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FUNDAMENTAL SOLUTIONS IN THE THEORY OF THERMOELASTIC DIFFUSIVE MATERIALS WITH MICROTEMPERATURES AND MICROCONCENTRATIONS

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Abstract. The main aim of this paper is to construct the fundamental solutions of a system of equations for isotropic thermoelastic diffusive materials with microtemperatures and microconcentrations in the case of steady oscillations in terms of elementary functions. In addition to this, the fundamental solutions of the system of equations of equilibrium theory of isotropic thermoelastic diffusivity materials with microtemperatures and microconcentrations are also established.

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1. Introduction

Eringen and his co-workers [1–7] formulated the theories of micromorphic continua. In these theories, the particles of a continuous body are assumed to be composed of microelements which undergo homogeneous deformations called microdeformations. The system of differential equations and boundary conditions governing a continuum with microstructure are deduced from the principles of conservation of mass, conservation of microinertia, balance of linear momentum, balance of first moment of momentum, and the balance of energy. The theory of thermodynamics of elastic bodies with microstructure was extended by [8] with the assumption that the microelements have different temperatures. He modified the Clausius–Duhem inequality to include microtemperatures and added first-order moment of energy equations to the basic balance laws for determining the microtemperatures of a continuum. Iesan and Quintanilla [9] constructed a linear theory for elastic materials with an inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. They established the continuous dependence of initial data and body loads and proved an existence theorem for initial boundary value problems using semigroup theory. The field equations of a theory of microstretch thermoelastic bodies with microtemperatures were established in [10], where Iesan proved a uniqueness theorem in the dynamic theory of anisotropic materials and then derived a linear theory of microstretch elastic solids with microtemperatures in which a microelement of a continuum is equipped with the mechanical degrees of freedom for rigid rotations and microdilatation in addition to the classical translation degrees of freedom [11]. He also established a uniqueness result in the dynamic theory of anisotropic bodies.

The mass transfer of a substance from a high concentration region to low-concentration regions is called diffusivity. Nowacki [12–15], Sherief and his co-workers [16], Aouadi [17] and Kansal [18] developed various thermoelastic diffusivity theories to describe coupled mechanical behavior among temperature, concentration, and strain fields in elastic solids. Aouadi et al. [19] developed the nonlinear theory of thermoelastic diffusivity materials with microtemperatures and microconcentrations. They also obtained the linear theory of thermoelastic diffusivity materials with microtemperatures and microconcentrations. They proved the well-posedness of a linear anisotropic problem with the help of the semigroup theory of linear operators and studied the asymptotic behaviour of the solutions. Bazarra et al. [20] introduced a numerical scheme in the linear theory of thermoelastic diffusivity materials with microtemperatures and microconcentrations based on the finite element method to approximate the spatial domain and the forwarded Euler scheme to discretize the time derivatives. They also deduced a priori error estimates for the approximative solutions, and obtained the linear convergence of the algorithm under suitable regularity assumptions. Chiril [21] derived the field equations and the consecutive equations of the linear theory of microstretch thermoelasticity for materials whose particles have microelements that are equipped with microtemperatures and microconcentrations.

There is a necessity to construct fundamental solutions for solving boundary value problems of elasticity and thermoelasticity by potential method [22]. The reason for constructing fundamental solutions is that an integral representation of the solution of a boundary value problem by fundamental solution is easily solved by numerical methods rather than a differential equation with specified boundary and initial conditions. Various authors [23, 24] and [25] constructed fundamental solutions in different theories of elasticity and thermoelasticity with microtemperatures.

In Section 2, the constitutive relations and field equations for isotropic thermoelastic diffusivity materials with microtemperatures and microconcentrations are written. The system of linearized equations for steady oscillations in the theory of thermoelastic diffusivity solids with microtemperatures and microconcentrations is obtained in Section 3. In Section 4, in terms of elementary functions, the fundamental solution of basic governing equations in the case of steady oscillations is constructed. Some basic properties of the fundamental matrix in the case of steady oscillations are discussed in Section 5. In Section 6, the fundamental solutions of basic governing equations in case of equilibrium are established.

2. Basic Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3, $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, $\mathbf{D_x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ and t denotes the time variable. Following [8, 10] and [19], the basic equations for an isotropic homogeneous thermoelastic diffusivity solid with microtemperatures and microconcentrations in the absence of body forces, heat sources, and mass diffusive sources are as follows:

Constitutive relations

$$t_{ij} = \lambda\, e_{ll}\delta_{ij} + 2\mu e_{ij} - \beta_1\theta\delta_{ij} - \beta_2 C\delta_{ij}, \tag{2.1}$$

$$\rho S = \beta_1 e_{ll} + \frac{\rho C_E}{T_0}\theta + \varpi C, \tag{2.2}$$

$$P = -\beta_2 e_{ll} - \varpi\,\theta + \chi C, \tag{2.3}$$

$$\rho\varepsilon_i = -c_1 T_i - \kappa_1 C_i, \tag{2.4}$$

$$\rho\Omega_i = -m_1 C_i - \kappa_1 T_i, \tag{2.5}$$

$$q_{ij} = -k_4 T_{l,l}\delta_{ij} - k_5 T_{i,j} - k_6 T_{j,i}, \tag{2.6}$$

$$q_i = k\theta_{,i} + k_1 T_i, \tag{2.7}$$

$$\tilde{\varsigma}_i = (k - k_3)\theta_{,i} + (k_1 - k_2)T_i, \tag{2.8}$$

$$\eta_{ij} = -h_4 C_{l,l}\delta_{ij} - h_5 C_{i,j} - h_6 C_{j,i}, \tag{2.9}$$

$$\sigma_i = (h - h_3)P_{,i} + (h_1 - h_2)C_i, \tag{2.10}$$

$$\eta_i = hP_{,i} + h_1 C_i, \tag{2.11}$$

$$\rho T_0\dot{S} = q_{i,i}, \tag{2.12}$$

$$\eta_{j,j} = \dot{C}, \tag{2.13}$$

Equations of motion

$$t_{ij,j} = \rho\ddot{u}_i, \tag{2.14}$$

Balance of first moment of energy

$$\rho\dot{\varepsilon}_i = q_{ji,i} + q_i - \tilde{\varsigma}_i, \tag{2.15}$$

Balance of first moment of mass diffusivity

$$\rho\dot{\Omega}_i = \eta_{ji,j} + \eta_i - \sigma_i, \tag{2.16}$$

where t_{ij} are the stress tensor components, $e_{ll} = u_{l,l}$ are the strain tensor components, and u_i are the displacement vector components. Lame's constants are λ and μ, $\beta_1 = (3\lambda + 2\mu)\alpha_t, \beta_2 = (3\lambda + 2\mu)\alpha_c$, α_t is the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusivity expansion, δ_{ij} is Kronecker's delta. The temperature is represented by $\theta = T - T_0$. The absolute temperature is T. In the reference configuration, the absolute temperature is T_0. C represents the concentration of diffusive material, ρ represents density, S represents entropy, C_E represents specific heat at constant strain, and P represents chemical potential. The first moments of energy vector and mass diffusivity vector are ε_i and Ω_i, respectively. T_i and C_i are microtemperature and microconcentration components, respectively. The microheat flux average is $\tilde{\varsigma}_i$. q_{ij}, η_{ij} are the first moment of heat flux and mass diffusivity flux tensors, respectively; σ_i is the micromass diffusivity flux average; q_i are the heat flux vector components; and η_i are the mass diffusivity flux vector components. The material constants are $\varpi, \chi, c_1, m_1, \kappa_1, k, k_1, ..., k_6$ and $h, h_1, ..., h_6$.

The governing equations for homogeneous isotropic thermoelastic diffusivity solid with microtemperatures and microconcentrations are obtained using equation (2.1) in (2.14), equations (2.4), (2.6)-(2.8) in (2.15), equations (2.5), (2.9)-(2.11) in (2.16), equations (2.2) and (2.7) in (2.12) and equations (2.3) and (2.11) in (2.13), as follows

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \beta_1\,\mathrm{grad}\,\theta - \beta_2\,\mathrm{grad}\,C = \rho\ddot{\mathbf{u}},\\
&k_6\Delta\mathbf{v} + (k_4+k_5)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v} - k_2\mathbf{v} - k_3\,\mathrm{grad}\,\theta = c_1\dot{\mathbf{v}} + \kappa_1\dot{\mathbf{w}},\\
&h_6\Delta\mathbf{w} + (h_4+h_5)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{w} - h_2\mathbf{w} - h_3\,\mathrm{grad}\,P = \kappa_1\dot{\mathbf{v}} + m_1\dot{\mathbf{w}},\\
&\beta_1 T_0\,\mathrm{div}\,\dot{\mathbf{u}} + \rho C_E\dot{\theta} + \varpi T_0\dot{C} = k\,\Delta\,\theta + k_1\,\mathrm{div}\,\mathbf{v},\\
&h\Delta[-\beta_2\,\mathrm{div}\,\mathbf{u} - \varpi\,\theta + \chi\,C] + h_1\,\mathrm{div}\,\mathbf{w} = \dot{C},
\end{aligned} \tag{2.17}$$

where Δ is the Laplacian operator, $\mathbf{v} = (T_1, T_2, T_3)$ and $\mathbf{w} = (C_1, C_2, C_3)$.

In the upcoming sections, the chemical potential has been used as a state variable rather than concentration. Therefore, the system of equations (2.17) with the help of equation (2.3) becomes

$$\begin{aligned}
&[\mu\Delta + (\lambda_0+\mu)\,\mathrm{grad}\,\mathrm{div}]\mathbf{u} - \rho\,\ddot{\mathbf{u}} - \gamma_1\,\mathrm{grad}\,\theta - \gamma_2\,\mathrm{grad}\,P = \mathbf{0},\\
&[k_6\Delta + (k_4+k_5)\,\mathrm{grad}\,\mathrm{div} - k_2]\mathbf{v} - c_1\,\dot{\mathbf{v}} - \kappa_1\dot{\mathbf{w}} - k_3\,\mathrm{grad}\,\theta = \mathbf{0},\\
&-\kappa_1\dot{\mathbf{v}} + [h_6\Delta + (h_4+h_5)\,\mathrm{grad}\,\mathrm{div} - h_2]\mathbf{w} - m_1\,\dot{\mathbf{w}} - h_3\,\mathrm{grad}\,P = \mathbf{0},\\
&-\gamma_1 T_0\,\mathrm{div}\,\dot{\mathbf{u}} + k_1\,\mathrm{div}\,\mathbf{v} + k\,\Delta\,\theta - cT_0\,\dot{\theta} - \kappa T_0\dot{P} = 0,\\
&-\gamma_2\,\mathrm{div}\,\dot{\mathbf{u}} + h_1\,\mathrm{div}\,\mathbf{w} - \kappa\,\dot{\theta} + h\Delta\,P - m\,\dot{P} = 0.
\end{aligned} \tag{2.18}$$

The coefficients $m, \kappa, \gamma_1, \gamma_2, \lambda_0$, and c are given in Appendix A.

3. Steady Oscillations

The displacement vector, microtemperature, microconcentration, temperature change, and chemical potential functions are assumed as:

$$\Big[\mathbf{u}(\mathbf{x},t), \mathbf{v}(\mathbf{x},t), \mathbf{w}(\mathbf{x},t), \theta(\mathbf{x},t), P(\mathbf{x},t)\Big] = \mathrm{Re}\Big[(\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*, \theta^*, P^*)e^{-\iota\omega t}\Big], \tag{3.1}$$

where ω is the frequency of oscillation.

Using equation (3.1) in the system of equations (2.18) and omitting the asterisk (*) for simplicity, the system of equations of steady oscillations is obtained as:

$$\begin{aligned}
&[\mu\Delta + (\lambda_0+\mu)\,\mathrm{grad}\,\mathrm{div} + \rho\omega^2]\mathbf{u} - \gamma_1\,\mathrm{grad}\,\theta - \gamma_2\,\mathrm{grad}\,P = \mathbf{0},\\
&[k_6\Delta + (k_4+k_5)\,\mathrm{grad}\,\mathrm{div} - k_2 + \iota\omega c_1]\mathbf{v} + \iota\omega\kappa_1\mathbf{w} - k_3\,\mathrm{grad}\,\theta = \mathbf{0},\\
&\iota\omega\kappa_1\mathbf{v} + [h_6\Delta + (h_4+h_5)\,\mathrm{grad}\,\mathrm{div} - h_2 + \iota\omega m_1]\mathbf{w} - h_3\,\mathrm{grad}\,P = \mathbf{0},\\
&\iota\omega\gamma_1 T_0\,\mathrm{div}\,\mathbf{u} + k_1\,\mathrm{div}\,\mathbf{v} + [k\,\Delta + \iota\omega cT_0]\theta + \iota\omega\kappa T_0 P = 0,\\
&\iota\omega\gamma_2\,\mathrm{div}\,\mathbf{u} + h_1\,\mathrm{div}\,\mathbf{w} + \iota\omega\kappa\,\theta + [h\,\Delta + \iota\omega m]P = 0.
\end{aligned} \tag{3.2}$$

We introduce the second-order matrix differential operators with constant coefficients

$$\mathbf{F}(\mathbf{D}_\mathbf{x}) = \Big(F_{gl}(\mathbf{D}_\mathbf{x})\Big)_{11\times 11},$$

where

$$F_{pq}(\mathbf{D_x}) = [\mu\Delta + \rho\omega^2]\delta_{pq} + (\lambda_0 + \mu)\frac{\partial^2}{\partial x_p \partial x_q}, F_{p;q+3}(\mathbf{D_x}) = F_{p+3;q}(\mathbf{D_x}) = 0,$$

$$F_{p;q+6}(\mathbf{D_x}) = F_{p+6;q}(\mathbf{D_x}) = 0, F_{p;10}(\mathbf{D_x}) = -\gamma_1\frac{\partial}{\partial x_p}, F_{p;11}(\mathbf{D_x}) = -\gamma_2\frac{\partial}{\partial x_p},$$

$$F_{p+3;q+3}(\mathbf{D_x}) = [k_6\Delta - k_2 + \iota\omega c_1]\delta_{pq} + (k_4 + k_5)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$F_{p+3;q+6}(\mathbf{D_x}) = F_{p+6;q+3}(\mathbf{D_x}) = \iota\omega\kappa_1\delta_{pq}, F_{p+3;10}(\mathbf{D_x}) = -k_3\frac{\partial}{\partial x_p},$$

$$F_{p+3;11}(\mathbf{D_x}) = F_{11;p+3}(\mathbf{D_x}) = 0, F_{p+6;q+6}(\mathbf{D_x}) =$$

$$= [h_6\Delta - h_2 + \iota\omega m_1]\delta_{pq} + (h_4 + h_5)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$F_{p+6;10}(\mathbf{D_x}) = F_{10;p+6}(\mathbf{D_x}) = 0, F_{p+6;11}(\mathbf{D_x}) = -h_3\frac{\partial}{\partial x_p}, F_{10;q}(\mathbf{D_x}) = \iota\omega\gamma_1 T_0\frac{\partial}{\partial x_q},$$

$$F_{10;q+3}(\mathbf{D_x}) = k_1\frac{\partial}{\partial x_q}, F_{10;10}(\mathbf{D_x}) = k\Delta + \iota\omega c T_0, F_{10;11}(\mathbf{D_x}) = \iota\omega\kappa T_0,$$

$$F_{11;q} = \iota\omega\gamma_2\frac{\partial}{\partial x_q}, F_{11;q+6}(\mathbf{D_x}) = h_1\frac{\partial}{\partial x_q}, F_{11;10}(\mathbf{D_x}) = \iota\omega\kappa,$$

$$F_{11;11}(\mathbf{D_x}) = h\Delta + \iota\omega m\,, \qquad p,q = 1,2,3.$$

and

$$\tilde{\mathbf{F}}(\mathbf{D_x}) = \Big(\tilde{F}_{gl}(\mathbf{D_x})\Big)_{11\times 11},$$

where

$$\tilde{F}_{pq}(\mathbf{D_x}) = \mu\Delta\delta_{pq} + (\lambda_0 + \mu)\frac{\partial^2}{\partial x_p \partial x_q}, \tilde{F}_{p+3;q+3}(\mathbf{D_x}) = k_6\Delta\delta_{pq} + (k_4 + k_5)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$\tilde{F}_{p+6;q+6}(\mathbf{D_x}) = h_6\Delta\delta_{pq} + (h_4 + h_5)\frac{\partial^2}{\partial x_p \partial x_q}, \tilde{F}_{10;10}(\mathbf{D_x}) = k\Delta, \tilde{F}_{11;11}(\mathbf{D_x}) = h\Delta,$$

$$\tilde{F}_{p;q+3}(\mathbf{D_x}) = \tilde{F}_{p;q+6}(\mathbf{D_x}) = \tilde{F}_{p+3;q}(\mathbf{D_x}) = \tilde{F}_{p+6;q}(\mathbf{D_x}) = 0,$$

$$\tilde{F}_{p+3;q+6}(\mathbf{D_x}) = \tilde{F}_{p+6;q+3}(\mathbf{D_x}) = \tilde{F}_{ie}(\mathbf{D_x}) = \tilde{F}_{ei}(\mathbf{D_x}) = 0,$$

$$\tilde{F}_{10;11}(\mathbf{D_x}) = \tilde{F}_{11;10}(\mathbf{D_x}) = 0\,, \quad p,q = 1,2,3; \quad e = 10,11; \quad i = 1,..,9.$$

The system of equations (3.2) can be represented as

$$\mathbf{F}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \theta, P)$ is a vector function with eleven components on E^3. The matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of operator $\mathbf{F}(\mathbf{D_x})$.

Definition 1: The operator $\mathbf{F}(\mathbf{D_x})$ is said to be elliptic if $|\tilde{\mathbf{F}}(\boldsymbol{k})| \neq 0$, $\boldsymbol{k} = (\mu_1, \mu_2, \mu_3)$.

Since $|\tilde{\mathbf{F}}(\boldsymbol{k})| = \mu^2\tilde{\lambda}kk_6k_7hh_6h_7|\boldsymbol{k}|^{22}$, $\tilde{\lambda} = \lambda_0 + 2\mu$, $k_7 = k_4 + k_5 + k_6$, $h_7 = h_4 + h_5 + h_6$.

Therefore, operator $\mathbf{F}(\mathbf{D_x})$ is an elliptic differential operator iff

$$\mu\tilde{\lambda}kk_6k_7hh_6h_7 \neq 0. \tag{3.3}$$

Definition 2: The fundamental solution of the system of equations (3.2) (the fundamental operator matrix $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = \Big(G_{gl}(\mathbf{x})\Big)_{11\times 11}$ satisfying condition

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}), \tag{3.4}$$

where $\delta(\mathbf{x})$ represents the Dirac delta, $\mathbf{I} = (\delta_{gl})_{11\times 11}$ is the unit matrix, and $\mathbf{x} \in \mathrm{E}^3$.

4. Construction of $\mathbf{G}(\mathbf{x})$ in Terms of Elementary Functions

Let us consider the system of non-homogeneous equations

$$[\mu\Delta + (\lambda_0+\mu)\,\mathrm{grad\,div} + \rho\omega^2]\mathbf{u} + \iota\omega\gamma_1 T_0\,\mathrm{grad}\,\theta + \iota\omega\gamma_2\,\mathrm{grad}\,P = \mathbf{H}, \tag{4.1}$$

$$[k_6\Delta + (k_4+k_5)\,\mathrm{grad\,div} + k_8]\mathbf{v} + \iota\omega\kappa_1\mathbf{w} + k_1\,\mathrm{grad}\,\theta = \mathbf{V}, \tag{4.2}$$

$$\iota\omega\kappa_1\mathbf{v} + [h_6\Delta + (h_4+h_5)\,\mathrm{grad\,div} + h_8]\mathbf{w} + h_1\,\mathrm{grad}\,P = \mathbf{W}, \tag{4.3}$$

$$-\gamma_1\,\mathrm{div}\,\mathbf{u} - k_3\,\mathrm{div}\,\mathbf{v} + [k\,\Delta + \iota\omega c T_0]\theta + \iota\omega\kappa P = Z, \tag{4.4}$$

$$-\gamma_2\,\mathrm{div}\,\mathbf{u} - h_3\,\mathrm{div}\,\mathbf{w} + \iota\omega\kappa T_0\,\theta + [h\,\Delta + \iota\omega m]P = X, \tag{4.5}$$

where $k_8 = -k_2 + \iota\omega c_1, h_8 = -h_2 + \iota\omega m_1$; $\mathbf{H}, \mathbf{V}, \mathbf{W}$ are vector functions with three components on E^3; Z and X are scalar functions on E^3.

Equations (4.1)-(4.5) can also be written as

$$\mathbf{F}^{tr}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{4.6}$$

where $\mathbf{F}^{tr}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q} = (\mathbf{H}, \mathbf{V}, \mathbf{W}, Z, X)$ and $\mathbf{x} \in \mathrm{E}^3$.

Using the divergence (div) operator on the equations (4.1) -(4.3), we get

$$[\tilde{\lambda}\Delta + \rho\omega^2]\,\mathrm{div}\,\mathbf{u} + \iota\omega\gamma_1 T_0\Delta\,\theta + \iota\omega\gamma_2\Delta\,P = \mathrm{div}\,\mathbf{H}, \tag{4.7}$$

$$(k_7\Delta + k_8)\,\mathrm{div}\,\mathbf{v} + \iota\omega\kappa_1\,\mathrm{div}\,\mathbf{w} + k_1\Delta\theta = \mathrm{div}\,\mathbf{V}, \tag{4.8}$$

$$\iota\omega\kappa_1\,\mathrm{div}\,\mathbf{v} + (h_7\Delta + h_8)\,\mathrm{div}\,\mathbf{w} + h_1\Delta P = \mathrm{div}\,\mathbf{W}. \tag{4.9}$$

The equations (4.4), (4.5) and (4.7)-(4.9) can be expressed as

$$\mathbf{N}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}}\,, \tag{4.10}$$

where $\mathbf{S}, \tilde{\mathbf{Q}}$, and $\mathbf{N}(\Delta)$ are given in Appendix A.

The equation (4.10) can be written in determinant form as

$$\Gamma_1(\Delta)\mathbf{S} = \mathbf{\Psi}, \tag{4.11}$$

where $\Gamma_1(\Delta)$, and $\mathbf{\Psi}$ are given in Appendix A.

On expanding $\Gamma_1(\Delta)$, we see that

$$\Gamma_1(\Delta) = \prod_{i=1}^{5}(\Delta + \lambda_i^2),$$

where $\lambda_i^2,\ i = 1, ..., 5$ are the roots of the equation $\Gamma_1(-\xi) = 0$(with respect to ξ).

Applying operator $\Gamma_1(\Delta)$ to the equation (4.1), we get

$$\Gamma_1(\Delta)(\Delta + \lambda_6^2)\mathbf{u} = \mathbf{\Psi}', \tag{4.12}$$

where λ_6^2, and $\mathbf{\Psi}'$ are given in Appendix A.

Multiplying equations (4.2) and (4.3) by $h_6\Delta + h_8$ and $\iota\omega\kappa_1$ respectively, we obtain

$$\begin{aligned}(h_6\Delta + h_8)[k_6\Delta + (k_4 + k_5)\,\mathrm{grad\,div} + k_8]\mathbf{v} + (h_6\Delta + h_8)\iota\omega\kappa_1\mathbf{w}\\ = (h_6\Delta + h_8)[\mathbf{V} - k_1\,\mathrm{grad}\,\theta],\end{aligned} \tag{4.13}$$

and

$$(\iota\omega\kappa_1)^2\mathbf{v} + \iota\omega\kappa_1[h_6\Delta + (h_4 + h_5\,)\mathrm{grad\,div} + h_8]\mathbf{w} = \iota\omega\kappa_1[\mathbf{W} - h_1\,\mathrm{grad}\,P]. \tag{4.14}$$

Using equation (4.14) in equation (4.13), we obtain

$$\begin{aligned}[(h_6\Delta + h_8)(k_6\Delta + k_8) - (\iota\omega\kappa_1)^2]\mathbf{v} = \iota\omega\kappa_1(h_4 + h_5)\,\mathrm{grad\,div}\,\mathbf{w}\\ +(h_6\Delta + h_8)[\mathbf{V} - k_1\,\mathrm{grad}\,\theta - (k_4 + k_5)\,\mathrm{grad\,div}\,\mathbf{v}] - \iota\omega\kappa_1[\mathbf{W} - h_1\,\mathrm{grad}\,P].\end{aligned} \tag{4.15}$$

Applying operator $\Gamma_1(\Delta)$ to the equation (4.15) and using equation (4.11), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{v} = \mathbf{\Psi}'', \tag{4.16}$$

where $\Gamma_2(\Delta)$, and $\mathbf{\Psi}''$ are given in Appendix A.

It can be seen that

$$\Gamma_2(\Delta) = (\Delta + \lambda_7^2)(\Delta + \lambda_8^2),$$

where λ_7^2, λ_8^2 are the roots of the equation $\Gamma_2(-\xi) = 0$(with respect to ξ).

Multiplying equations (4.2) and (4.3) by $\iota\omega\kappa_1$ and $k_6\Delta + k_8$ respectively, we obtain

$$(\iota\omega\kappa_1)[k_6\Delta + (k_4 + k_5)\,\mathrm{grad\,div} + k_8]\mathbf{v} + (\iota\omega\kappa_1)^2\mathbf{w} = (\iota\omega\kappa_1)[\mathbf{V} - k_1\,\mathrm{grad}\,\theta], \tag{4.17}$$

and

$$\begin{aligned}(\iota\omega\kappa_1)(k_6\Delta + k_8)\mathbf{v} + (k_6\Delta + k_8)[h_6\Delta + (h_4 + h_5\,)\mathrm{grad\,div} + h_8]\mathbf{w} =\\ = (k_6\Delta + k_8)[\mathbf{W} - h_1\,\mathrm{grad}\,P].\end{aligned} \tag{4.18}$$

Utilizing equation (4.17) in equation (4.18), we obtain

$$\begin{aligned}[(h_6\Delta + h_8)(k_6\Delta + k_8) - (\iota\omega\kappa_1)^2]\mathbf{w} = \iota\omega\kappa_1(k_4 + k_5)\,\mathrm{grad\,div}\,\mathbf{v}\\ +(k_6\Delta + k_8)[\mathbf{W} - h_1\,\mathrm{grad}\,P - (h_4 + h_5)\,\mathrm{grad\,div}\,\mathbf{w}] - \iota\omega\kappa_1[\mathbf{V} - k_1\,\mathrm{grad}\,\theta].\end{aligned} \tag{4.19}$$

Applying operator $\Gamma_1(\Delta)$ to the equation (4.19) and using equation (4.11), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{w} = \mathbf{\Psi}''', \tag{4.20}$$

where $\mathbf{\Psi}'''$ is given in Appendix A.

From equations (4.11), (4.12), (4.16) and (4.20), we obtain

$$\mathbf{\Theta}(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\mathbf{\Psi}}(\mathbf{x}), \tag{4.21}$$

where $\hat{\mathbf{\Psi}}$, and $\mathbf{\Theta}(\Delta)$ are given in Appendix B.

The expressions for $\mathbf{\Psi}', \mathbf{\Psi}'', \mathbf{\Psi}'''$ and Ψ_p, $(p = 4, 5)$ can be rewritten in the form

$$\mathbf{\Psi}' = \frac{1}{\mu}\Big[\Gamma_1(\Delta)\mathbf{J} + w_{11}(\Delta)\text{ grad div}\Big]\mathbf{H} + \sum_{i=2}^{5} w_{i1}(\Delta)\,\text{grad}\, w_i, \tag{4.22a}$$

$$\begin{aligned}\mathbf{\Psi}'' &= \Big[\frac{1}{N^*}(h_6\Delta + h_8)\Gamma_1(\Delta)\mathbf{J} + w_{22}(\Delta)\,\text{grad div}\Big]\mathbf{V} + w_{12}(\Delta)\,\text{grad div}\,\mathbf{H}\\ &+ w_{42}(\Delta)\,\text{grad}\,Z + w_{52}(\Delta)\,\text{grad}\,X + \Big[-\frac{1}{N^*}\iota\omega\kappa_1\Gamma_1(\Delta)\mathbf{J} + w_{32}(\Delta)\,\text{grad div}\Big]\mathbf{W},\end{aligned} \tag{4.22b}$$

$$\begin{aligned}\mathbf{\Psi}''' &= \Big[\frac{1}{N^*}(k_6\Delta + k_8)\Gamma_1(\Delta)\mathbf{J} + w_{33}(\Delta)\,\text{grad div}\Big]\mathbf{W} + w_{13}(\Delta)\,\text{grad div}\,\mathbf{H}\\ &+ w_{43}(\Delta)\,\text{grad}\,Z + w_{53}(\Delta)\,\text{grad}\,X + \Big[-\frac{1}{N^*}\iota\omega\kappa_1\Gamma_1(\Delta)\mathbf{J} + w_{23}(\Delta)\,\text{grad div}\Big]\mathbf{V},\end{aligned} \tag{4.22c}$$

$$\Psi_p = w_{1p}(\Delta)\,\text{div}\,\mathbf{H} + w_{2p}(\Delta)\,\text{div}\,\mathbf{V} + w_{3p}(\Delta)\,\text{div}\,\mathbf{W} + w_{4p}(\Delta)\,Z + w_{5p}(\Delta)\,X, \tag{4.22d}$$

where $\mathbf{J} = (\delta_{gh})_{3\times 3}$ is the unit matrix and the coefficients w_{pi}, $p, i = 1, ..., 5$ are given in Appendix B.

From equations (4.22), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{tr}(\mathbf{D_x})\mathbf{Q}(\mathbf{x}), \tag{4.23}$$

where the matrix $\mathbf{R}(\mathbf{D_x})$ is given in Appendix B.

From equations (4.6), (4.21) and (4.23), we obtain

$$\mathbf{\Theta}\mathbf{U} = \mathbf{R}^{tr}\mathbf{F}^{tr}\mathbf{U}.$$

The above relation implies

$$\mathbf{R}^{tr}\mathbf{F}^{tr} = \mathbf{\Theta}.$$

Therefore, we obtain

$$\mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x}) = \mathbf{\Theta}(\Delta). \tag{4.24}$$

We assume that
$$\lambda_p^2 \neq \lambda_q^2 \neq 0 \ p,q = 1,.....,8 \ p \neq q.$$
Let
$$\mathbf{Y}(\mathbf{x}) = \Big(Y_{ij}(\mathbf{x})\Big)_{11\times 11}, \ Y_{pp}(\mathbf{x}) = \sum_{g=1}^{6} r_{1g}\varsigma_g(\mathbf{x}),$$
$$Y_{p+3;p+3}(\mathbf{x}) = Y_{p+6;p+6}(\mathbf{x}) = \sum_{g=1,g\neq 6}^{8} r_{2g}\varsigma_g(\mathbf{x}),$$
$$Y_{ll}(\mathbf{x}) = \sum_{g=1}^{5} r_{3g}\varsigma_g(\mathbf{x}), Y_{qz}(\mathbf{x}) = 0 \ p = 1,2,3 \ l = 10,11 \ q,z = 1,.....,11 \ q \neq z$$
where $\varsigma_g(\mathbf{x})$, $g = 1,.....,8$, r_{1p}, $p = 1,.....,6$, r_{2l}, $l = 1,....7,8$, and r_{3q}, $q = 1,...,5$ are given in Appendix C.

Lemma 1: The matrix $\mathbf{Y}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}(\Delta)$, i.e.
$$\mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}). \tag{4.25}$$

Proof: To prove the lemma, it is sufficient to prove that
$$\Gamma_1(\Delta)(\Delta + \lambda_6^2)Y_{11}(\mathbf{x}) = \delta(\mathbf{x}), \tag{4.26}$$
$$\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{44}(\mathbf{x}) = \delta(\mathbf{x}), \tag{4.27}$$
$$\Gamma_1(\Delta)Y_{10;10}(\mathbf{x}) = \delta(\mathbf{x}). \tag{4.28}$$
Let us consider a sum
$$\sum_{i=1}^{6} r_{1i} = \frac{\sum_{j=1}^{6}(-1)^j z_j}{z_7},$$
where z_j, $j = 1,....,7$ are given in Appendix C.

On simplifying the right hand side of above relation, we obtain
$$\sum_{i=1}^{6} r_{1i} = 0. \tag{4.29}$$
Similarly, we find that
$$\sum_{i=2}^{6} r_{1i}(\lambda_1^2 - \lambda_i^2) = 0, \sum_{i=3}^{6} r_{1i}\Big[\prod_{j=1}^{2}(\lambda_j^2 - \lambda_i^2)\Big] = 0,$$
$$\sum_{i=4}^{6} r_{1i}\Big[\prod_{j=1}^{3}(\lambda_j^2 - \lambda_i^2)\Big] = 0, \sum_{i=5}^{6} r_{1i}\Big[\prod_{j=1}^{4}(\lambda_j^2 - \lambda_i^2)\Big] = 0,$$
$$\prod_{j=1}^{5} r_{16}(\lambda_j^2 - \lambda_6^2) = 1. \tag{4.30}$$

Also,

$$(\Delta+\lambda_p^2)\varsigma_g(\mathbf{x})=\delta(\mathbf{x})+(\lambda_p^2-\lambda_g^2)\varsigma_g(\mathbf{x}),\qquad p,g=1,...,8. \tag{4.31}$$

Now, consider

$$\Gamma_1(\Delta)(\Delta+\lambda_6^2)Y_{11}(\mathbf{x})=\prod_{i=1}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6}r_{1g}\varsigma_g(\mathbf{x}).$$

Using equations (4.29)-(4.31) in the above relation, we obtain

$$\begin{aligned}
\Gamma_1(\Delta)(\Delta+\lambda_6^2)Y_{11}(\mathbf{x}) &= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6}r_{1g}\Big[\delta(\mathbf{x})+(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\\
&= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\Big[\delta(\mathbf{x})\sum_{g=1}^{6}r_{1g}+\sum_{g=2}^{6}r_{1g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\\
&= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=2}^{6}r_{1g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\\
&= \prod_{i=3}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=2}^{6}r_{1g}(\lambda_1^2-\lambda_g^2)\Big[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]\\
&= \prod_{i=3}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{6}r_{1g}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]\\
&= \prod_{i=4}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{6}r_{1g}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]\\
&= \prod_{i=4}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{6}r_{1g}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]\\
&= \prod_{i=5}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{6}r_{1g}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]\\
&= \prod_{i=5}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{6}r_{1g}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]\\
&= (\Delta+\lambda_6^2)\Big[\sum_{g=5}^{6}r_{1g}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]\\
&= (\Delta+\lambda_6^2)\varsigma_6(\mathbf{x})=\delta(\mathbf{x}).
\end{aligned}$$

Equations (4.27) and (4.28) can be proved in a similar way.

We introduce the matrix

$$\mathbf{G}(\mathbf{x})=\mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}). \tag{4.32}$$

From equations (4.24), (4.25) and (4.32), we obtain

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) = \mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}(\mathbf{x})$ is a solution to the equation (3.4).

Theorem 1: If condition (3.3) is satisfied, then the fundamental solution of the system of equations (3.2) is the matrix $\mathbf{G}(\mathbf{x})$ given by equation (4.32) and it is represented as follows:

$$G_{gl}(\mathbf{x}) = R_{gl}(\mathbf{D_x})Y_{11}(\mathbf{x}), G_{gq}(\mathbf{x}) = R_{gq}(\mathbf{D_x})Y_{44}(\mathbf{x}), G_{gj}(\mathbf{x}) = R_{gj}(\mathbf{D_x})Y_{10;10}(\mathbf{x}),$$
$$g = 1, \ldots\ldots, 11; \quad l = 1, 2, 3; \quad q = 4, \ldots\ldots, 9; \quad j = 10, 11.$$

5. Basic Properties of Matrix $\mathbf{G}(\mathbf{x})$

Theorem 2: Each column of the matrix $\mathbf{G}(\mathbf{x})$ is a solution of system of equations (3.2) at every point $\mathbf{x} \in \mathrm{E}^3$ except at the origin.

Theorem 3: If the condition (3.3) is satisfied, then the fundamental solution of the system $\tilde{\mathbf{F}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ is the matrix

$$\mathbf{B}(\mathbf{x}) = \Big(B_{rz}(\mathbf{x})\Big)_{11\times 11},$$

$$B_{ij}(\mathbf{x}) = \left[\frac{1}{\tilde{\lambda}}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{\mu}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$B_{i+3;j+3}(\mathbf{x}) = \left[\frac{1}{k_7}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{k_6}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$B_{i+6;j+6}(\mathbf{x}) = \left[\frac{1}{h_7}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{h_6}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$B_{10;10} = \frac{\varsigma_1^*(\mathbf{x})}{k}, B_{11;11} = \frac{\varsigma_1^*(\mathbf{x})}{h}, B_{iq} = B_{qi} = 0, B_{i+3;l} = B_{l;i+3} = 0,$$

$$B_{i+6;d} = B_{d;i+6} = 0, B_{10;11} = B_{11;10} = 0, \varsigma_1^* = -\frac{1}{4\pi|\mathbf{x}|}, \varsigma_2^* = -\frac{|\mathbf{x}|}{8\pi},$$

$$\tilde{R}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} - \Delta\delta_{ij}, \quad i, j = 1, 2, 3; \quad q = 4, \ldots., 11; \quad l = 7, \ldots, 11; \quad d = 10, 11.$$

6. Fundamental Solutions of System of Equations in Equilibrium Theory

If we put $\omega = 0$ in the system of equations (3.2), we obtain the system of equations in equilibrium theory of thermoelastic diffusivity with microtemperatures and microconcentrations as:

$$[\mu\Delta + (\lambda_0 + \mu)\,\mathrm{grad\,div}]\mathbf{u} - \gamma_1\,\mathrm{grad}\,\theta - \gamma_2\,\mathrm{grad}\,P = \mathbf{0},$$

$$[k_6\Delta + (k_4 + k_5)\,\mathrm{grad\,div} - k_2]\mathbf{v} - k_3\,\mathrm{grad}\,\theta = \mathbf{0},$$

$$[h_6\Delta + (h_4 + h_5)\,\mathrm{grad\,div} - h_2]\mathbf{w} - h_3\,\mathrm{grad}\,P = \mathbf{0},$$

$$k_1 \operatorname{div} \mathbf{v} + k \, \Delta \, \theta = 0,$$
$$h_1 \operatorname{div} \mathbf{w} + h \, \Delta \, P = 0. \tag{6.1}$$

The second-order matrix differential operator with constant coefficients is introduced as:

$$\mathbf{E}(\mathbf{D_x}) = \Big(E_{gl}(\mathbf{D_x})\Big)_{11 \times 11},$$

where matrix $\mathbf{E}(\mathbf{D_x})$ can be obtained from $\mathbf{F}(\mathbf{D_x})$ by taking $\omega = 0$.

The system of equations (6.1) can be represented as

$$\mathbf{E}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}.$$

Definition 3: The operator $\mathbf{E}(\mathbf{D_x})$ is said to be elliptic differential operator iff equation (3.3) is satisfied.

Definition 4: The fundamental solution of the system of equations (6.1) (the fundamental matrix of operator $\mathbf{E}$) is the matrix $\mathbf{G}'(\mathbf{x}) = \Big(G'_{gl}(\mathbf{x})\Big)_{11 \times 11}$ satisfying condition

$$\mathbf{E}(\mathbf{D_x})\mathbf{G}'(\mathbf{x}) = \delta(\mathbf{x}) \, \mathbf{I}(\mathbf{x}). \tag{6.2}$$

We consider the system of non-homogeneous equations

$$[\mu\Delta + (\lambda_0 + \mu) \operatorname{grad} \operatorname{div}]\mathbf{u} = \mathbf{H}', \tag{6.3}$$
$$[k_6\Delta + (k_4 + k_5) \operatorname{grad} \operatorname{div} - k_2]\mathbf{v} + k_1 \operatorname{grad} \theta = \mathbf{V}', \tag{6.4}$$
$$[h_6\Delta + (h_4 + h_5) \operatorname{grad} \operatorname{div} - h_2]\mathbf{w} + h_1 \operatorname{grad} P = \mathbf{W}', \tag{6.5}$$
$$-\gamma_1 \operatorname{div} \mathbf{u} - k_3 \operatorname{div} \mathbf{v} + k \, \Delta \, \theta = Z', \tag{6.6}$$
$$-\gamma_2 \operatorname{div} \mathbf{u} - h_3 \operatorname{div} \mathbf{w} + h \, \Delta \, P = X', \tag{6.7}$$

where $\mathbf{H}', \mathbf{V}', \mathbf{W}'$ are vector functions with three components on E^3; Z' and X' are scalar functions on E^3.

The system of equations (6.3)-(6.7) can also be written in the form

$$\mathbf{E}^{tr}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}'(\mathbf{x}), \tag{6.8}$$

where $\mathbf{E}^{tr}$ is the transpose of matrix $\mathbf{E}$ and $\mathbf{Q}'(\mathbf{x}) = (\mathbf{H}', \mathbf{V}', \mathbf{W}', Z', X')$.

Applying operator div to the equations (6.3)-(6.5), we obtain

$$\Delta \operatorname{div} \mathbf{u} = \frac{1}{\tilde{\lambda}} \operatorname{div} \mathbf{H}' = \Phi_1, \tag{6.9}$$
$$(k_7\Delta - k_2) \operatorname{div} \mathbf{v} + k_1 \, \Delta \, \theta = \operatorname{div} \mathbf{V}', \tag{6.10}$$
$$(h_7\Delta - h_2) \operatorname{div} \mathbf{w} + h_1 \, \Delta \, P = \operatorname{div} \mathbf{W}'. \tag{6.11}$$

Using equation (6.6) in the equation (6.10), we get

$$\Delta(\Delta - D^2) \operatorname{div} \mathbf{v} = \Phi_2, \tag{6.12}$$

where D^2, and Φ_2 are given in Appendix D.

Using equation (6.7) in equation (6.11), we get

$$\Delta(\Delta - L^2)\,\mathrm{div}\,\mathbf{w} = \Phi_3, \tag{6.13}$$

where L^2, and Φ_3 are given in Appendix D.

Applying operators $\Delta(\Delta - D^2)$ and $\Delta(\Delta - L^2)$ to the equations (6.6) and (6.7), respectively and using equations (6.12) and (6.13), we get

$$\Delta^2(\Delta - D^2)\,\theta = \Phi_4, \tag{6.14}$$

$$\Delta^2(\Delta - L^2)\,P = \Phi_5, \tag{6.15}$$

where Φ_4, and Φ_5 are given in Appendix D.

Applying the operators $\Delta, \Delta^2(\Delta - D^2), \Delta^2(\Delta - L^2)$ to the equations (6.3), (6.4) and (6.5) respectively and using equations (6.9) and (6.12)-(6.15), we obtain

$$\begin{gathered}\Delta^2\,\mathbf{u} = \mathbf{\Phi}',\\ \Delta^2(\Delta - D^2)\left(\Delta - \frac{k_2}{k_6}\right)\mathbf{v} = \mathbf{\Phi}'',\\ \Delta^2(\Delta - L^2)\left(\Delta - \frac{h_2}{h_6}\right)\mathbf{v} = \mathbf{\Phi}''',\end{gathered} \tag{6.16}$$

where $\mathbf{\Phi}'$, $\mathbf{\Phi}''$, and $\mathbf{\Phi}'''$ are given in Appendix D.

From equations (6.14)-(6.16), we get

$$\mathbf{\Lambda}(\Delta)\,\mathbf{U}(\mathbf{x}) = \hat{\mathbf{\Phi}}(\mathbf{x}), \tag{6.17}$$

where $\mathbf{\Lambda}(\Delta)$, and $\hat{\mathbf{\Phi}}(\mathbf{x})$ are given in Appendix D.

The expressions for $\mathbf{\Phi}'$, $\mathbf{\Phi}''$, $\mathbf{\Phi}'''$, and Φ_p, $p = 4, 5$ can be rewritten as

$$\hat{\mathbf{\Phi}}(\mathbf{x}) = \mathbf{T}^{tr}(\mathbf{D_x})\mathbf{Q}'(\mathbf{x}), \tag{6.18}$$

where matrix $\mathbf{T}(\mathbf{D_x})$ is given in Appendix E.

From equations (6.8), (6.17) and (6.18), we get

$$\mathbf{E}(\mathbf{D_x})\mathbf{T}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta). \tag{6.19}$$

Let

$$\begin{gathered}\mathbf{Y}'(\mathbf{x}) = \Big(Y'_{ij}(\mathbf{x})\Big)_{11\times 11},\\ Y'_{pp}(\mathbf{x}) = \varsigma_2^*(\mathbf{x}),\ Y'_{p+3;p+3}(\mathbf{x}) = r'_{11}\varsigma_2^*(\mathbf{x}) + r'_{12}\varsigma_1^*(\mathbf{x}) + r'_{13}\varsigma_3^*(\mathbf{x}) + r'_{15}\varsigma_5^*(\mathbf{x}),\\ Y'_{p+6;p+6}(\mathbf{x}) = r'_{21}\varsigma_2^*(\mathbf{x}) + r'_{22}\varsigma_1^*(\mathbf{x}) + r'_{24}\varsigma_4^*(\mathbf{x}) + r'_{26}\varsigma_6^*(\mathbf{x}),\\ Y'_{10;10}(\mathbf{x}) = r'_{31}\varsigma_2^*(\mathbf{x}) + r'_{32}\varsigma_1^*(\mathbf{x}) + r'_{33}\varsigma_3^*(\mathbf{x}),\\ Y'_{11;11}(\mathbf{x}) = r'_{41}\varsigma_2^*(\mathbf{x}) + r'_{42}\varsigma_1^*(\mathbf{x}) + r'_{44}\varsigma_4^*(\mathbf{x}),\\ Y_{qz}(\mathbf{x}) = 0\ p = 1, 2, 3\ q, z = 1, \ldots.., 11\ q \neq z,\end{gathered}$$

where $\varsigma_i^*(\mathbf{x})$, $i = 3, ..., 6$, $r'_{1j}, j = 1, 2, 3, 5$, $r'_{2q}, q = 1, 2, 4, 6$, $r'_{3z}, z = 1, 2, 3$, and $r'_{1p}, p = 1, 2, 4$ are given in Appendix F.

Lemma 2: The matrix $\mathbf{Y}'$ defined above is the fundamental matrix of operator $\mathbf{\Lambda}(\Delta)$, i.e.

$$\mathbf{\Lambda}(\Delta)\mathbf{Y}'(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}). \tag{6.20}$$

Proof: To prove the lemma, it is sufficient to prove that

$$\Delta^2 Y'_{11}(\mathbf{x}) = \delta(\mathbf{x}), \Delta^2(\Delta - D^2)(\Delta - \tau_1^2)Y'_{44}(\mathbf{x}) = \delta(\mathbf{x}), \Delta^2(\Delta - L^2)(\Delta - \tau_2^2)Y'_{77}(\mathbf{x}) = \delta(\mathbf{x}),$$

$$\Delta^2(\Delta - D^2)Y'_{10;10}(\mathbf{x}) = \delta(\mathbf{x}), \Delta^2(\Delta - L^2)Y'_{11;11}(\mathbf{x}) = \delta(\mathbf{x}). \tag{6.21}$$

It is very easy to prove equations (6.21). This has been left for the reader.

We introduce the matrix

$$\mathbf{G}'(\mathbf{x}) = \mathbf{T}(\mathbf{D_x})\mathbf{Y}'(\mathbf{x}). \tag{6.22}$$

From equations (6.19), (6.20) and (6.22), we obtain

$$\mathbf{E}(\mathbf{D_x})\mathbf{G}'(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}).$$

Hence $\mathbf{G}'(\mathbf{x})$ is a solution to the equation (6.2).

Theorem 4: If the condition (3.3) is met, then the fundamental solution of the system of equations (6.1) is the matrix $\mathbf{G}'(\mathbf{x})$ given by the equation (6.22) and it can be represented in the following form:

$$G'_{gl}(\mathbf{x}) = T_{gl}(\mathbf{D_x})Y'_{11}(\mathbf{x}), G'_{g;l+3}(\mathbf{x}) = T_{g;l+3}(\mathbf{D_x})Y'_{44}(\mathbf{x}),$$

$$G'_{g;l+6}(\mathbf{x}) = T_{g;l+6}(\mathbf{D_x})Y'_{77}(\mathbf{x}), G'_{gj}(\mathbf{x}) = T_{gj}(\mathbf{D_x})Y'_{jj}(\mathbf{x}),$$

$$g = 1, ..., 11 \; l = 1, 2, 3 \; j = 10, 11.$$

7. Conclusions

In terms of elementary functions, the fundamental solution of system of equations in the theory of thermoelastic diffusive materials with microtemperatures and microconcentrations in the case of steady oscillations has been constructed. By potential method, the fundamental solution to the system of equations makes it possible to investigate three-dimensional boundary value problems of theory of thermoelastic diffusive materials with microtemperatures and microconcentrations. Some basic properties of the fundamental matrix are also discussed.

Appendix A

$$m = \frac{1}{\chi}, \; \kappa = m\varpi, \; \gamma_1 = \beta_1 + \beta_2\kappa, \; \gamma_2 = \beta_2 m, \; \lambda_0 = \lambda - \beta_2\gamma_2, \; c = \frac{\rho C_E}{T_0} + \varpi\kappa$$

$$\mathbf{S} = (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{w}, \theta, P), \tilde{\mathbf{Q}} = (w_1, ..., w_5) = (\operatorname{div}\mathbf{H}, \operatorname{div}\mathbf{V}, \operatorname{div}\mathbf{W}, Z, X),$$

$$\mathbf{N}(\Delta) = \Big(N_{gl}(\Delta)\Big)_{5\times 5} =$$

$$= \begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 & 0 & 0 & \iota\omega\gamma_1 T_0 \Delta & \iota\omega\gamma_2\Delta \\ 0 & k_7\Delta + k_8 & \iota\omega\kappa_1 & k_1\Delta & 0 \\ 0 & \iota\omega\kappa_1 & h_7\Delta + h_8 & 0 & h_1\Delta \\ -\gamma_1 & -k_3 & 0 & k\Delta + \iota\omega c T_0 & \iota\omega\kappa \\ -\gamma_2 & 0 & -h_3 & \iota\omega\kappa T_0 & h\Delta + \iota\omega m \end{pmatrix}_{5\times 5}$$

$$\mathbf{\Psi} = (\Psi_1, ..., \Psi_5), \Psi_p = \frac{1}{M^*}\sum_{i=1}^{5} N^*_{ip} w_i,$$

$$\Gamma_1(\Delta) = \frac{1}{M^*}\,|\,\mathbf{N}(\Delta)|, \;\; M^* = \tilde{\lambda} k k_7 h h_7, \quad p = 1, ..., 5$$

and N^*_{ip} is the cofactor of the element N_{ip} of the matrix $\mathbf{N}$.

$$\lambda_6^2 = \frac{\rho\omega^2}{\mu}, \mathbf{\Psi}' = \frac{1}{\mu}\Big[\Gamma_1(\Delta)\mathbf{H} - \mathrm{grad}[(\lambda_0+\mu)\Psi_1 + \iota\omega\gamma_1 T_0\Psi_4 + \iota\omega\gamma_2\Psi_5]\Big],$$

$$\Gamma_2(\Delta) = \frac{1}{N^*}\begin{vmatrix} k_6\Delta + k_8 & \iota\omega\kappa_1 \\ \iota\omega\kappa_1 & h_6\Delta + h_8 \end{vmatrix},$$

$$N^* = k_6 h_6, \mathbf{\Psi}'' = \frac{1}{N^*}\Big[(h_6\Delta + h_8)[\Gamma_1(\Delta)\mathbf{V} - k_1\,\mathrm{grad}\,\Psi_4 - (k_4+k_5)\,\mathrm{grad}\,\Psi_2]$$

$$-\iota\omega\kappa_1[\Gamma_1(\Delta)\mathbf{W} - h_1\,\mathrm{grad}\,\Psi_5 - (h_4+h_5)\,\mathrm{grad}\,\Psi_3]\Big],$$

$$\mathbf{\Psi}''' = \frac{1}{N^*}\Big[(k_6\Delta + k_8)[\Gamma_1(\Delta)\mathbf{W} - h_1\,\mathrm{grad}\,\Psi_5 - (h_4+h_5)\,\mathrm{grad}\,\Psi_3]$$

$$-\iota\omega\kappa_1[\Gamma_1(\Delta)\mathbf{V} - k_1\,\mathrm{grad}\,\Psi_4 - (k_4+k_5)\,\mathrm{grad}\,\Psi_2]\Big]$$

Appendix B

$$\hat{\mathbf{\Psi}} = (\mathbf{\Psi}', \mathbf{\Psi}'', \mathbf{\Psi}''', \Psi_4, \Psi_5),$$

$$\mathbf{\Theta}(\Delta) = \Big(\Theta_{gq}(\Delta)\Big)_{11\times 11}$$

$$\Theta_{pp}(\Delta) = \Gamma_1(\Delta)(\Delta + \lambda_6^2) = \prod_{i=1}^{6}(\Delta + \lambda_i^2),$$

$$\Theta_{p+3;p+3}(\Delta) = \Theta_{p+6;p+6}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{i=1, i\neq 6}^{8}(\Delta + \lambda_i^2),$$

$$\Theta_{jj}(\Delta) = \Gamma_1(\Delta) = \prod_{i=1}^{5}(\Delta + \lambda_i^2),\; \Theta_{gq}(\Delta) = 0,$$

$$p = 1, 2, 3; \quad g, q = 1, ..., 11; \quad j = 10, 11; \quad g \neq q$$

$$w_{p1}(\Delta) = -\frac{1}{M^*\mu}\Big[(\lambda_0+\mu)N^*_{p1}(\Delta) + \iota\omega\gamma_1 T_0 N^*_{p4}(\Delta) + \iota\omega\gamma_2 N^*_{p5}(\Delta)\Big]$$

$$w_{p2}(\Delta)=-\frac{1}{M^*N^*}\Big[(h_6\Delta+h_8)[(k_4+k_5)N^*_{p2}+k_1N^*_{p4}]-\iota\omega\kappa_1h_1N^*_{p5}-\iota\omega\kappa_1(h_4+h_5)N^*_{p3}\Big]$$

$$w_{p3}(\Delta)=-\frac{1}{M^*N^*}\Big[(k_6\Delta+k_8)[(h_4+h_5)N^*_{p3}+h_1N^*_{p5}]-\iota\omega\kappa_1k_1N^*_{p4}-\iota\omega\kappa_1(k_4+k_5)N^*_{p2}\Big]$$

$$w_{p4}(\Delta)=\frac{N^*_{p4}}{M^*},\ w_{p5}(\Delta)=\frac{N^*_{p5}}{M^*},\quad p=1,.....,5$$

$$\mathbf{R}(\mathbf{D_x})=\Big(R_{gq}(\mathbf{D_x})\Big)_{11\times 11}$$

$$R_{ij}(\mathbf{D_x})=\frac{1}{\mu}\Gamma_1(\Delta)\delta_{ij}+w_{11}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i+3;j+3}(\mathbf{D_x})=\frac{1}{N^*}(h_6\Delta+h_8)\Gamma_1(\Delta)\delta_{ij}+w_{22}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i+6;j+6}(\mathbf{D_x})=\frac{1}{N^*}(k_6\Delta+k_8)\Gamma_1(\Delta)\delta_{ij}+w_{33}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i;j+3}(\mathbf{D_x})=w_{12}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},R_{i;j+6}(\mathbf{D_x})=w_{13}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i;p+6}(\mathbf{D_x})=w_{1p}(\Delta)\frac{\partial}{\partial x_i},R_{i+3;j}(\mathbf{D_x})=w_{21}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i+3;j+6}(\mathbf{D_x})=w_{23}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j}-\frac{1}{N^*}\iota\omega\kappa_1\Gamma_1(\Delta)\delta_{ij},$$

$$R_{i+3;p+6}(\mathbf{D_x})=w_{2p}(\Delta)\frac{\partial}{\partial x_i},R_{i+6;j}=w_{31}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i+6;j+3}(\mathbf{D_x})=w_{32}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j}-\frac{1}{N^*}\iota\omega\kappa_1\Gamma_1(\Delta)\delta_{ij},$$

$$R_{i+6;p+6}(\mathbf{D_x})=w_{3p}(\Delta)\frac{\partial}{\partial x_i},R_{p+6;i}(\mathbf{D_x})=w_{p1}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{p+6;i+3}(\mathbf{D_x})=w_{p2}(\Delta)\frac{\partial}{\partial x_i},R_{p+6;i+6}(\mathbf{D_x})=w_{p3}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{p+6;l+6}=w_{pl}(\Delta),\ \ i,j=1,2,3;\ \ p,l=4,5$$

Appendix C

$$\varsigma_g(\mathbf{x})=-\frac{e^{\iota\lambda_g|\mathbf{x}|}}{4\pi|\mathbf{x}|},r_{1p}=\prod_{i=1,i\neq p}^{6}(\lambda_i^2-\lambda_p^2)^{-1},r_{2l}=\prod_{i=1,i\neq 6,i\neq l}^{8}(\lambda_i^2-\lambda_l^2)^{-1},$$

$$r_{3q}=\prod_{i=1,i\neq q}^{5}(\lambda_i^2-\lambda_q^2)^{-1},\ \ p=1,.....,6;\ \ g=1,.....,8;\ \ l=1,....7,8;\ \ q=1,...,5$$

$$z_1=\prod_{i=3}^{6}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2-\lambda_l^2)(\lambda_5^2-\lambda_6^2),$$

$$z_2 = \prod_{i=3}^{6}(\lambda_1^2 - \lambda_i^2) \prod_{j=4}^{6}(\lambda_3^2 - \lambda_j^2) \prod_{l=5}^{6}(\lambda_4^2 - \lambda_l^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_3 = \prod_{i=2, i\neq 3}^{6}(\lambda_1^2 - \lambda_i^2) \prod_{j=4}^{6}(\lambda_2^2 - \lambda_j^2) \prod_{l=5}^{6}(\lambda_4^2 - \lambda_l^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_4 = \prod_{i=2, i\neq 4}^{6}(\lambda_1^2 - \lambda_i^2) \prod_{j=3, j\neq 4}^{6}(\lambda_2^2 - \lambda_j^2) \prod_{l=5}^{6}(\lambda_3^2 - \lambda_l^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_5 = \prod_{i=2, i\neq 5}^{6}(\lambda_1^2 - \lambda_i^2) \prod_{j=3, j\neq 5}^{6}(\lambda_2^2 - \lambda_j^2) \prod_{l=4, l\neq 5}^{6}(\lambda_3^2 - \lambda_l^2)(\lambda_4^2 - \lambda_6^2),$$

$$z_6 = \prod_{i=2}^{5}(\lambda_1^2 - \lambda_i^2) \prod_{j=3}^{5}(\lambda_2^2 - \lambda_j^2) \prod_{l=4}^{5}(\lambda_3^2 - \lambda_l^2)(\lambda_4^2 - \lambda_5^2),$$

$$z_7 = \prod_{i=2}^{6}(\lambda_1^2 - \lambda_i^2) \prod_{j=3}^{6}(\lambda_2^2 - \lambda_j^2) \prod_{l=4}^{6}(\lambda_3^2 - \lambda_l^2) \prod_{p=5}^{6}(\lambda_4^2 - \lambda_p^2)(\lambda_5^2 - \lambda_6^2).$$

Appendix D

$$D^2 = \frac{1}{k\,k_7}(k\,k_2 - k_3 k_1), \Phi_2 = \frac{1}{k\,k_7}[k\,\Delta\,\mathrm{div}\,\mathbf{V}' - k_1\gamma_1\Phi_1 - k_1\,\Delta\,Z'],$$

$$L^2 = \frac{1}{h\,h_7}(h\,h_2 - h_3 h_1), \Phi_3 = \frac{1}{h\,h_7}[h\,\Delta\,\mathrm{div}\,\mathbf{W}' - h_1\gamma_2\Phi_1 - h_1\,\Delta\,X'],$$

$$\Phi_4 = \frac{1}{k}[k_3\,\Phi_2 + (\Delta - D^2)(\Delta\,Z' + \gamma_1\Phi_1)], \Phi_5 = \frac{1}{h}[h_3\,\Phi_3 + (\Delta - L^2)(\Delta\,X' + \gamma_2\Phi_1)],$$

$$\mathbf{\Phi}' = \frac{1}{\mu}[\Delta\,\mathbf{H}' - (\lambda_0 + \mu)\,\mathrm{grad}\,\Phi_1]$$

$$\mathbf{\Phi}'' = -\frac{\gamma_1 k_1}{k\,k_7}\left(\Delta - \frac{k_2}{k_6}\right)\mathrm{grad}\,\Phi_1 - \frac{k_1}{k\,k_7}\left(\Delta - \frac{k_2}{k_6}\right)\mathrm{grad}\,\Delta\,Z' +$$

$$\frac{1}{k_6}\left[\Delta^2(\Delta - D^2) - \frac{1}{k\,k_7}\left\{(k_4 + k_5)k\,\Delta + k_1 k_3\right\}\Delta\,\mathrm{grad}\,\mathrm{div}\right]\mathbf{V}'$$

$$\mathbf{\Phi}''' = -\frac{\gamma_2 h_1}{h\,h_7}\left(\Delta - \frac{h_2}{h_6}\right)\mathrm{grad}\,\Phi_1 - \frac{h_1}{h\,h_7}\left(\Delta - \frac{h_2}{h_6}\right)\mathrm{grad}\,\Delta\,X' +$$

$$\frac{1}{h_6}\left[\Delta^2(\Delta - L^2) - \frac{1}{h\,h_7}\left\{(h_4 + h_5)h\,\Delta + h_1 h_3\right\}\Delta\,\mathrm{grad}\,\mathrm{div}\right]\mathbf{W}',$$

$$\hat{\mathbf{\Phi}}(\mathbf{x}) = (\mathbf{\Phi}', \mathbf{\Phi}'', \mathbf{\Phi}''', \Phi_4, \Phi_5), \mathbf{\Lambda}(\Delta) = \Big(\Lambda_{pq}(\Delta)\Big)_{11\times 11}$$

$$\Lambda_{ii}(\Delta) = \Delta^2, \Lambda_{i+3;i+3}(\Delta) = \Delta^2(\Delta - D^2)\left(\Delta - \frac{k_2}{k_6}\right),$$

$$\Lambda_{i+6;i+6}(\Delta) = \Delta^2(\Delta - L^2)\left(\Delta - \frac{h_2}{h_6}\right), \quad \Lambda_{10;10} = \Delta^2(\Delta - D^2),$$

$$\Lambda_{11;11} = \Delta^2(\Delta - L^2), \quad \Lambda_{lj} = 0, \quad i, j = 1, 2, 3; \quad l, j = 1, ..., 11; \quad l \neq j$$

Appendix E

$$\mathbf{T}(\mathbf{D_x}) = \Big(T_{gl}(\mathbf{D_x})\Big)_{11\times 11}$$

$$T_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\,\Delta\,\delta_{ij} + m_{11}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$T_{i+3;j+3}(\mathbf{D_x}) = \frac{1}{k_6}\,\Delta^2(\Delta - D^2)\,\delta_{ij} + m_{22}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$T_{i+6;j+6}(\mathbf{D_x}) = \frac{1}{h_6}\,\Delta^2(\Delta - L^2)\,\delta_{ij} + m_{22}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$T_{10;10}(\mathbf{D_x}) = m_{44}(\Delta),\ T_{11;11}(\mathbf{D_x}) = m_{55}(\Delta),\ T_{i;j+3}(\mathbf{D_x}) = m_{12}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$T_{i;j+6}(\mathbf{D_x}) = m_{13}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},\ T_{i;10}(\mathbf{D_x}) = m_{14}(\Delta)\frac{\partial}{\partial x_i},\ T_{i;11}(\mathbf{D_x}) = m_{15}(\Delta)\frac{\partial}{\partial x_i},$$

$$T_{i+3;j}(\mathbf{D_x}) = T_{i+3;j+6}(\mathbf{D_x}) = T_{i+3;11}(\mathbf{D_x}) = 0, \quad T_{i+3;10}(\mathbf{D_x}) = m_{24}(\Delta)\frac{\partial}{\partial x_i},$$

$$T_{i+6;j}(\mathbf{D_x}) = T_{i+6;j+3}(\mathbf{D_x}) = T_{i+6;10}(\mathbf{D_x}) = 0, \quad T_{i+6;11}(\mathbf{D_x}) = m_{35}(\Delta)\frac{\partial}{\partial x_i},$$

$$T_{10;i}(\mathbf{D_x}) = T_{10;i+6}(\mathbf{D_x}) = T_{10;11}(\mathbf{D_x}) = 0, \quad T_{10;i+3}(\mathbf{D_x}) = m_{42}(\Delta)\frac{\partial}{\partial x_i},$$

$$T_{11;i}(\mathbf{D_x}) = T_{11;i+3}(\mathbf{D_x}) = T_{11;10}(\mathbf{D_x}) = 0, \quad T_{11;i+6}(\mathbf{D_x}) = m_{53}(\Delta)\frac{\partial}{\partial x_i},$$

$$m_{11}(\Delta) = -\frac{\lambda_0 + \mu}{\mu\tilde{\lambda}}, \quad m_{22}(\Delta) = -\frac{\Delta[k(k_4 + k_5)\Delta + k_1 k_3]}{k\,k_6\,k_7},$$

$$m_{33}(\Delta) = -\frac{\Delta[h(h_4 + h_5)\Delta + h_1 h_3]}{h\,h_6\,h_7}, \quad m_{44}(\Delta) = \frac{\Delta(k_7\Delta - k_2)}{k\,k_7},$$

$$m_{55}(\Delta) = \frac{\Delta(h_7\Delta - h_2)}{h\,h_7}, \quad m_{12}(\Delta) = -\frac{k_1\gamma_1(\Delta - \frac{k_2}{k_6})}{\tilde{\lambda}\,k\,k_7},$$

$$m_{13}(\Delta) = -\frac{h_1\gamma_2(\Delta - \frac{h_2}{h_6})}{\tilde{\lambda}\,h\,h_7}, \quad m_{14}(\Delta) = \frac{\gamma_1(k_7\,\Delta - k_2)}{\tilde{\lambda}\,k\,k_7},$$

$$m_{15}(\Delta) = \frac{\gamma_2(h_7\,\Delta - h_2)}{\tilde{\lambda}\,h\,h_7}, \quad m_{24}(\Delta) = \frac{k_3\,\Delta}{k\,k_7}, \quad m_{35}(\Delta) = \frac{h_3\,\Delta}{h\,h_7},$$

$$m_{42}(\Delta) = -\frac{k\,\Delta(\Delta - \frac{k_2}{k_6})}{k\,k_7}, \quad m_{53}(\Delta) = -\frac{h\,\Delta(\Delta - \frac{h_2}{h_6})}{h\,h_7}, \quad i,j = 1,2,3$$

Appendix F

$$\varsigma_3^*(\mathbf{x}) = -\frac{e^{-D|\mathbf{x}|}}{4\pi|\mathbf{x}|},\ \varsigma_4^*(\mathbf{x}) = -\frac{e^{-L|\mathbf{x}|}}{4\pi|\mathbf{x}|},\ \varsigma_5^*(\mathbf{x}) = -\frac{e^{-\tau_1|\mathbf{x}|}}{4\pi|\mathbf{x}|},\ \varsigma_6^*(\mathbf{x}) = -\frac{e^{-\tau_2|\mathbf{x}|}}{4\pi|\mathbf{x}|},$$

$$r'_{11} = \frac{1}{D^2\tau_1^2},\ r'_{12} = \frac{D^2 + \tau_1^2}{D^4\tau_1^4},\ r'_{13} = \frac{1}{D^4(D^2 - \tau_1^2)},\ r'_{15} = \frac{1}{\tau_1^4(\tau_1^2 - D^2)},$$

$$r'_{21} = \frac{1}{L^2\tau_2^2},\ r'_{22} = \frac{L^2+\tau_2^2}{L^4\tau_2^4},\ r'_{24} = \frac{1}{L^4(L^2-\tau_2^2)},\ r'_{26} = \frac{1}{\tau_2^4(\tau_2^2-L^2)},$$

$$r'_{31} = -\frac{1}{D^2},\ r'_{32} = -r'_{33} = -\frac{1}{D^4},\ r'_{41} = -\frac{1}{L^2},\ r'_{42} = -r'_{44} = -\frac{1}{L^4},$$

$$\tau_1^2 = \frac{k_2}{k_6},\ \tau_2^2 = \frac{h_2}{h_6}.$$

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