinh p_2 b & p_2 \cosh p_2 b & 0 & -1 & 0 & -p_1 \cosh p_1 b \\ 0 & 0 & \alpha p_2^2 \cosh p_2 b & \alpha p_2^2 \sinh p_2 b & 0 & 0 & 0 & -p_1^2 \sinh p_1 b \\ 0 & 0 & \alpha p_2^3 \sinh p_2 b & -\alpha p_2^3 \cosh p_2 b & 0 & 0 & 0 & -p_1^3 \cosh p_1 b \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & \ell & \cosh p_2 \ell & \sinh p_2 \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh p_2 \ell & \sinh p_2 \ell & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4II} \\ c_{1II} \\ c_{2II} \\ c_{3II} \\ c_{4II} \end{bmatrix} =$$

$$= \frac{1}{2p_2^3} \begin{bmatrix} p_2\xi - p_2 b + \sinh p_2 (b-\xi) \\ p_2 \cosh p_2 (b-\xi) - p_2 \\ \alpha p_2^2 \sinh p_2 (b-\xi) \\ \alpha p_2^3 \cosh p_2 (b-\xi) \\ 0 \\ 0 \\ -p_2\xi + p_2\ell - \sinh p_2 (\ell-\xi) \\ -\sinh p_2 (\ell-\xi) \end{bmatrix} \tag{67}$$

The closed form solutions for the unknown functions $a_{11}(\xi), \ldots, c_{42}(\xi)$ obtained by solving equations (64) and (67) are again very long formulae and for this reason they are not presented here.

Remark 12. The Green function $G_t(x\xi)$ is symmetric, i.e., it satisfies the symmetry condition $G_t(x,\xi) = G_t(\xi,\xi)$. Fulfillment of the symmetry condition was verified by numerical computations.

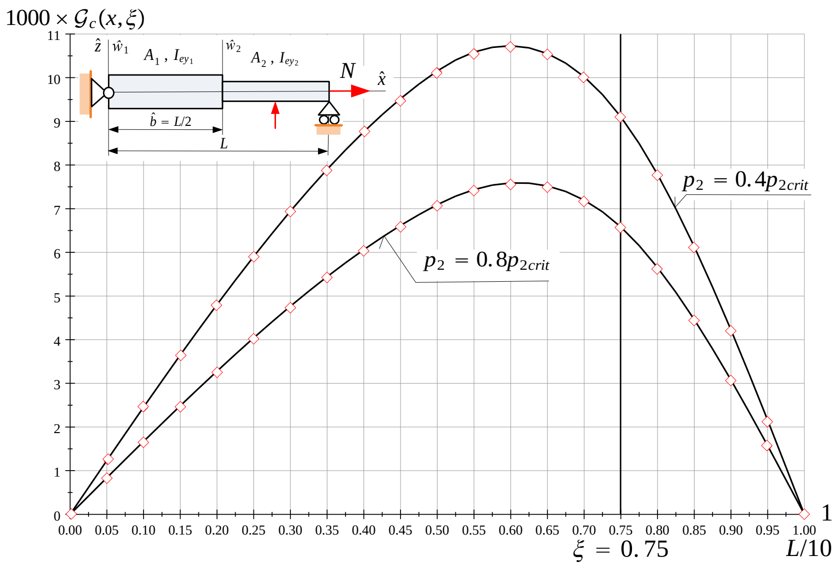

Figure 7. The Green function of a PPStp beam subjected to a tensile force

Remark 13. The dimensionless Green functions $\mathcal{G}_t(x,\xi)$ can be calculated by utilizing equation (45). $\mathcal{G}_t(x,\xi)$ fulfills the symmetry conditions (46).

Figure 7 shows the dimensionless Green function $\mathcal{G}_t(x,\xi)$ utilizing the data given in Remark 11.

Remark 14. It is clear from [Figure 6] {Figure 7} that the deflections are [greater] {smaller} if p_2 is [greater] {greater}. The fulfillment of these relationships is a natural requirement for the Green functions considered.

8. Axial load and eigenfrequencies of stepped beams

8.1. **Governing equations for the eigenvalue problem.** In this section it is our main objective to clarify the effect of the axial load on the eigenfrequencies of PPStp beams. Making use of the dimensionless Green functions the eigenvalue problems to be solved are governed by the homogeneous Fredholm integral equations for the case of a compressive force

$$w(x)=\chi\int_0^\ell \mathcal{G}_c(x,\xi)w(\xi)\left\{\begin{array}{ll}1 & \text{if } \xi\in[0,b],\\ \kappa & \text{if } \xi\in[b,\ell].\end{array}\right\}\mathrm{d}\xi\,, \tag{68}$$

and for the case of a tensile force

$$w(x)=\chi\int_0^\ell \mathcal{G}_t(x,\xi)w(\xi)\left\{\begin{array}{ll}1 & \text{if } \xi\in[0,b],\\ \kappa & \text{if } \xi\in[b,\ell].\end{array}\right\}\mathrm{d}\xi\,. \tag{69}$$

Here χ, i.e., the eigenvalue sought, and κ are given by equation (48). In the sequel we shall seek numerical solutions for the above problems utilizing the data related to the stepped beams that are considered in Subsection 6.3.

In the following, we shall need the value of the smallest critical force for the mentioned stepped beams. The solution to the corresponding eigenvalue problem is given in Appendix A – see Figure 8.

8.2. **Example 2.** Two problems are solved numerically. For the first problem it is assumed that $\nu=0.90$; then $\alpha=0.65610000$, and $\kappa=1.234586718$. For the second problem $\nu=0.80$, $\alpha=0.40960000$, and $\kappa=1.562500000$. These data are taken from Table 1. The first eigenfrequency and the critical force can be calculated by utilizing the data presented in Tables 3 and 4 – see Figures 5 and 8 for a comparison. Tables 3 and 4 contain some further data that are also utilized in the computations.

Table 3. Values of χ_1

	$\sqrt{\chi_1(b)}/\pi^2$				
ν	$b=0.2$	$b=0.4$	$b=0.5$	$b=0.6$	$b=0.8$
0.9	0.90273411	0.92130879	0.93858272	0.95892739	0.99240078
0.8	0.80179239	0.82483041	0.85314059	0.89300215	0.97673594

Table 4. Critical force

	$\sqrt{N_{2\,\mathrm{crit}}(b)}$				
ν	$b=0.2$	$b=0.4$	$b=0.5$	$b=0.6$	$b=0.8$
0.9	3.16728280	3.30994880	3.43419178	3.58174237	3.82743853
0.8	3.18497550	3.43128449	3.66658411	3.98927283	4.72167938

Let us denote the first eigenfrequency for [compression] {tension} by [ω_{1c}] {ω_{1t}}. The first eigenfrequency of the unloaded beam is ω_1.

Tables 5–9 contain the computed results for Problem 1, 10–14 for Problem 2. In both cases the values of b are 0.2, 0.4, 0.5, 0.6, and 0.8. The first column in each table is a list of the values the quotient $\mathcal{N}_2/\mathcal{N}_{2crit}$ has, the second and fourth columns contain those values of ω_{1c} and ω_{1t} which belong to $\mathcal{N}_2/\mathcal{N}_{2crit}$. The third and fifth columns show the differences between two consecutive values of ω_{1c} and ω_{1t}. If these differences are constants then the functions $\omega_{1c}(\mathcal{N}_2/\mathcal{N}_{2crit})$ and $\omega_{1t}(\mathcal{N}_2/\mathcal{N}_{2crit})$ are in principle linear functions.

Each table is followed by two equations. The first is a quadratic approximation of the function $\omega_{1c}(\mathcal{N}_2/\mathcal{N}_{2crit})$, the second is a quadratic approximation of the function $\omega_{1c}(\mathcal{N}_2/\mathcal{N}_{2crit})$. See equations (70)–(79) for details. The quadratic approximations fit to the values of these functions an accuracy of four to five digits.

8.2.1. *Solutions to Problem 1.*

Table 5. Computational results for $\nu = 0.9$ and $b = 0.2$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega^2_{1c}/\omega^2_{1\ \text{no load}}$	Differences	$\omega^2_{1t}/\omega^2_{1\ \text{no load}}$	Differences
0.000	0.99985049		1.00014949	
0.100	0.90001388	0.09983661	1.09998352	0.09983402
0.200	0.80002506	0.09998882	1.19996457	0.09998105
0.300	0.70003341	0.09999164	1.29994325	0.09997868
0.400	0.60003881	0.09999461	1.39991964	0.09997640
0.500	0.50004109	0.09999771	1.49989385	0.09997421
0.600	0.40004012	0.10000098	1.59986595	0.09997210
0.700	0.30003571	0.10000441	1.69983602	0.09997007
0.800	0.20002769	0.10000802	1.79980415	0.09996812
0.900	0.10001586	0.10001183	1.89977039	0.09996624
1.000	0.00000000	0.10001586	1.99973481	0.09996443

The quadratic approximations fit to the data presented in Table 5 with four-digit accuracy.

$$\frac{\omega^2_{1c}}{\omega^2_{1\ \text{no load}}} = -3.331\,337\,633 \times 10^{-4}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.999\,625\,6857\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,945\,6098, \tag{70a}$$

$$\frac{\omega^2_{1t}}{\omega^2_{1\ \text{no load}}} = 6.307\,884\,335\times 10^{-5}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.999\,631\,7373\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,053\,866. \tag{70b}$$

Table 6. Computational results for $\nu = 0.9$ and $b = 0.4$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99986338		1.00013660	
0.100	0.90020466	0.09965872	1.09975821	0.09962161
0.200	0.80037027	0.09983439	1.19948114	0.09972293
0.300	0.70049473	0.09987554	1.29917048	0.09968934
0.400	0.60057578	0.09991895	1.39882781	0.09965733
0.500	0.50061099	0.09996479	1.49845461	0.09962680
0.600	0.40059775	0.10001324	1.59805228	0.09959767
0.700	0.30053323	0.10006452	1.69762213	0.09956984
0.800	0.20041436	0.10011887	1.79716537	0.09954324
0.900	0.10023782	0.10017654	1.89668315	0.09951779
1.000	0.00000000	0.10023782	1.99617657	0.09949342

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -2.596\,816\,904 \times 10^{-3} \frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.997\,328\,4065 \frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,932\,6019, \tag{71a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -1.310\,482\,273 \times 10^{-3} \frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.997\,432\,2238 \frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,058\,716. \tag{71b}$$

Table 7. Computational results for $\nu = 0.9$ and $b = 0.5$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99987327		1.00012673	
0.100	0.90032899	0.09954428	1.09961298	0.09948625
0.200	0.80059649	0.09973249	1.19917114	0.09955816
0.300	0.70079878	0.09979771	1.29867743	0.09950629
0.400	0.60093180	0.09986698	1.39813462	0.09945719
0.500	0.50099116	0.09994064	1.49754526	0.09941064
0.600	0.40097208	0.10001908	1.59691174	0.09936648
0.700	0.30086936	0.10010271	1.69623628	0.09932454
0.800	0.20067734	0.10019202	1.79552097	0.09928468
0.900	0.10038982	0.10028753	1.89476772	0.09924676
1.000	0.00000000	0.10038982	1.99397837	0.09921065

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -4.104\,840\,441 \times 10^{-3} \frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.995\,790\,1871 \frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,920\,6831, \tag{72a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -2.080\,081\,369 \times 10^{-3} \frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.9959923923 \frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,062431. \tag{72b}$$

Table 8. Computational results for $\nu = 0.9$ and $b = 0.6$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99988350		1.00011651	
0.100	0.90032837	0.09955513	1.09961538	0.09949887
0.200	0.80059672	0.09973165	1.19917793	0.09956255
0.300	0.70080097	0.09979575	1.29869081	0.09951288
0.400	0.60093665	0.09986431	1.39815694	0.09946613
0.500	0.50099889	0.09993777	1.49757901	0.09942207
0.600	0.40098229	0.10001659	1.59695951	0.09938050
0.700	0.30088097	0.10010132	1.69630074	0.09934123
0.800	0.20068841	0.10019256	1.79560483	0.09930409
0.900	0.10039741	0.10029100	1.89487377	0.09926894
1.000	0.00000000	0.10039741	1.99410940	0.09923563

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -4.123\,396\,713\times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.995\,764\,61\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,919\,997\,9, \quad \text{(73a)}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -1.964\,394\,841\times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.996\,009\,0117\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,059\,584. \quad \text{(73b)}$$

Table 9. Computational results for $\nu = 0.9$ and $b = 0.8$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99989764		1.00010236	
0.100	0.90003396	0.09986368	1.09996006	0.09985771
0.200	0.80006160	0.09997236	1.19991449	0.09995442
0.300	0.70008252	0.09997908	1.29986357	0.09994909
0.400	0.60009631	0.09998621	1.39980760	0.09994403
0.500	0.50010250	0.09999381	1.49974684	0.09993923
0.600	0.40010058	0.10000191	1.59968151	0.09993468
0.700	0.30009002	0.10001057	1.69961186	0.09993035
0.800	0.20007019	0.10001983	1.79953809	0.09992623
0.900	0.10004044	0.10002975	1.89946039	0.09992230
1.000	0.00000002	0.10004042	1.99937895	0.09991856

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -5.252\,032\,453\times 10^{-4}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.999\,440\,542\,7\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,959\,9196, \quad \text{(74a)}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -1.137\,211\,210\times 10^{-4}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.999\,462\,7271\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,038\,224. \quad \text{(74b)}$$

8.2.2. *Solutions to Problem 2.*

Table 10. Computational results for $\nu = 0.8$ and $b = 0.2$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99985219		1.00014779	
0.100	0.90004447	0.09980773	1.09994714	0.09979935
0.200	0.80008021	0.09996426	1.19988624	0.09993910
0.300	0.70010685	0.09997336	1.29981761	0.09993137
0.400	0.60012397	0.09998288	1.39974153	0.09992392
0.500	0.50013114	0.09999283	1.49965827	0.09991675
0.600	0.40012789	0.10000325	1.59956811	0.09990983
0.700	0.30011372	0.10001417	1.69947127	0.09990316
0.800	0.20008809	0.10002564	1.79936799	0.09989672
0.900	0.10005039	0.10003770	1.89925848	0.09989049
1.000	0.00000000	0.10005039	1.99914295	0.09988447

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -6.912\,436\,031 \times 10^{-4}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.999\,263\,3476\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,943\,9839, \tag{75a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -1.802\,051\,770 \times 10^{-4}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.999\,281\,1886\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,054\,509. \tag{75b}$$

Table 11. Computational results for $\nu = 0.8$ and $b = 0.4$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99987346		1.00012652	
0.100	0.90062605	0.09924741	1.09925682	0.09913030
0.200	0.80112977	0.09949628	1.19840145	0.09914463
0.300	0.70150559	0.09962418	1.29743850	0.09903705
0.400	0.60174757	0.09975802	1.39637232	0.09893382
0.500	0.50184936	0.09989821	1.49520701	0.09883468
0.600	0.40180417	0.10004519	1.59394643	0.09873942
0.700	0.30160473	0.10019944	1.69259424	0.09864781
0.800	0.20124325	0.10036148	1.79115389	0.09855966
0.900	0.10071134	0.10053191	1.88962866	0.09847477
1.000	0.00000000	0.10071134	1.98802165	0.09839298

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -7.537\,051\,383 \times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.992\,328\,968\,6\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,906\,4673, \tag{76a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -4.640\,413\,266 \times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.992\,575\,2561\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,072\,981. \tag{76b}$$

Table 12. Computational results for $\nu = 0.8$ and $b = 0.5$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99988983		1.00011019	
0.100	0.90114665	0.09874319	1.09864583	0.09853564
0.200	0.80207470	0.09907195	1.19709441	0.09844858
0.300	0.70277232	0.09930238	1.29535536	0.09826095
0.400	0.60322679	0.09954553	1.39343768	0.09808232
0.500	0.50342440	0.09980239	1.49134978	0.09791211
0.600	0.40335040	0.10007400	1.58909957	0.09774979
0.700	0.30298888	0.10036152	1.68669445	0.09759488
0.800	0.20232264	0.10066624	1.78414136	0.09744691
0.900	0.10133306	0.10098957	1.88144685	0.09730549
1.000	0.00000000	0.10133306	1.97861707	0.09717021

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -1.381\,407\,376 \times 10^{-2}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.985\,936\,3720\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,855\,375\,9, \tag{77a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -8.037\,238\,551 \times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.986\,515\,837\,5\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,096\,317. \tag{77b}$$

Table 13. Computational results for $\nu = 0.8$ and $b = 0.6$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99990727		1.00009273	
0.100	0.90148845	0.09841882	1.09825700	0.09816427
0.200	0.80270523	0.09878321	1.19627516	0.09801816
0.300	0.70363169	0.09907354	1.29406894	0.09779378
0.400	0.60424744	0.09938425	1.39165166	0.09758272
0.500	0.50453016	0.09971728	1.48903562	0.09738396
0.600	0.40445537	0.10007479	1.58623218	0.09719656
0.700	0.30399619	0.10045919	1.68325185	0.09701967
0.800	0.20312300	0.10087319	1.78010438	0.09685253
0.900	0.10180313	0.10131987	1.87679882	0.09669444
1.000	0.00000046	0.10180267	1.97334357	0.09654475

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -1.820\,778\,680 \times 10^{-2}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.981\,395\,528\,8\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,791\,5525, \tag{78a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -9.\,344\,847\,633 \times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.982\,506\,6770\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,115\,576. \tag{78b}$$

Table 14. Computational results for $\nu = 0.8$ and $b = 0.8$

$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{ no load}}^2$	Differences	$\omega_{1t}^2/\omega_{1\text{ no load}}^2$	Differences
0.000	0.99993292		1.00006707	
0.100	0.90028412	0.09964880	1.09966903	0.09960196
0.200	0.80051788	0.09976624	1.19929442	0.09962540
0.300	0.70069738	0.09982051	1.29887909	0.09958467
0.400	0.60081828	0.09987910	1.39842568	0.09954659
0.500	0.50087577	0.09994251	1.49793660	0.09951093
0.600	0.40086450	0.10001127	1.59741409	0.09947748
0.700	0.30077846	0.10008604	1.69686015	0.09944607
0.800	0.20061092	0.10016754	1.79627667	0.09941652
0.900	0.10035429	0.10025663	1.89566536	0.09938869
1.000	0.00000000	0.10035429	1.99502779	0.09936244

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = -3.\,574\,398\,585 \times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.996\,325\,2147\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999\,937\,4502, \tag{79a}$$

$$\frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = -1.613\,827\,513 \times 10^{-3}\frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.996\,593\,5135\frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000\,040\,048. \tag{79b}$$

Remark 15. The differences listed in the tables vary very little as a function of N/N_{crit}: they can be considered practically constant. It is also worth noting that the largest change in value (which is still a very small change in value) concerning the differences is for the value $b = 0.5$, where the change in cross section is at the central point of the beam.

9. Concluding remarks

Making use of the definition presented in [18] for the Green functions of coupled boundary value problems, the paper has presented the Green functions of pinned-pinned stepped beams with heterogeneous cross section provided that (a) no axial load is exerted on the beam, (b) the beam is subjected to a compressive axial force, and (c) a tensile axial force is exerted on the beam. The eigenvalue problem related to the free vibrations of the pinned-pinned stepped beams is reduced to an eigenvalue problem governed by a homogeneous Fredholm integral equation. The vibration problems of the axially loaded stepped beams are also reduced to two Fredholm integral equations. Then these eigenvalue problems are solved numerically and the computational results are presented. It is a well known result that the equations

$$\frac{\omega_{1c}^2}{\omega_{1\text{ no load}}^2} = 1.0 - \frac{\mathcal{N}}{\mathcal{N}_{crit}}, \qquad \frac{\omega_{1t}^2}{\omega_{1\text{ no load}}^2} = 1.0 + \frac{\mathcal{N}}{\mathcal{N}_{crit}} \tag{80}$$

are the solutions to a similar problem for simply supported homogeneous and heterogeneous beams – in the second case cross sectional heterogeneity is assumed. According to our computational results, equations (80) provide very good solutions for both pinned-pinned and stepped beams – the maximum of the relative error in $\omega_{1t}^2/\omega_{1\ \text{no load}}^2$ for $\nu = 0.8$, $b = 0.5$ and $\mathcal{N}/\mathcal{N}_{crit} = 1.0$ is 1.069% (see Table 12).

Appendix A. Stability problem of pinned-pinned stepped beams

The stability problem of stepped beams is governed by ODEs (51) associated with boundary and continuity conditions (27). Making use of the solutions given by equations (54), the eigenvalue problem (51), (27) with $p_2 = \sqrt{\mathcal{N}_2}$ as the eigenvalue yields the following homogeneous equation system for the unknown integration constants:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & b & \cos bp_2\gamma & \sin bp_2\gamma & -1 & -b & -\cos p_2 b & -\sin p_2 b \\ 0 & 1 & -p_2\gamma \sin bp_2\gamma & p_2\gamma \cos bp_2\gamma & 0 & -1 & p_2 \sin p_2 b & -p_2 \cos p_2 b \\ 0 & 0 & -\cos bp_2\gamma & -\sin bp_2\gamma & 0 & 0 & \cos p_2 b & \sin p_2 b \\ 0 & 0 & \gamma \sin bp_2\gamma & -\gamma \cos bp_2\gamma & 0 & 0 & -\sin p_2 b & \cos p_2 b \\ 0 & 0 & 0 & 0 & 1 & \ell & \cos p_2\ell & \sin p_2\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos p_2\ell & \sin p_2\ell \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{81}$$

$$\gamma = \sqrt{\alpha}$$

The characteristic equation is the determinant of the coefficient matrix

$$\mathcal{D} = -\gamma\ell \cos b\gamma p_2 \sin p_2 (\ell - b) - \ell \sin b\gamma p_2 \cos p_2 (\ell - b) = 0. \tag{82}$$

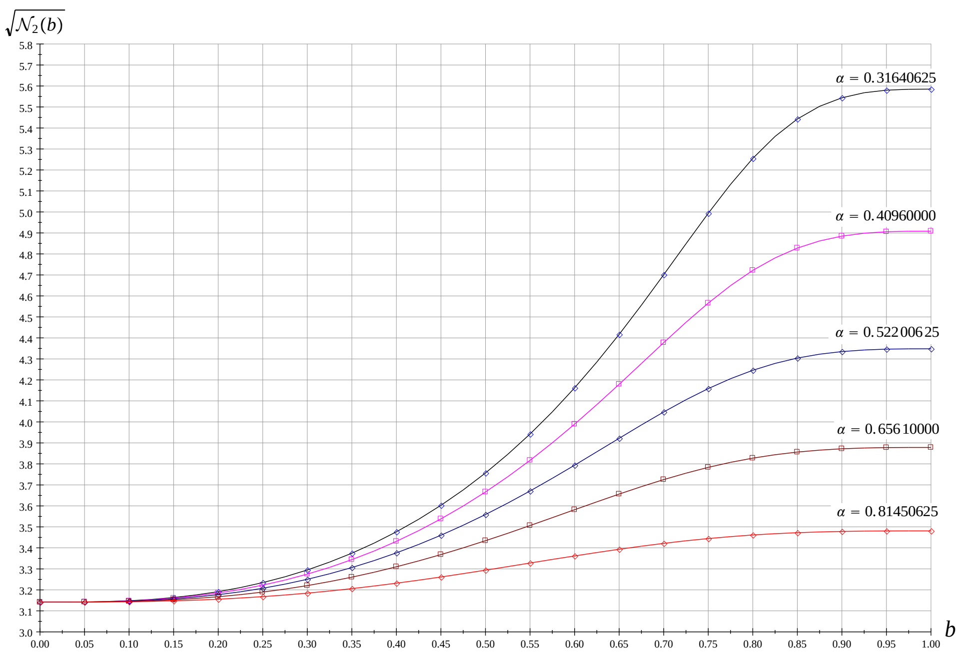

Figure 8. Critical force against b, α is a parameter

REMARK 16. Assume that $\alpha = b = \ell = 1$ and $p_2 = p$. Then we get the characteristic equation for a uniform fixed-fixed beam

$$\mathcal{D} = \sin p = 0 \tag{83}$$

where $p = \pi$ is the smallest root for p.

Equation (82) has been solved numerically. Figure 8 shows the critical force $\sqrt{N_{2\,\mathrm{crit}}(b)}$ against b obtained from the numerical solution mentioned.

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