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SOLUTIONS FOR THE VIBRATION AND STABILITY PROBLEMS OF HETEROGENOUS BEAMS WITH THREE SUPPORTS USING GREEN FUNCTIONS

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Abstract. The goal of this study is to calculate the eigenvalues that provide the eigenfrequencies and the critical loads for two heterogeneous beams with three supports: the (first) [second] beam is (fixed)[pinned] at the left end, the intermediate support is a roller while the right end of the beams can move vertically but the rotation is prevented there. The beams are referred to as FrsRp and PrsRp beams. Determination of the (eigenfrequencies) [critical loads] leads to three point eigenvalue problems associated with homogeneous boundary conditions. With the Green functions that belong to these eigenvalue problems we can transform them into eigenvalue problems governed by homogeneous Fredholm integral equations. The eigenvalue problems can then be reduced to algebraic eigenvalue problems that are solvable numerically by utilizing effective solution algorithms.

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1. Introduction

Since beam buckling can be a prevalent cause of failure in engineering applications, it has been the focus of research for a long time. The Swiss mathematician Leonhard Euler was a pioneer in this subject, publishing his well-known formula for the critical (buckling) load of straight bars under compression in 1759. There are multiple sources about shells, columns, arches and other structures [1–3]. For example, the books [3, 4] provide extremely thorough information about solutions to a wide range of engineering problems, as well as applications. Article [5] investigates experimentally, analytically and numerically the static and dynamic stability problem of columns under self-weight. In [6] both geometrical and load imperfections are considerred in the buckling studies of columns.

Furthermore, the first concept of the Green function was published by George Green in 1828. His book [7] presents, discusses, and demonstrates how to use the Green function approach to electrostatic issues governed by partial differential equations. In the publication [8], the Green function for two-point boundary value problems

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governed by ordinary differential equations was established. In 1926, the first book [9] that comprehensively covered the notion of the Green function was published.

The results published in [10] were generalized for degenerated ordinary differential equation systems in 1975 [11, 12].

In the publication [13], the existence proof for several three-point boundary value issues linked to third-order nonlinear differential equations is presented by using Green functions. The related Green functions for some three-point boundary value problems governed by linear ordinary differential equations of order two are provided in article [14].

The free vibration and buckling problems of two heterogeneous beams are solved in this article based on the aforementioned literature. Cross-sectional inhomogeneity refers to the fact that the material is linearly elastic, isotropic, and the material distribution can change throughout the cross-section. Free vibration and stability equations are given for three-point boundary value issues. These are subsequently replaced with Fredholm integral equations using the Kernel function. A formulation of the Green function for three-point boundary value issues with homogeneous boundary conditions is also included. The boundary element approach is used to provide numerical solutions to integral equations, and algebraic equations are introduced in this manner. The eigenvalues of free vibration and the linear buckling loads are affected significantly by the location of the middle support in general. The results are compared to the results of some finite element calculations and high correlation is found.

2. Differential equations

2.1. Governing equations. The considerred heterogeneous FrsRp and PrsRp beams are shown in Figure 1. The axial force N acting on the beams is compressive. The cross section of the beams is uniform throughout their length. The axis $\hat{x}$ of the coordinate system $\hat{x}, \hat{y}, \hat{z}$ coincide with the E-weighted center line of the beams. Its origin is located at the left end of the beam. The beams are symmetric with respect to the coordinate plane $\hat{x}\hat{z}$. It is assumed that the modulus of elasticity E satisfies

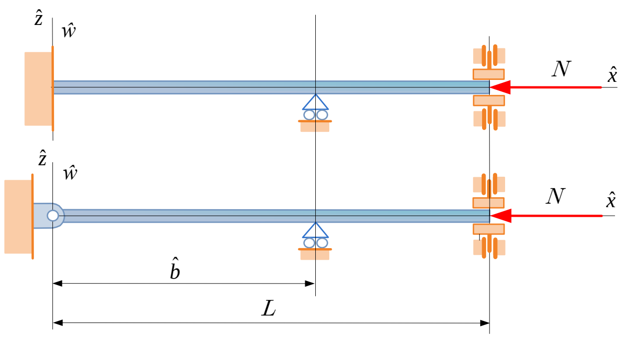

Figure 1. FrsRp and PrsRp beams

the condition $E(\hat{y},\hat{z}) = E(-\hat{y},\hat{z})$ over the cross section A, i.e., it is independent of the coordinate $\hat{z}$. In this case the beam has cross sectional heterogeneity [15]. L is the length of the beams while $\hat{b}$ gives the position of the middle roller support.

The E-weighted first moment $Q_{\hat{y}}$ is zero in this coordinate system:

$$Q_{\hat{y}} = \int_A \hat{z}E(\hat{y},\hat{z})\mathrm{d}A = 0\,. \tag{2.1}$$

Equilibrium problems of beams with cross sectional heterogeneity – the axial force N is zero – are governed by the ordinary differential equation [15]:

$$\frac{\mathrm{d}^4\hat{w}}{\mathrm{d}\hat{x}^4} = \frac{\hat{f}_z}{I_{ey}}\,, \tag{2.2}$$

where $\hat{w}(x)$ is the vertical displacement of the material points on the E-weighted center line, $\hat{f}_z(x)$ is the intensity of the verical distributed load acing on the beam. The E-weighted moment of inertia I_{ey} is defined by the equation

$$I_{ey} = \int_A E(\hat{y},\hat{z})z^2\,\mathrm{d}A\,. \tag{2.3}$$

If the beam is homogeneous the modulus of elasticity E is constant. Hence

$$I_{ey} = IE, \qquad I = \int_A z^2\,\mathrm{d}A \tag{2.4}$$

in which I is the moment of inertia.

In what follows we shall use dimensionless variables defined by the following relations [16]

$$\begin{aligned} x &= \hat{x}/L, \qquad \xi = \hat{\xi}/L, \qquad w = \hat{w}/L,\\ y &= \frac{\mathrm{d}\hat{w}}{\mathrm{d}\hat{x}} = \frac{\mathrm{d}w}{\mathrm{d}x}, \qquad b = \hat{b}/\hat{\ell}\,, \qquad \ell = \frac{x}{L}\Big|_{x=L} = 1\,, \end{aligned} \tag{2.5}$$

where $\hat{\xi}$ is also a coordinate measured on the axis $\hat{x}$ with the same origin as for $\hat{x}$. Applying dimensionless quantities to equation (2.2) we have

$$w^{(4)} = f_z\,, \quad w^{(0)} = w\,, \quad w^{(k)} = \frac{\mathrm{d}^k w}{\mathrm{d}x^k}, \quad (k = 1,\dots,4); \quad f_z = \frac{L^3\hat{f}_z}{I_{ey}} \tag{2.6}$$

Table 1.

Boundary conditions	
FrsRp beams	PrsRp beam
$w(0)=0\,,\ w^{(1)}(0)=0$	$w(0)=0\,,\ w^{(2)}(0)=0$
$w^{(1)}(\ell)=0\,,\ w^{(3)}(\ell)=0$	$w^{(1)}(\ell)=0\,,\ w^{(3)}(\ell)=0$
Continuity conditions	
$w(b-0)=w(b+0)=0\,,$ $w^{(1)}(b-0)=w^{(1)}(b+0)\,,$ $w^{(2)}(b-0)=w^{(2)}(b+0)\,,$	

The ordinary differential equation $(2.6)_1$ (ODE) is associated with the boundary and continuity conditions presented in Table 1.

The general solution for the homogeneous ODE

$$w^{(4)}=0 \tag{2.7}$$

is very simple:

$$w=\sum_{n=0}^{n=4}a_nw_n=a_n+a_1x^1+a_2x^2+a_3x^3+a_4x^4\,, \tag{2.8}$$

in which a_k $(k=0,\dots,4)$ are undetermined integration constants.

Making use of the Green functions that belongs to the boundary value problems determined by ODE (2.6) and the corresponding boundary and continuity conditions presented in Table 1 the solution for the dimensionless deflection w is given by the integral

$$w(x)=\int_0^\ell G(x,\xi)f_z(\xi)\,\mathrm{d}\xi\,. \tag{2.9}$$

where $G(x,\xi)$ stand for the Green functions in question.

The Green functions we shall need are presented in Section 3.

2.2. **Vibration problem.** The dimensionless amplitude for the free vibrations of FrsRp and PrsRp beams will also be denoted by w. It should fulfill the the following homogeneous ODE

$$\frac{\mathrm{d}^4w}{\mathrm{d}x^4}=\lambda w\,,\qquad \lambda=\frac{\rho_aA\omega^2L^4}{I_{ey}}\,, \tag{2.10}$$

where λ is the eigenvalue sought, ρ_a is the average density over the cross section while ω is the circular frequency of the vibrations.

Substituting $\lambda w(\xi)$ for $f(\xi)$ in (2.9) yields the homogeneous Fredholm integral equation

$$w(x)=\lambda\int_{\xi=0}^{\ell=1}G(x,\xi)\,w(\xi)\,\mathrm{d}\xi\,. \tag{2.11}$$

In this approach, the three point eigenvalue problem determined by ODE (2.10) and the boundary and continuity conditions presented in Table 1 is reduced to an eigenvalue problem governed by the homogeneous Fredholm integral equation (2.11).

2.3. **Stability problem.** If the uniform heterogeneous beams shown in Figure 1 are subjected to an axial force N the corresponding equilibrium problems are governed by ODE

$$w^{(4)} \pm \mathcal{N}\, w^{(2)} = f_z, \qquad \mathcal{N} = L^2 \frac{N}{I_{ey}}, \tag{2.12}$$

where the axial force N is constant ($N > 0$) while the sign of $\mathcal{N}$ is [positive] (negative) if the axial force is [compressive] (tensile).

If the stability problem is considered the axial force is compressive and $f_z = 0$. We have, therefore, two eigenvalue problems (one for each beam shown in Figure 1) – the eigenvalue sought is $\mathcal{N}$ – determined by ODE

$$w^{(4)} = -\mathcal{N}\, w^{(2)} \tag{2.13}$$

and the boundary and continuity conditions in Table 1. If we write $-\mathcal{N}\, w^{(2)}$ for f_z in (2.9) we get

$$w(x) = -\mathcal{N} \int_0^\ell G(x,\xi) \frac{\mathrm{d}^2 w(\xi)}{\mathrm{d}\xi^2}\, \mathrm{d}\xi = -\mathcal{N} \left(G(x,\xi) \frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi}\bigg|_{\xi=0}^{\ell} - \int_0^\ell \frac{\partial G(x,\xi)}{\partial \xi} \frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi}\, \mathrm{d}\xi \right)$$

where

$$G(x,\xi) \frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi}\bigg|_{\xi=0}^{\ell} = 0$$

since $G(x,0)$ is zero and the derivative $\mathrm{d}w(\xi)/\mathrm{d}\xi$ is also zero if $\xi = \ell = 1$. Hence

$$w(x) = \mathcal{N} \int_0^\ell \frac{\partial G(x,\xi)}{\partial \xi} \frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi}\, \mathrm{d}\xi\,. \tag{2.14}$$

Introduce the notations

$$\frac{\mathrm{d}w}{\mathrm{d}x} = y, \qquad \frac{\partial^2 G(x,\xi)}{\partial x\, \partial \xi} = \mathcal{K}(x,\xi)$$

and derive equation (2.14) with respect to x. In this way we get a homogeneous Fredholm integral equation:

$$y(x) = \mathcal{N} \int_0^\ell \mathcal{K}(x,\xi)\, y(\xi)\, d\xi\,. \tag{2.15}$$

Consequently, the eigenvalue problems determined by ODE (2.13) and the homogeneous boundary and continuity conditions presented in Table 1 are reduced to eigenvalue problems governed by homogeneous Fredholm integral equations. It should be mentioned that the above line of thought is based on book [17] and paper [16].

3. Green function for three-point boundary value problems

3.1. **Definition.** In this subsection we present the definition that provides the main properties of the Green function for ODEs. The definition is based on book [18].

Consider the inhomogeneous ordinary differential equation

$$L[y(x)] = \sum_{n=0}^{2k} p_n(x) y^{(n)}(x) = r(x)\,, \tag{3.1}$$

where k is a natural number, the functions $p_n(x)$ and $r(x)$ are continuous and $p_{2k}(x) \neq 0$ if $x \in [0, \ell]$ ($\ell > 0$). Moreover let b an inner point in the interval $[0, \ell]$: $b = \ell_1$, $\ell - b = \ell_2$ and $\ell_1 + \ell_2 = \ell$.

The inhomogeneous differential equation (3.1) is associated with the following homogeneous boundary and continuity conditions:

$$\begin{aligned}
\sum_{n=0}^{2k} \alpha_{nrI} y_I^{(n-1)}(0) &= 0\,, \qquad r = 1, 2, \ldots, k \\
\sum_{n=0}^{2k} \beta_{nrI} y_I^{(n-1)}(b) - \sum_{n=0}^{2k} \beta_{nrII} y_{II}^{(n-1)}(b) &= 0\,, \qquad r = 1, 2, ..., 2k \\
\sum_{n=0}^{2k} \gamma_{nrII} y_{II}^{(n-1)}(\ell) &= 0\,. \qquad r = 1, 2, \ldots, k
\end{aligned} \tag{3.2}$$

The Roman numeral I and II belong to the intervals $[0, b]$ and $[b, \ell]$: y_I and y_{II} are the solutions to the differential equation in the intervals I and II. It is assumed that α_{nrI}, β_{nrI}, β_{nrII} and γ_{nrII} are arbitrary constants.

The Green function $G(x, \xi)$ that belongs to the three point boundary value problem (3.1), and (3.2) is defined by the following formulas and properties [18]:

Formulas:

$$G(x,\xi) = \begin{cases} G_{1I}(x,\xi) & \text{if } x, \xi \in [0, \ell], \\ G_{2I}(x,\xi) & \text{if } x \in [b, \ell] \text{ and } \xi \in [0, \ell], \\ G_{1II}(x,\xi) & \text{if } x \in [0, b] \text{ and } \xi \in [b, \ell], \\ G_{2II}(x,\xi) & \text{if } x, \xi \in [b, \ell]. \end{cases} \tag{3.3}$$

Properties:

1. The function $G_{1I}(x, \xi)$ is a continuous function of x and ξ if $0 \leq x \leq \xi \leq b$ and $0 \leq \xi \leq x \leq b$. In addition it is $2k$ times differentiable with respect to x and the derivatives

$$\frac{\partial^n G_{1I}(x,\xi)}{\partial x^n} = G_{1I}(x,\xi)^{(n)}(x,\xi)\,, \qquad n = 1, 2, \ldots, 2k \tag{3.4}$$

are also continuous functions of x and ξ in the triangles $0 \leq x \leq \xi \leq b$ and $0 \leq \xi \leq x \leq b$.

2. Let ξ be fixed in $[0, b]$. Then the function $G_{1I}(x, \xi)$ and its derivatives

$$G_{1I}^{(n)}(x,\xi) = \frac{\partial^n G_{1I}(x,\xi)}{\partial x^n}\,, \qquad n = 1, 2, \ldots, 2k - 2 \tag{3.5}$$

should be continuous for $x = \xi$:

$$G_{1I}^{(n)}(\xi+0,\xi) - G_{1I}^{(n)}(\xi-0,\xi) = 0\,, \quad n = 0,1,2,\ldots 2k-2 \tag{3.6a}$$

The derivative $G_{1I}^{(2k-1)}(x,\xi)$ should, however, have a jump if $x = \xi$:

$$G_{1I}^{(2k-1)}(\xi+0,\xi) - G_{1I}^{(2k-1)}(\xi-0,\xi) = \frac{1}{p_{2k}(\xi)}\,. \tag{3.6b}$$

In contrast to this, $G_{2I}(x,\xi)$ and its derivatives

$$G_{2I}^{(n)}(x,\xi) = \frac{\partial^n G_{2I}(x,\xi)}{\partial x^n}\,, \qquad n = 1,2,\ldots,2k \tag{3.7}$$

are all continuous functions for any x in $[b,\ell]$.
3. Let ξ be fixed in $[b,\ell]$. The function $G_{1II}(x,\xi)$ and its derivatives

$$G_{1II}^{(n)}(x,\xi) = \frac{\partial^n G_{1II}(x,\xi)}{\partial x^n}\,, \qquad n = 1,2,\ldots,2k \tag{3.8}$$

are all continuous functions for any x in $[0,b]$.
4. Though the function $G_{2II}(x,\xi)$ and its derivatives

$$G_{2II}^{(n)}(x,\xi) = \frac{\partial^n G_{2II}(x,\xi)}{\partial x^n}\,, \qquad n = 1,2,\ldots,2k-2 \tag{3.9}$$

should also be continuous for $x = \xi$:

$$G_{2II}^{(n)}(\xi+0,\xi) - G_{2II}^{(n)}(\xi-0,\xi) = 0\,, \quad n = 0,1,2,\ldots 2k-2 \tag{3.10a}$$

the derivative $G_{2II}^{(2k-1)}(x,\xi)$ should, however, have a jump if $x = \xi$:

$$G_{2II}^{(2k-1)}(\xi+0,\xi) - G_{2II}^{(2k-1)}(\xi-0,\xi) = \frac{1}{p_{2k}(\xi)}\,. \tag{3.10b}$$

5. Let α be an arbitrary but finite non-zero constant. For a fixed $\xi \in [0,\ell]$ the product $G(x,\xi)\alpha$ as a function of x ($x \neq \xi$) should satisfy the homogeneous differential equation

$$M\left[G(x,\xi)\alpha\right] = 0\,.$$

6. The product $G(x,\xi)\alpha$ as a function of x should satisfy both the boundary conditions and the continuity conditions

$$\begin{aligned} &\textstyle\sum_{n=1}^{2k} \alpha_{nrI}\, G^{(n-1)}(0) = 0\,, && r = 1,\ldots,k \\ &\textstyle\sum_{n=1}^{2\kappa} \big(\beta_{nrI}\, G^{(n-1)}(b-0) - \beta_{nrII}\, G^{(n-1)}(b+0)\big) = 0\,, && r = 1,\ldots,2k \\ &\textstyle\sum_{n=1}^{2k} \gamma_{nrII}\, G^{(n-1)}(\ell) = 0\,. && r = 1,\ldots,k \end{aligned} \tag{3.11}$$

The above continuity conditions should be satisfied by the function pairs $G_{1I}(x,\xi)$, $G_{2I}(x,\xi)$ and $G_{1II}(x,\xi)$, $G_{2II}(x,\xi)$ as well.

Remark 1. It can be proved – see paper [18] for details – that the solution of the three-point boundary value problem (3.1), and (3.2) has the form

$$y(x) = \int_0^\ell G(x,\xi) r(\xi) d\xi\,. \tag{3.12}$$

Remark 2. If the boundary value problem defined by (3.1) and (3.2) is self adjoint then the Green function is symmetric [18]:

$$G(x,\xi) = G(\xi,x)\,. \tag{3.13}$$

In Subsections 3.2 and 3.3 we present the Green functions that belong to differential equation (2.6) under the boundary and continuity conditions presented in Table 1. The calculations are detailed for FrsRp beams only. As regards PrsRp beams we shall give the final formulae only.

3.2. **Green function for FrsRp beams.**

3.2.1. *Calculation of the Green function if $\xi \in [0,b]$.* We shall assume that $G_{1I}(x,\xi)$ has the following form:

$$\begin{aligned} G_{1I}(x,\xi) &= \sum_{m=1}^{4}(a_{mI}(\xi)+b_{mI}(\xi))w_m(x), \qquad x<\xi \\ G_{1I}(x,\xi) &= \sum_{m=1}^{4}(a_{mI}(\xi)-b_{mI}(\xi))w_m(x), \qquad x>\xi \end{aligned} \tag{3.14}$$

if $x \in [0,b]$. On the contrary, we search $G_{2I}(x,\xi)$ as

$$G_{2I}(x,\xi) = \sum_{m=1}^{4} c_{mI}(\xi)w_m(x), \tag{3.15}$$

if $x \in [b,\ell]$. The coefficients $a_{mI}(\xi), b_{mI}(\xi)$ and $c_{mI}(\xi)$ are unknown functions, $w_m(x)$ is given by (2.8).

Note that representation (3.14) and (3.15) for $G_{1I}(x,\xi)$ and $G_{2I}(x,\xi)$ ensure the fulfillment of Properties 1 and 5 of the definition.

Continuity and discontinuity conditions (3.6) result in the following equations

$$\sum_{m=1}^{4} b_{mI}(\xi)w_m^{(n)}(\xi) = 0, \qquad n=0,1,2 \tag{3.16a}$$

and

$$\sum_{m=1}^{4} b_{mI}(\xi)w_m^{(3)}(\xi) = -\frac{1}{2}. \tag{3.16b}$$

For FrsRp beams equations (3.16a) and (3.16b) assume the form

$$\begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \\ 0 & 1 & 2\xi & 3\xi^2 \\ 0 & 0 & 2 & 6\xi \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} b_{1I} \\ b_{2I} \\ b_{3I} \\ b_{4I} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}. \tag{3.17}$$

Hence

$$b_{1I} = \frac{\xi^3}{12}, \quad b_{2I} = -\frac{\xi^2}{4}, \quad b_{3I} = \frac{\xi}{4}, \quad b_{4I} = \frac{1}{12}\,. \tag{3.18}$$

REMARK 3. Note that (a) the determination of b_{mI} ensures the fulfillment of Property 2 of the Green function; (b) the results obtained for b_{mI} are independent of the boundary and continuity conditions.

According to Property 6 of the definition $G_{1I}(x,\xi)$ and $G_{2I}(x,\xi)$ should satisfy the boundary and continuity conditions in Table 1. Utilizing them we get:

(a) Boundary conditions at $x=0$:

$$\sum_{k=1}^{4} a_{kI} w_k(0) = -\sum_{k=1}^{4} b_{kI} w_k(0)\,, \tag{3.19a}$$

$$\sum_{k=1}^{4} a_{kI} w_k^{(1)}(0) = -\sum_{k=1}^{4} b_{kI} w_k^{(1)}(0)\,. \tag{3.19b}$$

(b) Continuity conditions at $x=b$:

$$\sum_{k=1}^{4} a_{kI} w_k(b) = \sum_{k=1}^{4} b_{kI} w_k(b)\,, \tag{3.19c}$$

$$\sum_{k=1}^{4} c_{kI} w_k(b) = 0\,, \tag{3.19d}$$

$$\sum_{k=1}^{4} a_{kI} w_k^{(1)}(b) - \sum_{k=1}^{4} c_{kI} w_k^{(1)}(b) = \sum_{k=1}^{4} b_{kI} w_k^{(1)}(b)\,, \tag{3.19e}$$

$$\sum_{k=1}^{4} a_{kI} w_k^{(2)}(b) - \sum_{k=1}^{4} c_{kI} w_k^{(2)}(b) = \sum_{k=1}^{4} b_{kI} w_k^{(2)}(b)\,. \tag{3.19f}$$

(c) Boundary conditions at $x=\ell$:

$$\sum_{k=1}^{4} c_{kI} w_k^{(1)}(\ell) = 0\,, \tag{3.19g}$$

$$\sum_{k=1}^{4} c_{kI} w_k^{(2)}(\ell) = 0\,. \tag{3.19h}$$

The previous linear equations can be given in matrix form as well:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & b & b^2 & b^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 & 0 & -1 & -2b & -3b^2 \\ 0 & 0 & 2 & 6b & 0 & 0 & -2 & -6b \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\ell & 3\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1I} \\ a_{2I} \\ a_{3I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} -\xi^3 \\ 3\xi^2 \\ \xi^3 - 3\xi^2 b + 3\xi b^2 - b^3 \\ 0 \\ -3\xi^2 + 6\xi b - 3b^2 \\ 6\xi - 6b \\ 0 \\ 0 \end{bmatrix}. \tag{3.20}$$

After solving the linear equation system (3.20) the following relationship is obtained for $G_{1I}(x,\xi)$:

$$G_{1I}(x,\xi)=\sum_{\ell=1}^{4}\left(a_{\ell I}(\xi)\pm b_{\ell I}(\xi)\right)w_\ell(x)=-\frac{1}{12}\xi^3\pm\frac{1}{12}\xi^3+\left(\frac{3\xi^2}{12}\pm\left(-\frac{3\xi^2}{12}\right)\right)x+$$
$$+\left(\frac{3\xi}{12b^2\left(4\ell-3b\right)}\left(8\xi b^2-12\xi b\ell+4\ell\xi^2-2\xi^2 b-3b^3+4b^2\ell\right)\pm\frac{3\xi}{12}\right)x^2+$$
$$+\left(-\frac{1}{12b^3\left(4\ell-3b\right)}\left(6\xi^2b^2-12\xi^2b\ell-3b^4+4b^3\ell+4\ell\xi^3\right)\pm\frac{-1}{12}\right)x^3 \quad (3.21\text{a})$$

As regards $G_{2I}(x,\xi)$ we have

$$G_{2I}(x,\xi)=\sum_{\ell=1}^{4}c_{\ell I}(\xi)w_\ell(x)=\xi^2\frac{\left(\xi-b\right)\left(x-b\right)\left(2\ell-x-b\right)}{2b\left(4\ell-3b\right)} \quad (3.21\text{b})$$

3.2.2. *Calculation of the Green function if $\xi\in[b,\ell]$.* The assumptions that are used are similar to those presented in Subsection 3.2.1:

If $x\in[b,\ell]$ then

$$\begin{aligned} G_{2II}(x,\xi)&=\sum_{m=1}^{4}(a_{mII}(\xi)+b_{mII}(\xi))w_m(x), \qquad x<\xi\\ G_{2II}(x,\xi)&=\sum_{m=1}^{4}(a_{mII}(\xi)-b_{mII}(\xi))w_m(x), \qquad x>\xi \end{aligned} \quad (3.22)$$

however, if $x\in[0,b]$ then

$$G_{1II}(x,\xi)=\sum_{m=1}^{4}c_{mII}(\xi)w_m(x). \quad (3.23)$$

Here the coefficients $a_{mII}(\xi), b_{mII}(\xi)$ and $c_{mII}(\xi)$ are again unknown functions.

We remind the reader of the fact that the above representations for $G_{1II}(x,\xi)$ and $G_{2II}(x,\xi)$ ensure the fulfillment of Property 1 and 5 of the definition.

Continuity and discontinuity conditions (3.10) lead again to equation system (3.17) in which now the coefficients $b_{mII}(\xi)$, $m=1,2,3,4$ are the unknowns. Hence $b_{mII}(\xi)=b_{mI}(\xi)$.

It's worth noting that determining the coefficients b_{mII} assures that the Green function's Properties 3 and 4 are satisfied. Making use of the boundary and continuity conditions given in Table 1 equations again the following equations can be obtained for the eight unknown coefficients $a_{mII}(\xi)$ and $c_{mII}(\xi)$:

(a) Boundary conditions at $x=0$:

$$\sum_{k=1}^{4}c_{kII}\,w_k(0)=0\,, \quad (3.24\text{a})$$

$$\sum_{k=1}^{4}c_{kII}\,w_k^{(1)}(0)=0\,, \quad (3.24\text{b})$$

(b) Continuity conditions at $x = b$:

$$\sum_{k=1}^{4} c_{kII}\, w_k(b) = 0\,, \tag{3.24c}$$

$$\sum_{k=1}^{4} a_{kII}\, w_k(b) = -\sum_{k=1}^{4} b_{kII}\, w_k(b)\,, \tag{3.24d}$$

$$\sum_{k=1}^{4} a^{(1)}_{kII}\, w_k(b) - \sum_{k=1}^{4} c^{(1)}_{kII}\, w_k(b) = -\sum_{k=1}^{4} b^{(1)}_{kII}\, w_k(b)\,, \tag{3.24e}$$

$$\sum_{k=1}^{4} a^{(2)}_{kII}\, w_k(b) - \sum_{k=1}^{4} c^{(2)}_{kII}\, w_k(b) = -\sum_{k=1}^{4} b^{(2)}_{kII}\, w_k(b)\,. \tag{3.24f}$$

(c) Boundary conditions at $x = \ell$:

$$\sum_{k=1}^{4} a^{(1)}_{kII}\, w_k(\ell) = \sum_{k=1}^{4} b^{(1)}_{kII}\, w_k(\ell)\,, \tag{3.24g}$$

$$\sum_{k=1}^{4} a^{(3)}_{kII}\, w_k(\ell) = \sum_{k=1}^{4} b^{(3)}_{kII}\, w_k(\ell)\,, \tag{3.24h}$$

Since $c_{1II} = c_{2II} = 0$ the final equation system has the following form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b^2 & b^3 \\ 1 & b & b^2 & b^3 & 0 & 0 \\ 0 & 1 & 2b & 3b^2 & -2b & -3b^2 \\ 0 & 0 & 2 & 6b & -2 & -6b \\ 0 & 1 & 2\ell & 3\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4II} \\ c_{3II} \\ c_{4II} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 0 \\ -\xi^3 + 3b\xi^2 - 3b^2\xi + b^3 \\ 3\xi^2 - 6b\xi + 3b^2 \\ -6\xi + 6b \\ -3\xi^2 + 6\xi\ell - 3\ell^2 \\ -1 \end{bmatrix} \tag{3.25}$$

After having solved the previous equation system substitution of the results obtained into equations (3.22), (3.23) and using some algebra yield:

$$G_{1II}(x,\xi) = \sum_{\ell=1}^{4} c_{\ell II}(\xi) w_\ell(x) = x^2 \frac{(x-b)\,(\xi-b)\,(2\ell-\xi-b)}{2b\,(4\ell-3b)} \tag{3.26a}$$

and

$$\begin{aligned} G_{2II}(x,\xi) &= \sum_{\ell=1}^{4} \left(a_{\ell II}(\xi) \pm b_{\ell II}(\xi)\right) w_\ell(x) = \\ &= -\frac{1}{12\,(4\ell-3b)} \left(4b^3\ell - 12b^2\xi\ell + 6\xi^2 b^2 + 4\ell\xi^3 - 3\xi^3 b\right) \pm \frac{\xi^3}{12} + \\ &+ \left(\frac{3}{12\,(4\ell-3b)} \left(4b^2\ell - 12\xi b\ell + 3\xi^2 b + 4\ell\xi^2\right) \pm \frac{-3\xi^2}{12}\right) x + \\ &+ \left(\frac{3}{12\,(4\ell-3b)} \left(-2b^2 + 4\xi\ell - 4\xi^2 + 3\xi b\right) \pm \frac{3\xi}{12}\right) x^2 + \left(-\frac{1}{12} \pm \frac{-1}{12}\right) x^3 \end{aligned} \tag{3.26b}$$

Note that the calculation of the functions $a_{\ell II}$ and $c_{\ell II}$ is based on Property 6 of the definition.

Figure 2 depicts the Green function for an FrsRp beam. It is assumed that $L = 100$ mm, $\hat{b} = 50$ mm and $\hat{\xi} = 75$ mm. The computed points are drawn by red diamonds and the function itself is shown using a continuous line. This notation convention will be applied to the other figures in the present paper. The Green function shown in Figure 2 is the dimensionless displacement due to a dimensionless vertical unit force exerted on the beam at the point $\xi = 0.75$.

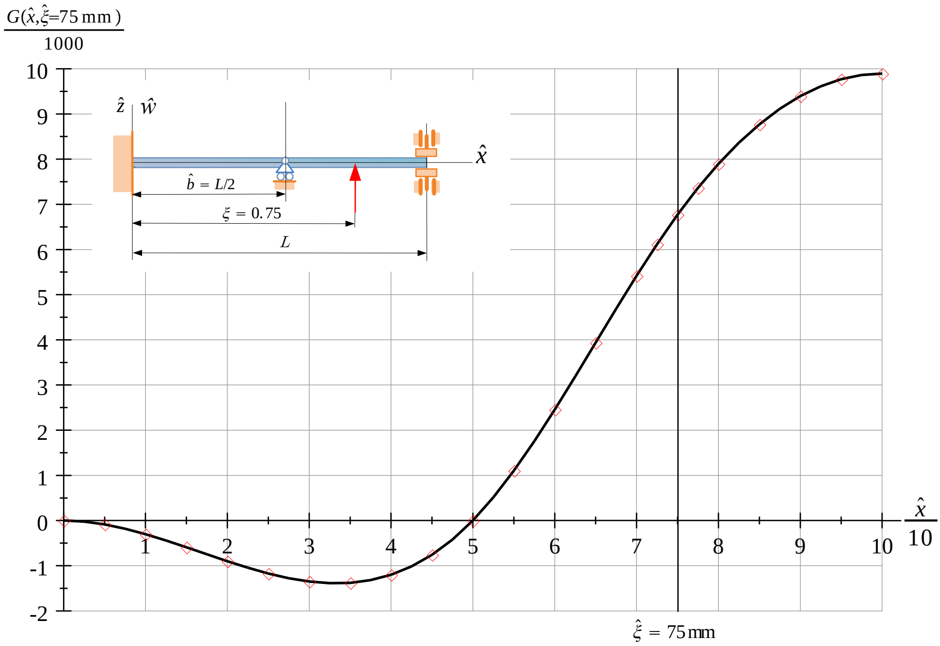

Figure 2. The Green function of an FrsRp beam

3.3. **Green function for PrsRp beams.** Repeating the calculations steps presented in Subsection 3.2 for PrsRp beams yields the following four elements for the corresponding Green function – the calculation details are all omitted here.

$$
\begin{aligned}
G_{1I}(x,\xi) &= \sum_{\ell=1}^{4} \left(a_{\ell I}(\xi) \pm b_{\ell I}(\xi)\right) w_\ell(x) = \left(-\frac{1}{12}\xi^3 \pm \frac{1}{12}\xi^3\right) + \\
&+ \left(-\frac{1}{12b\,(2b-3\ell)}\left(-9b^3\xi+6b^2\xi^2+12\ell b^2\xi-3b\xi^3-9\ell b\xi^2+6\ell\xi^3\right) \pm \left(-\frac{3\xi^2}{12}\right)\right) x + \\
&+ \left(-\frac{3}{12}\xi \pm \frac{3}{12}\xi\right) x^2 + \left(\frac{1}{12b^2\,(2b-3\ell)}\left(-2b^3+3b^2\xi+3\ell b^2-6\ell b\xi+\xi^3\right) \pm \frac{-1}{12}\right) x^3,
\end{aligned}
\tag{3.27a}
$$

$$G_{2I}(x,\xi) = \sum_{\ell=1}^{4} c_{\ell I}(\xi)w_\ell(x) = \frac{1}{4b}\frac{\xi}{3\ell - 2b}(b-x)\left(\xi^2 - b^2\right)(b + x - 2\ell), \tag{3.27b}$$

$$G_{1II}(x,\xi) = \sum_{\ell=1}^{4} c_{\ell II}(\xi)w_\ell(x) = \frac{1}{4b}\frac{x}{3\ell - 2b}(b-\xi)\left(x^2 - b^2\right)(b + \xi - 2\ell), \tag{3.27c}$$

$$\begin{aligned} G_{2II}(x,\xi) &= \sum_{\ell=1}^{4} \left(a_{\ell II}(\xi) \pm b_{\ell II}(\xi)\right) z_\ell(x) = \\ &= \frac{1}{12(3\ell - 2b)}\left(-b^4 + 6b^2\xi\ell - 3b^2\xi^2 - 3\xi^3\ell + 2\xi^3 b\right) \pm \frac{\xi^3}{12} + \\ &+ \left(\frac{3}{12(3\ell - 2b)}\left(2b^2\ell - 8b\xi\ell + 2b\xi^2 + 3\xi^2\ell\right) \pm \frac{-3\xi^2}{12}\right)x + \\ &+ \left(\frac{3}{12(3\ell - 2b)}\left(-b^2 + 3\xi\ell - 3\xi^2 + 2b\xi\right) \pm \frac{3\xi}{12}\right)x^2 + \left(-\frac{1}{12} \pm \frac{-1}{12}\right)x^3. \end{aligned} \tag{3.27d}$$

Figure 3 shows the Green function of a PrsRp beam under the same conditions as Figure 2 depicts the Green function of an FrsRp beam.

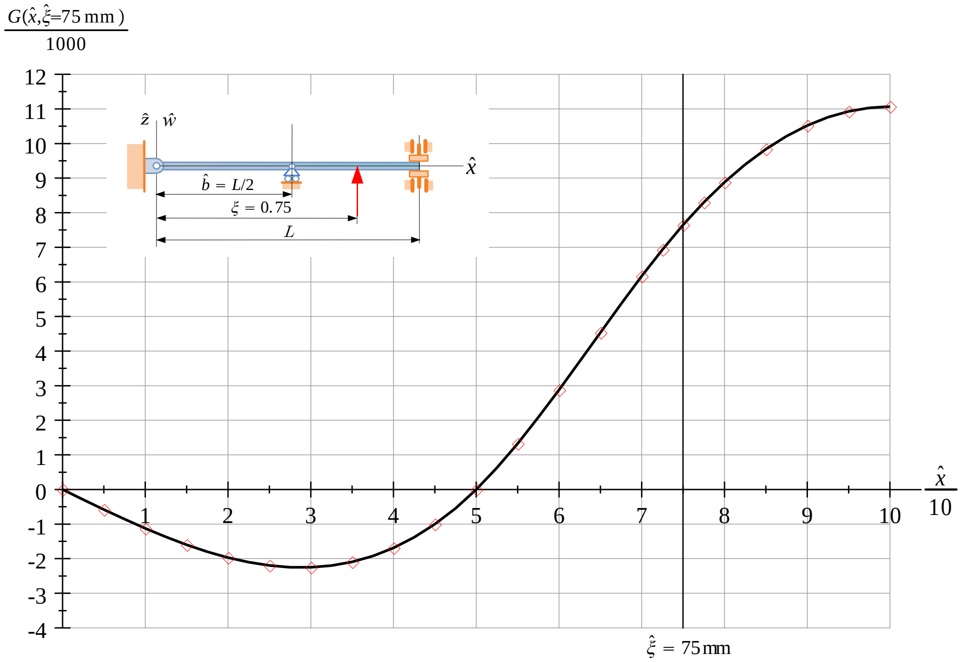

Figure 3. The Green function of an PrsRp beam

Remark 4. The Green function given by equations (3.21) and (3.26) (FrsRp beams), (3.27) (PrsRp beams), should satisfy symmetry condition (3.13). It can be proved by paper and pencil calculations that this condition is really fulfilled. Note that for G_{2I}

and G_{1II} a comparison of (3.21b) and (3.26a) as well as that of (3.27b) and (3.27c) clearly shows the fulfillment of the symmetry condition.

Remark 5. The Green functions (3.21), (3.26) and (3.27) are dimensionless quantities. By substituting $\hat{b}$, L, $\hat{x}$ and $\hat{\xi}$ for b, ℓ, x andξ in (3.21), (3.26) and (3.27) we obtain the Green functions for a selected length unit. Then the unit of the Green function is the cube of the length unit selected.

4. Numerical solutions for the free vibration and stability problems

4.1. **The free vibration of FrsRp and PrsRp beams.** Making use of the algorithm detailed in Subsection 7.2 of the book [18] a Fortran 90 program was developed for solving eigenvalue problem (2.11), i.e., for computing the eigenvalues λ (the natural circular frequencies) of the freely vibrating FrsRp and PrsRp beams (the axial force is now zero) shown in Figure 1. Table 2 and Table 3 present the values of $\lambda_i/4.730042^2$, $(i = 1; 2; 3)$ for FrsRp and PrsRp against 21 uniformly increasing b values in the interval $[0.0, 1.0]$.

Table 2. Solutions for the eigenvalues λ of FrsRp beams

b	$\frac{\sqrt{\lambda_1}}{4.73004^2}$	$\frac{\sqrt{\lambda_2}}{4.73004^2}$	$\frac{\sqrt{\lambda_3}}{4.73004^2}$
0.000	0.2545	1.3707	3.3876
0.050	0.2751	1.4832	3.6692
0.100	0.2989	1.6159	4.0085
0.150	0.3264	1.7737	4.4165
0.200	0.3587	1.9626	4.9052
0.250	0.3970	2.1896	5.4794
0.300	0.4428	2.4631	6.0887
0.350	0.4983	2.7884	6.1999
0.400	0.5667	3.1487	5.3372
0.450	0.6520	3.3867	4.7648
0.500	0.7599	3.1710	5.0303
0.550	0.8973	2.7863	5.8037
0.600	1.0688	2.4707	6.2988
0.650	1.2566	2.2989	5.7913
0.700	1.3675	2.4086	5.1793
0.750	1.3348	2.9030	4.7797
0.800	1.2429	3.3735	5.1642
0.850	1.1494	3.2747	6.2815
0.900	1.0727	3.0380	6.0457
0.950	1.0203	2.8423	5.6192
1.000	1.0000	2.7568	5.4059

Polynomials (4.1), (4.2) and (4.2) are fitted onto the computed discrete values of $\sqrt{\lambda_k}/4.73004^2(k = 1, 2, 3)$ presented in Table 2.

Polynomials for the first eigenvalue:

$$\frac{\sqrt{\lambda_1}}{4.73004^2} = -30.475\,109\,9b^6 + 55.991\,058\,8b^5 - 33.778\,406\,3b^4 + 10.464\,803\,2b^3 \\ - 0.743\,507\,022b^2 + 0.443\,726\,955\,b + 0.254\,099\,177, \quad b \in [0, 0.625] \quad (4.1a)$$

$$\frac{\sqrt{\lambda_1}}{4.73004^2} = 4172.10844b^6 - 20498.0817b^5 + 41635.1607b^4 - 44700.0752b^3 + 26718.0755b^2 - 8418.2836b + 1092.09696, \qquad b \in [0.625, 1] \tag{4.1b}$$

Polynomials for the second eigenvalue:

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -173.518714b^6 + 150.076835b^5 - 40.2464201b^4 + 11.0106042b^3 + 2.87051697b^2 + 2.08205616b + 1.37069202, \qquad b \in [0, 0.35] \tag{4.2a}$$

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -310557.518b^6 + 845065.5b^5 - 950123.189b^4 + 564838.656b^3 - 187284.725b^2 + 32856.2932b - 2381.76203, \qquad b \in [0.35, 0.55] \tag{4.2b}$$

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -30502.3232b^6 + 110600.102b^5 - 166009.888b^4 + 132140.138b^3 - 58844.5586b^2 + 13891.8131b - 1353.11155, \qquad b \in [0.55, 0.775] \tag{4.2c}$$

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -11948.2965\ b^6 + 71424.3166\ b^5 - 176869.672\ b^4 + 232479.487\ b^3 - 171191.813\ b^2 + 66992.1274\ b - 10883.3937, \qquad b \in [0.775, 1] \tag{4.2d}$$

Polynomials for the third eigenvalue:

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = 413332.622b^6 - 649900.151b^5 + 411344.981b^4 - 133672.324b^3 + 23342.1105b^2 - 2033.36718b + 70.5109770, \qquad b \in [0.25, 0.4] \tag{4.2e}$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = 48544.0853b^6 - 91571.4048b^5 + 48349.6875b^4 + 9960.13403b^3 - 18412.9201b^2 + 6294.34668b - 702.680179, \qquad b \in [0.4, 0.55] \tag{4.2f}$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = 2073702.86b^6 - 7913552.44x^5 + 12563013.6x^4 - 10618667.1x^3 + 5039221.87x^2 - 1272877.15x + 133685.395, \qquad b \in [0.55, 0.7] \tag{4.2g}$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = -243505.380x^6 + 882582.997b^5 - 1246562.71b^4 + 831781.698b^3 - 231920.390b^2 - 1744.47318b + 9178.32482, \qquad b \in [0.7, 0.825] \tag{4.2h}$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = -284198.684x^6 + 1619914.01x^5 - 3845547.04x^4 + 4867005.53x^3 - 3463802.03x^2 + 1314397.42x - 207763.794, \qquad b \in [0.825, 1.0] \tag{4.2i}$$

Figures 4, 5 and 6 show the graphs of the functions $\sqrt{\lambda_k(b)}/4.73004^2 (k = 1, 2, 3)$.

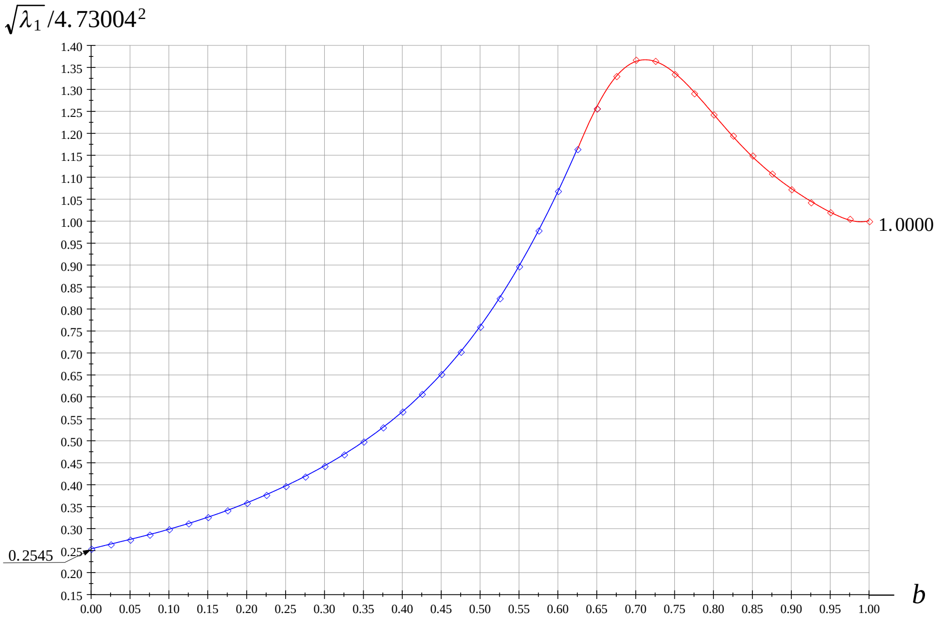

Figure 4. Function $\sqrt{\lambda_1}/4.73004^2$ against b

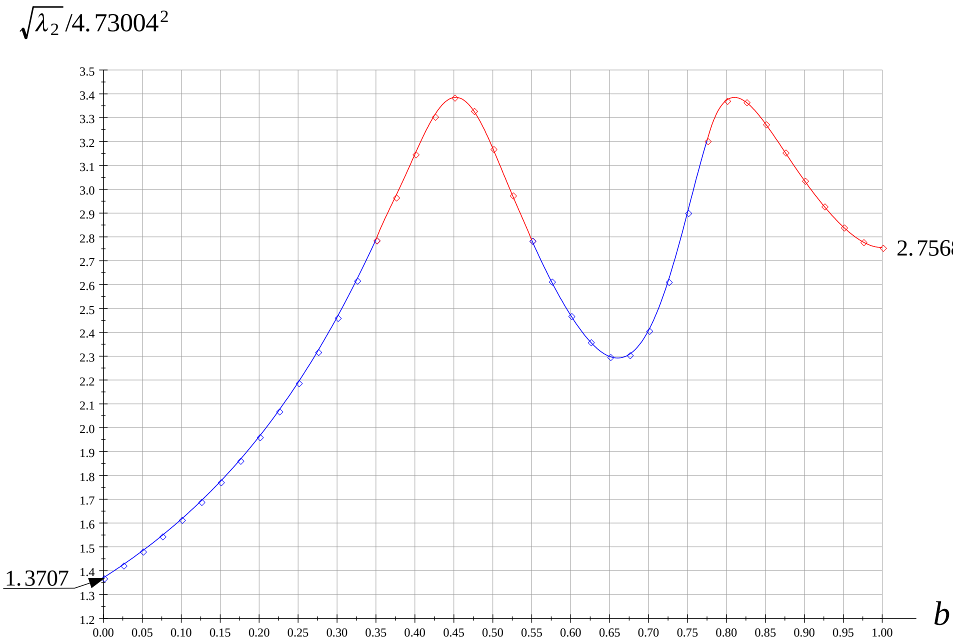

Figure 5. Function $\sqrt{\lambda_2}/4.73004^2$ against b

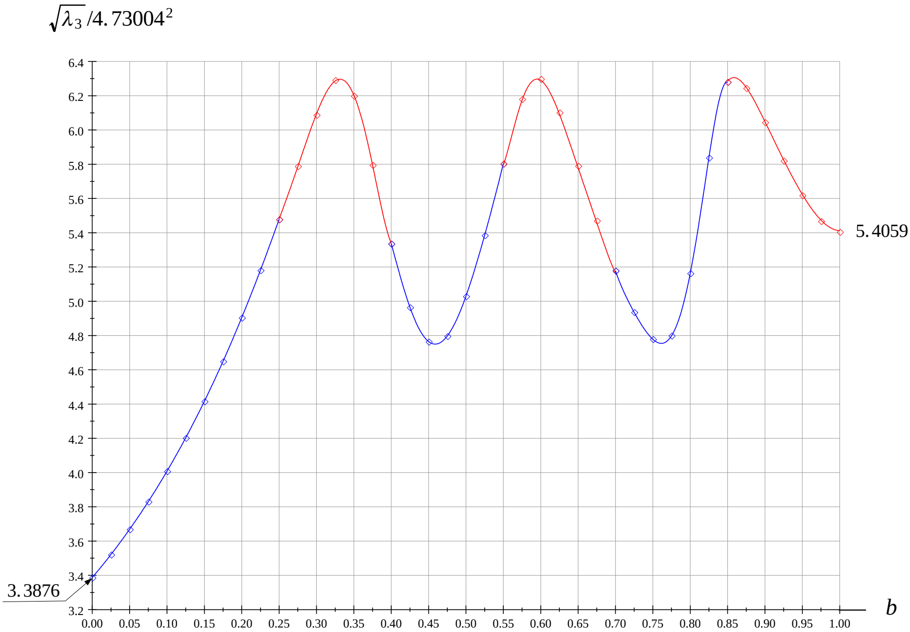

Figure 6. Function $\sqrt{\lambda_3}/4.73004^2$ against b

For PrsPRp beams Table 3 contains the computational results.

Table 3. Solutions for the eigenvalues λ of PrsRp beams

b	$\frac{\sqrt{\lambda_1}}{4.73004^2}$	$\frac{\sqrt{\lambda_2}}{4.73004^2}$	$\frac{\sqrt{\lambda_3}}{4.73004^2}$
0.000	0.2545	1.3707	3.3876
0.050	0.2729	1.4722	3.6446
0.100	0.2942	1.5951	3.9650
0.150	0.3191	1.7435	4.3553
0.200	0.3484	1.9225	4.8163
0.250	0.3836	2.1369	5.3049
0.300	0.4251	2.3877	5.4168
0.350	0.4756	2.6539	4.6226
0.400	0.5383	2.7983	4.0604
0.450	0.6153	2.5824	4.2051
0.500	0.7101	2.2442	4.7850
0.550	0.8229	1.9719	5.4203
0.600	0.9388	1.8167	5.2861
0.650	1.0050	1.8513	4.7224
0.700	0.9830	2.1426	4.2451
0.750	0.9163	2.6083	4.0641
0.800	0.8445	2.7945	4.7548
0.850	0.7819	2.6420	5.4909
0.900	0.7333	2.4480	5.1979
0.950	0.7012	2.2972	4.8356
1.000	0.6891	2.2338	4.6607

Polynomials (4.3), (4.4) and (4.5) are fitted onto the computed discrete values of $\sqrt{\lambda_k}/4.73004^2(k=1,2,3)$ presented in Table 3.

Polynomials for the first eigenvalue:

$$\frac{\sqrt{\lambda_1}}{4.73004^2} = -41.080\,096\,2b^6 + 63.147\,880\,9b^5 - 33.556\,975\,5b^4 + 9.262\,020\,48b^3 - \\ - 0.510\,379\,219b^2 + 0.383\,506\,985b + 0.254\,263\,737, \quad b \in [0, 0.575] \quad (4.3a)$$

$$\frac{\sqrt{\lambda_1}}{4.73004^2} = 1322.696\,25b^6 - 6268.407\,31b^5 + 12232.936\,9b^4 - 12555.559\,8b^3 + \\ + 7129.384\,25b^2 - 2117.638\,81b + 257.278\,716, \quad b \in [0.575, 1] \quad (4.3b)$$

Polynomials for the second eigenvalue:

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -295.492\,865b^6 + 201.661\,94b^5 - 47.755\,381\,3b^4 + 10.626\,197\,6b^3 + \\ + 3.195\,613\,34b^2 + 1.848\,783\,9b + 1.370\,702\,43, \quad b \in [0, 0.3] \quad (4.4a)$$

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -230\,912.0\,83b^6 + 557\,443.625b^5 - 554\,657.434b^4 + 291\,066.631b^3 - \\ - 84987.941\,7b^2 + 13104.515\,1b - 832.379\,11, \quad b \in [0.3, 0.5] \quad (4.4b)$$

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = -5813.451\,13b^6 + 18042.657\,9b^5 - 22715.726b^4 + 14807.845\,8b^3 - \\ - 5220.857\,11b^2 + 918.110\,040b - 55.842\,264\,2, \quad b \in [0.5, 0.75] \quad (4.4c)$$

$$\frac{\sqrt{\lambda_2}}{4.73004^2} = 4928.738\,89b^6 - 22244.172\,9b^5 + 39776.050\,7b^4 - 34844.077\,6b^3 + \\ + 14431.718\,2b^2 - 1786.725\,11b - 259.298\,607, \quad b \in [0.75, 1] \quad (4.4d)$$

Polynomials for the third eigenvalue:

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = -2623.305\,96b^6 + 1237.635\,68b^5 - 231.339\,925b^4 + 30.257\,014\,1b^3 + \\ + 10.385\,009\,6b^2 + 4.566\,326\,26b + 3.387\,600\,46, \quad b \in [0, 0.2] \quad (4.5a)$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = 1.131\,061\,48 \times 10^5 b^6 - 20897.459\,2b^5 - 90629.798b^4 + 66709.594\,1b^3 - \\ - 19457.354\,7b^2 + 2639.576\,61b - 134.025\,486, \quad b \in [0.2, 0.35] \quad (4.5b)$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = 29681.777\,8b^6 - 36384.972\,8b^5 - 6235.294\,22b^4 + 30141.818\,9b^3 - \\ - 18135.112\,4b^2 + 4465.631\,98b - 399.021\,738, \quad b \in [0.35, 0.5] \quad (4.5c)$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = -1\,479\,416.38b^6 + 500\,694.40b^5 - 7\,035\,081.66b^4 + 5\,252\,557.83 \times 10^6 b^3 - \\ - 2\,197\,910.20b^2 + 488\,766.44b - 45129.13, \quad b \in [0.5, 0.65] \quad (4.5d)$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = -743\,272.68b^6 + 3\,194\,975.26 \times 10^6 b^5 - 5\,713\,572.25b^4 + 5\,441\,676.39b^3 - \\ - 2\,911\,437.16b^2 + 829\,724.55b - 98399.24, \quad b \in [0.65, 0.8] \quad (4.5e)$$

$$\frac{\sqrt{\lambda_3}}{4.73004^2} = 372\,346.0b^6 - 2\,006\,546.6b^5 + 4\,496\,069.4b^4 - 5\,360\,883.7b^3 + \\ + 3\,586\,778.1b^2 - 1\,276\,523.3b + 188\,764.8 \quad b \in [0.8, 1.0] \quad (4.5f)$$

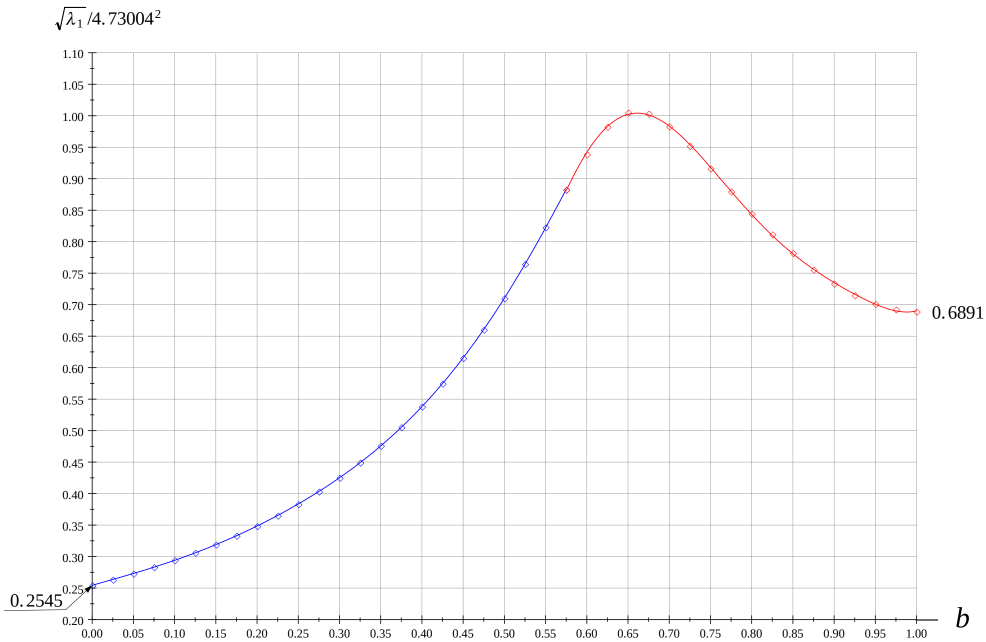

Figure 7. Function $\sqrt{\lambda_1}/4.73004^2$ against b

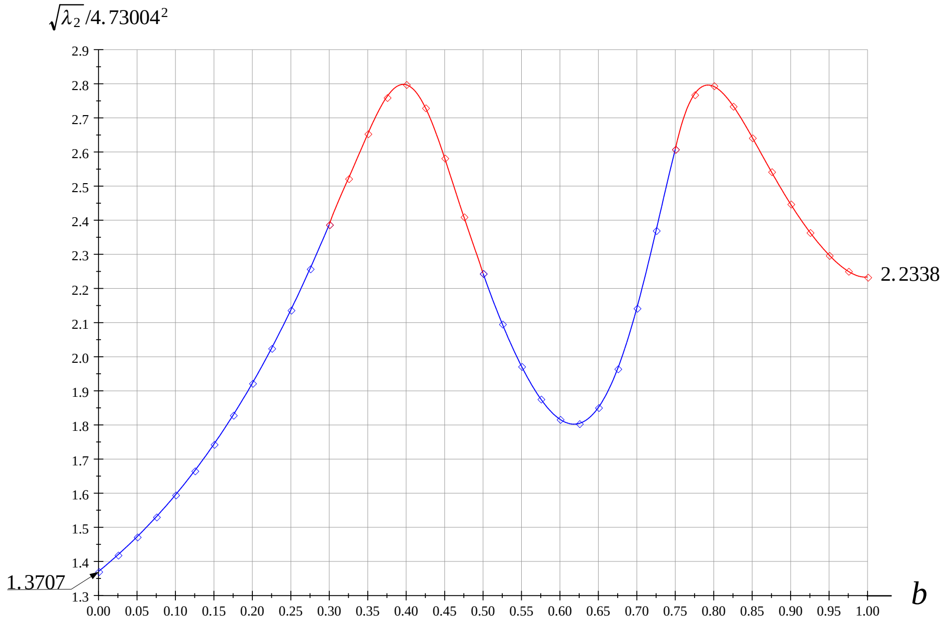

Figure 8. Function $\sqrt{\lambda_2}/4.73004^2$ against b

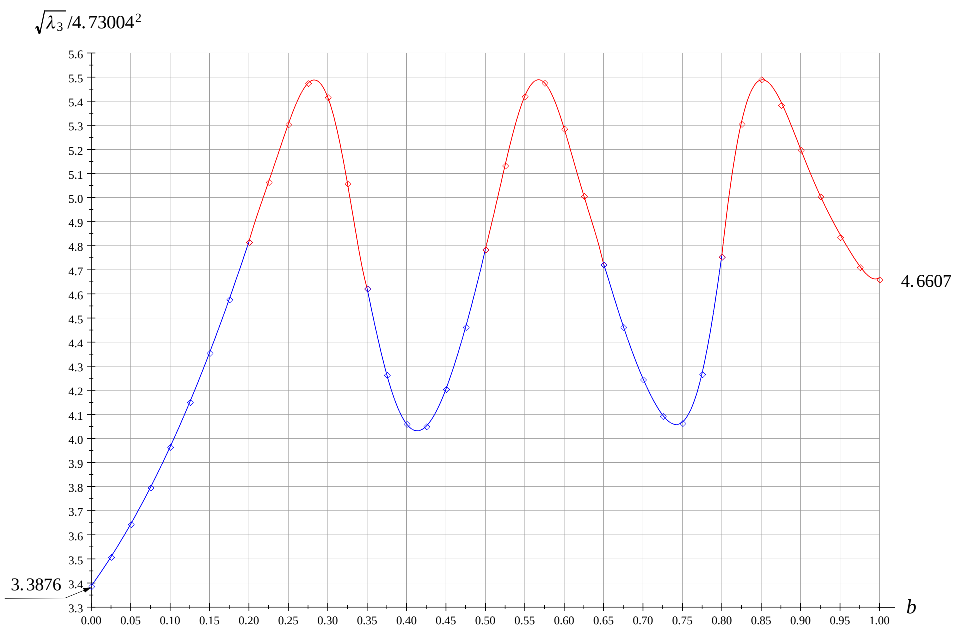

Figure 9. Function $\sqrt{\lambda_3}/4.73004^2$ against b

Figures 7,8, and 9 show the graphs of the functions $\sqrt{\lambda_k(b)}/4.73004^2 (k = 1, 2, 3)$. As regards Figures 4, 5, 6, 7, 8, and 9 the discrete point pairs are denoted by diamonds. The continuous lines are drawn by using polynomials (4.1), (4.2), (4.2), (4.3), (4.4) and (4.5) which fit onto the discrete point pairs with four digit accuracy.

4.2. **Stability problems of FrsRp and PrsRp beams.**

4.2.1. *Solution procedures.* There are various methods for calculating the critical load. (a) It is possible to solve the eigenvalue problem determined by the homogeneous Fredholm integral equation (2.15) numerically if we apply the boundary element technique. See for instance [16] which uses this technique for other support arrangements. (b) It is also possible to establish the nonlinear characteristic equations and then to solve them for the critical load In the present paper the boundary element approach will be preferred, and the numerical solution of the characteristic problem is used to validate the findings obtained using this approach. As regards the boundary element technique the solution steps are detailed in Subsection 8.15.2 in [12]. A Fortran 90 program was developed. The kernel in equation (2.15) has the following form

$$\mathcal{K}(x,\xi) = \begin{cases} \mathcal{K}_{1I}(x,\xi) & \text{if } x,\xi \in [0,\ell], \\ \mathcal{K}_{2I}(x,\xi) & \text{if } x \in [b,\ell] \text{ and } \xi \in [0,\ell], \\ \mathcal{K}_{1II}(x,\xi) & \text{if } x \in [0,b] \text{ and } \xi \in [b,\ell], \\ \mathcal{K}_{2II}(x,\xi) & \text{if } x,\xi \in [b,\ell], \end{cases} \tag{4.6a}$$

where

$$\begin{aligned} \mathcal{K}_{1I}(x,\xi) &= \frac{\partial^2 G_{1I}(x,\xi)}{\partial x\,\partial\xi}, & \mathcal{K}_{2I}(x,\xi) &= \frac{\partial^2 G_{2I}(x,\xi)}{\partial x\,\partial\xi}, \\ \mathcal{K}_{1II}(x,\xi) &= \frac{\partial^2 G_{1II}(x,\xi)}{\partial x\,\partial\xi}, & \mathcal{K}_{2II}(x,\xi) &= \frac{\partial^2 G_{2II}(x,\xi)}{\partial x\,\partial\xi}. \end{aligned} \tag{4.6b}$$

It is obvious from equations (4.6) that the determination of the kernel $\mathcal{K}(x,\xi)$ requires the calculation of second derivatives.

4.2.2. *The kernel function for FrsRp beams.* Making use of equations (3.21), (3.26) and (4.6) we get the elements of the kernel function for FrsRp beams in the following form:

$$\begin{aligned} \mathcal{K}_{1I}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{1I}(x,\xi) &= \left(\frac{6}{12}\xi \pm \left(-\frac{6}{12}\xi\right)\right) + \\ &+ \left(-\frac{6}{12b^2(4\ell-3b)}\left(3b^3 - 16b^2\xi - 4\ell b^2 + 6b\xi^2 + 24\ell b\xi - 12\ell\xi^2\right) \pm \frac{6}{12}\right) x + \\ &- \frac{36}{12b^3}\frac{\xi}{4\ell-3b}\left(b^2 - 2\ell b + \xi\ell\right) x^2, \end{aligned} \tag{4.7a}$$

$$\mathcal{K}_{2I}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{2I}(x,\xi) = \frac{1}{b}\xi \frac{2b-3\xi}{4\ell-3b}(x-\ell), \tag{4.7b}$$

$$\mathcal{K}_{1II}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{1II}(x,\xi) = \frac{1}{b} x \frac{2b-3x}{4\ell-3b}(\xi-\ell), \tag{4.7c}$$

$$\mathcal{K}_{2II}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{2II}(x,\xi) = \left(\frac{3}{12\,(4\ell - 3b)}\,(6b\xi - 12b\ell + 8\xi\ell) \pm \frac{-6\xi}{12} \right) + \\ + \left(\frac{6}{12\,(4\ell - 3b)}\,(3b - 8\xi + 4\ell) \pm \frac{6}{12} \right) x. \tag{4.7d}$$

Figure 10 depicts the kernel function of an FrsRp beam provided that $L = 100$ mm, $\hat{b} = 50$ mm and $\hat{\xi} = 75$ mm.

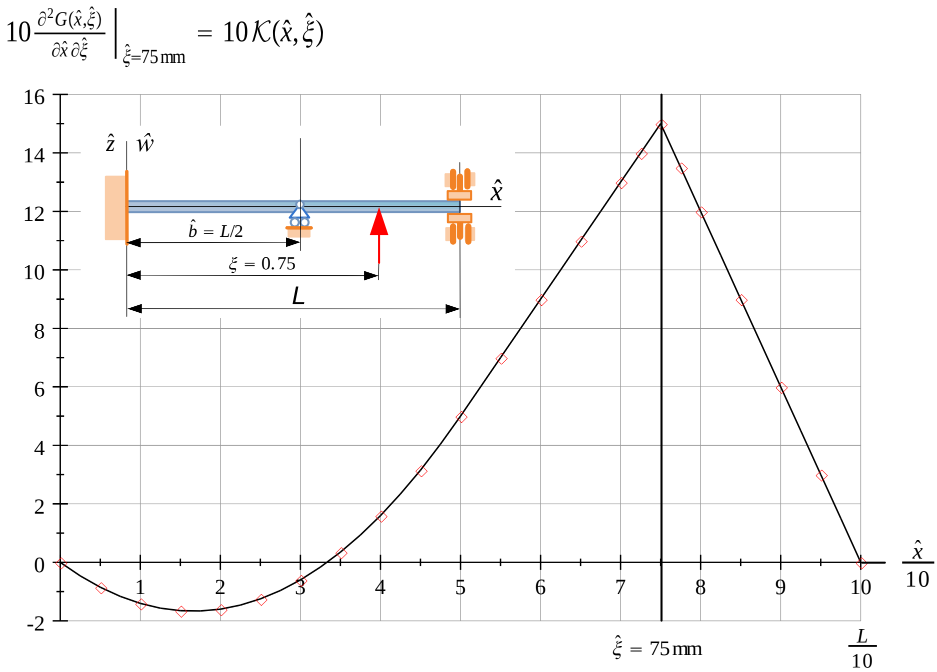

Figure 10. The kernel function of an FrsRp beam

4.2.3. *The kernel function for PrsRp beams.* Making use of equations (3.27) and (4.6) we can derive the elements of the kernel function for PrsRp beams:

$$\mathcal{K}_{1I}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{1I}(x,\xi) = \\ \left(-\frac{1}{12b\,(2b - 3\ell)}\left(-9b^3 + 12b^2\xi + 12\ell b^2 - 9b\xi^2 - 18\ell b\xi + 18\ell\xi^2\right) \pm \left(-\frac{6\xi}{12} \right) \right) \\ + \left(-\frac{6}{12} \pm \frac{6}{12} \right) x + \left(\frac{3}{12b^2\,(2b - 3\ell)}\left(3b^2 - 6\ell b + 3\xi^2\right) \right) x^2, \tag{4.8a}$$

$$\mathcal{K}_{2I}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{2I}(x,\xi) = -\frac{1}{2b\,(3\ell - 2b)}\left(3\xi^2 - b^2\right)(x - \ell)\,, \tag{4.8b}$$

$$\mathcal{K}_{1II}(x,\xi) = \frac{\partial^2}{\partial x \partial \xi} G_{1II}(x,\xi) = -\frac{1}{2b\,(3\ell - 2b)}\left(3x^2 - b^2\right)(\xi - \ell)\,, \tag{4.8c}$$

$$\mathcal{K}_{2II}(x,\xi)=\frac{\partial^2}{\partial x\partial\xi}G_{2II}(x,\xi)=\left(\frac{3}{12\,(3\ell-2b)}\,(4b\xi-8b\ell+6\xi\ell)\pm\frac{-6\xi}{12}\right)+$$
$$+\left(\frac{6}{12\,(3\ell-2b)}\,(2b-6\xi+3\ell)\pm\frac{6}{12}\right)x. \tag{4.8d}$$

Figure 11 shows the kernel function of a PrsRp beam assuming that $L=100$ mm, $\hat{b}=50$ mm and $\hat{\xi}=75$ mm.

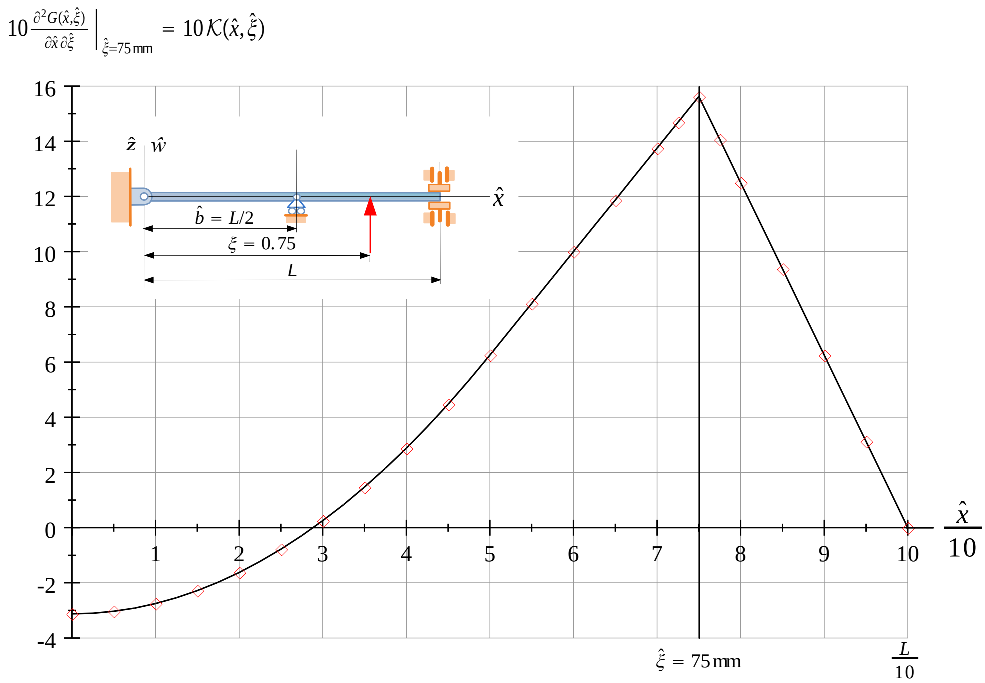

Figure 11. The kernel function of a PrsRp beam

Remark 6. The kernel functions given by equations (4.7) and (4.8) (FrsRp beams), (3.27) (PrsRp beams), satisfy the symmetry condition $\mathcal{K}(x,\xi)=\mathcal{K}(\xi,x)$. It can be proved by paper and pencil calculations that this condition is really fulfilled. As regards $\mathcal{K}_{2I}$ and $\mathcal{K}_{1II}$, however, a comparison of (4.7b) and (4.7b) as well as that of (4.8b) and (4.8c) clearly shows the fulfillment of the previous symmetry condition.

4.3. **Computational results.**

4.3.1. *FrsRp beams.* Tables 4 contain the values of the dimensionless critical force $\sqrt{\mathcal{N}_{\text{crit}}}/\pi$ as a function of b.

Table 4. The critical forces of FrsRp beam

b	$\sqrt{\mathcal{N}_{\text{crit}}}/\pi$	$\sqrt{\mathcal{N}(b)}/\pi$	b	$\sqrt{\mathcal{N}_{\text{crit}}}/\pi$	$\sqrt{\mathcal{N}(b)}/\pi$
0.000	1.00000	1.00003	0.500	1.57277	1.57282
0.025	1.01910	1.01908	0.525	1.61363	1.61370
0.050	1.03895	1.03893	0.550	1.65509	1.6541
0.075	1.05958	1.05957	0.575	1.69675	1.69650
0.100	1.08104	1.08105	0.600	1.73809	1.73862
0.125	1.10336	1.10338	0.625	1.77845	1.77940
0.150	1.12659	1.12661	0.650	1.81707	1.81793
0.175	1.15078	1.15079	0.675	1.85313	1.85350
0.200	1.17599	1.17599	0.700	1.88580	1.88552
0.225	1.20226	1.20224	0.725	1.91439	1.91360
0.250	1.22964	1.22962	0.750	1.93846	1.93750
0.275	1.25819	1.25817	0.775	1.95784	1.95711
0.300	1.28796	1.28794	0.800	1.97270	1.97252
0.325	1.31898	1.31898	0.825	1.98350	1.98393
0.350	1.35131	1.35132	0.850	1.99086	1.99171
0.375	1.38496	1.38499	0.875	1.99550	1.99638
0.400	1.41996	1.41998	0.900	1.99812	1.99859
0.425	1.45630	1.45629	0.925	1.99940	1.99914
0.450	1.49394	1.49391	0.950	1.99989	1.99897
0.475	1.53281	1.53277	0.975	2.00000	1.99915
0.500	1.57277	1.57282	1.000	2.00001	2.00087

The dimensionless parameter b in the first column shows the location of the middle roller support. The second column contains the critical value for the dimensionless compressive force, more precisely, the quantity $\sqrt{\mathcal{N}_{\text{crit}}}/\pi$ against the discrete values of b. The third column contains the approximations computed by using the polynomials $\sqrt{\mathcal{N}(b)}/\pi$ fitted onto the point pairs taken from the first two columns of Table 4:

$$\sqrt{\mathcal{N}_{\text{crit}}(b)}/\pi = -1.930\,307\,982 b^5 + 1.921\,741\,813 b^4 - 0.156\,388\,723\,6 b^3 + \\ + 0.637\,849\,474\,6 b^2 + 0.746\,170\,973\,5 b + 1.000\,037\,785\,, \quad b \text{in } [0, 0.5] \tag{4.9a}$$

$$\sqrt{\mathcal{N}_{\text{crit}}(b)}/\pi = -2.260\,967\,607 b^5 + 24.937\,737\,41 b^4 - 62.207\,343\,07 b^3 + \\ + 61.907\,897\,88 b^2 - 25.522\,238\,92 b + 5.145\,794\,214\,. \quad b \text{ in } [0.5, 1.0] \tag{4.9b}$$

Figure 12 depicts the dimensionless critical force against b. The discrete points are depicted by diamonds, while the corresponding polynomials are drawn using continuous lines.

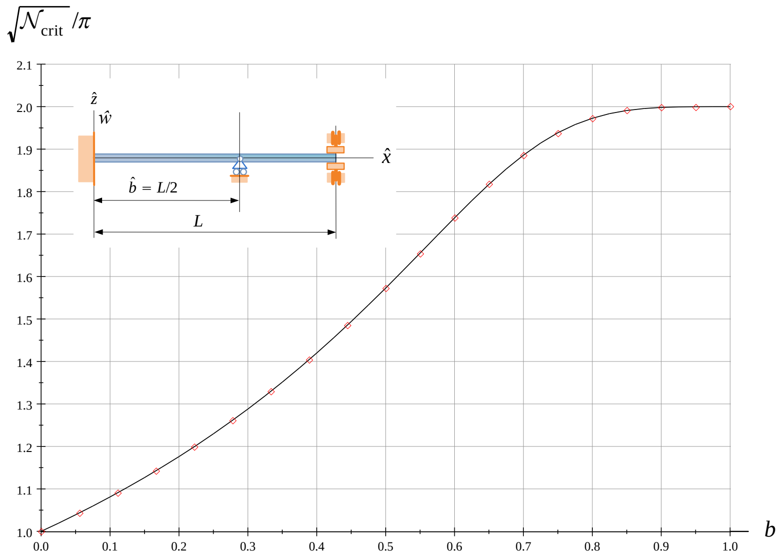

Figure 12. The dimensionless critical force for an FrsRp beam

4.3.2. *PrsRp beams.* Tables 5 contains the values of the dimensionless critical force $\sqrt{\mathcal{N}_{\rm crit}}/\pi$ as a function of b. The schemes of these tables are the same as those for Tables 4.

Table 5. The critical forces of PrsRp beam

b	$\sqrt{\mathcal{N}_{\rm crit}}/\pi$	$\sqrt{\mathcal{N}(b)}/\pi$	$x=b$	$\sqrt{\mathcal{N}_{\rm crit}}/\pi$	$\sqrt{\mathcal{N}(b)}/\pi$
0.000	1.00000	1.00004	0.500	1.43029	1.43033
0.025	1.01694	1.01692	0.525	1.44907	1.44919
0.050	1.03446	1.03444	0.550	1.46537	1.46560
0.075	1.05258	1.05257	0.575	1.47877	1.47889
0.100	1.07130	1.07131	0.600	1.48897	1.48889
0.125	1.09063	1.09066	0.625	1.49581	1.49559
0.150	1.11060	1.11062	0.650	1.49935	1.49912
0.175	1.13120	1.13121	0.675	1.49984	1.49971
0.200	1.15243	1.15242	0.700	1.49768	1.49771
0.225	1.17428	1.17426	0.725	1.49334	1.49352
0.250	1.19673	1.19670	0.750	1.48736	1.48760
0.275	1.21973	1.21971	0.775	1.48025	1.48046
0.300	1.24323	1.24323	0.800	1.47251	1.47258
0.325	1.26715	1.26717	0.825	1.46456	1.46447
0.350	1.29137	1.29139	0.850	1.45679	1.45657
0.375	1.31573	1.31575	0.875	1.44954	1.44930
0.400	1.34002	1.34003	0.900	1.44311	1.44297
0.425	1.36398	1.36397	0.925	1.43776	1.43783
0.450	1.38728	1.38725	0.950	1.43371	1.43397
0.475	1.40954	1.40952	0.975	1.43117	1.43139
0.500	1.43029	1.43033	1.000	1.43029	1.42989

The polynomials fitted onto the computational results are given below:

$$\sqrt{\mathcal{N}_{\text{crit}}(b)}\,/\pi = -4.332\,224\,892b^5 + 2.844\,038\,856b^4 - 0.646\,783\,773\,5b^3 + \\ + 0.551\,935\,872\,4x^2 + 0.661\,558\,617\,6b + 1.000\,046\,948\,, \quad bin[0, 0.5] \quad (4.10a)$$

$$\sqrt{\mathcal{N}_{\text{crit}}(b)}\,/\pi = -18.118\,635\,59x^5 + 68.962\,372\,98x^4 - 99.608\,791\,11x^3 + \\ + 66.823\,638\,2x^2 - 20.119\,893\,78x + 3.491\,203\,828\,, \quad bin[0.5, 1.0] \quad (4.10b)$$

Figure 13 depicts the dimensionless critical force against b. The continuous lines belong to polynomials (4.10). Note that the dimensionless critical force reaches its maximum if $b \in [0.65, 0.68]$.

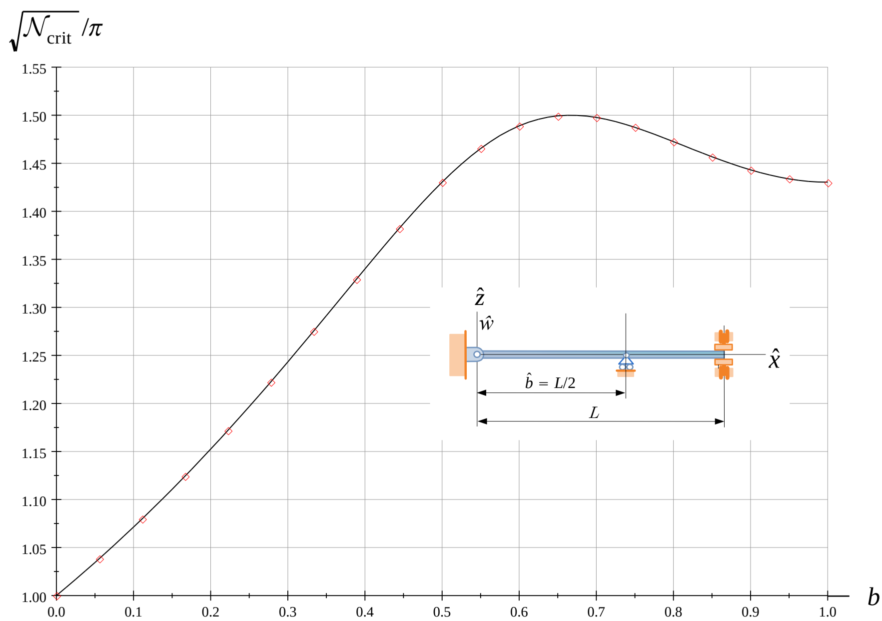

Figure 13. The dimensionless critical force for an PrsRp beam

Remark 7. The corresponding nonlinear characteristic equations are presented in Appendix A – see equations (A.4) and (A.5). They are also solved numerically. The results obtained coincide up to five to six digit accuracy with those presented in Tables 4 and 5.

5. Example

Consider an FrsRp beam with the cross section shown in Figure 14. It is assumed that $a = c = 100$ mm, $a_1 = a_2 = a/3$, $E_1 = E_{\text{aluminium}} \approx 7.0 \cdot 10^4$ N/mm^2 while $E_2 = E_{steel} \approx 2.1 \cdot 10^5$ N/mm^2. The length L of the beam is 3000 mm.

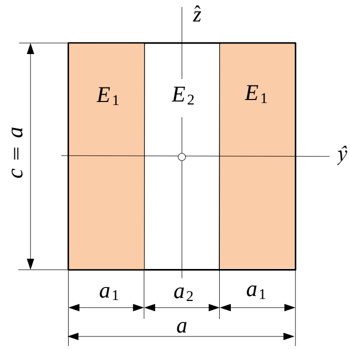

Figure 14. The cross section of an FrsRp beam

Under these conditions

$$I_{ey} = \frac{a^4}{12}(\frac{2E_1 + E_2}{3}) = \frac{100^4}{12}(\frac{2 \times 0.71 + 2.0}{3})10^5 =$$
$$= 9.5 \times 10^{11}\,\mathrm{Nmm}^2 = 9.5 \times 10^{14}\,\mathrm{kg\,mm}^3/\,\mathrm{sec}^2 \quad (5.1)$$

and

$$\rho_a = \frac{1}{A}\int_A \rho\,\mathrm{d}A = \frac{(2\rho_1 + \rho_2)\,A_1}{A} = \frac{(2 \times 2710 + 7850) \times 100 \times \frac{100}{3}}{10^9 \times 100^2} = \quad (5.2)$$
$$= 4.423333 \times 10^{-6}\ \mathrm{kg/mm}^3 \quad (5.3)$$

According to Table 2 the dimmensionless critical load for $b = 0.4$ is given by the equation $\sqrt{\mathcal{N}_{crit}}/\pi = 1.41996$ from where we get

$$\mathcal{N}_{crit} = (1.41996 \times 3.14)^2 = 19.879 \quad (5.4)$$

With $\mathcal{N}_{crit}$ equation (2.6) yields

$$N_{crit} = \frac{I_{ey}\mathcal{N}_{crit}}{L^2} = \frac{9.5 \times 10^{11} \times 19.879}{3000^2} = 2.0983 \times 10^6\ \mathrm{N} \quad (5.5)$$

As regards the first eigenvalue λ_1 concorning the free vibrations it follows from Table 2 that

$$\sqrt{\lambda_1|_{b=0.4}} = 0.5667 \times 4.73004^2 = 12.678 \quad (5.6)$$

With $\sqrt{\lambda_1|_{b=0.4}}$ equation (2.10) yields

$$\omega_1 = \frac{\sqrt{\lambda_1|_{b=0.4}}}{L^2}\sqrt{\frac{I_{ey}}{\rho_a A}}$$

from where substituting (5.1), (5.2) and (5.6) we obtain

$$\omega_1 = \frac{12.678}{3000^2} \times \sqrt{\frac{9.5 \times 10^{14}}{4.423333 \times 10^{-6} \times 100^2}} = 206.440\frac{\mathrm{rad}}{\mathrm{sec}} \quad (5.7)$$

The above results are validated by the commercial finite element program Ansys. 228 uniform hexahedral elements (SOLID185) were used to generate the geometry mesh. Table 6 shows a comparison.

Table 6. Comparison of the results

	Our solution	Ansys solution	Relative error
Critical load N_{crit}	2.0983×10^6	2.0731×10^6	1.2%
Eigenfrequency for the unloaded beam	$\frac{206.440}{2\pi} = 32.85$	32.30	1.67%

There is a good agreement between our solutions and the finite element findings.

6. Concluding remarks

Making use of the definition given in paper [18] the Green functions for the three point boundary value issues have been derived, which describe the mechanical behavior of a beam fixed at the left end and rotation prevented at the right end, and pinned beam at the left end and rotation prevented at the right end with an intermediate roller support. It is assumed that the beams have cross sectional heterogeneity [15].

Utilizing the Green functions the free vibration and linear stability problems of these beams are transformed into eigenvalue problems governed by the homogeneous Fredholm integral equation:

$$\begin{gathered} w(x) = \lambda \int_{\xi=0}^{\ell=1} G(x,\xi)\, w(\xi)\, \mathrm{d}\xi, \\ y(x) = \int_{0}^{\ell=1} \mathcal{K}(x,\xi) y(\xi) d\xi, \\ \mathcal{K}(x,\xi) = \frac{\partial^2 G(x,\xi)}{\partial x\, \partial \xi}\,, \qquad y(x) = \frac{\mathrm{d}w(x)}{\mathrm{d}x} \end{gathered} \tag{6.1}$$

It is clear from [Figure 4 – FrsRp beam] (Figure 7 – PrsRp beam) that the smallest eigenvalue λ_1 reaches its maximum if [$b \approx 0.7125$] ($b \approx 0.667$). It is also clear from Figure 14 – PrsRp beam – that the critical force has a maximum if $b \approx 0.674$.

The eigenvalue problem (6.1) is replaced by algebraic eigenvalue problems using the boundary element technique. The numerical solution of stability problems is compared to the solutions obtained numerically solving the corresponding characteristic equations presented for completeness in the Appendix A, The two solutions coincide with each other with the accuracy of four to five digits.

Appendix A. Characteristic equations

In this Appendix we present the characteristic equations. It is worthwhile to direct the reader to Table 2.8. in book [3].

If the axial force is not zero ($N \neq 0$) but a compressive force then, according to equations (2.13), the stability problem of beams are governed by the differential equation

$$w^{(4)} + p^2 w^{(2)} = 0\,, \qquad p^2 = \mathcal{N} = L^2 N / I_{ey}\,. \tag{A.1}$$

The general solutions are

$$w_r = a_1 + a_2 x + a_3 \cos px + a_4 \sin px \qquad x \in [0, b] \tag{A.2a}$$

and

$$w_\ell = c_1 + c_2 x + c_3 \cos px + c_4 \sin px \qquad x \in [b, \ell = 1] \tag{A.2b}$$

where a_k and c_k $(k = 1, \ldots, 4)$ are undermined integration constants. For FrsRp beams equation (A.1) is associated with the following boundary and continuity conditions:

$$w_r(0) = 0\,, \quad w_r^{(1)}(0) = 0\,; \qquad w_\ell^{(1)}(\ell) = 0\,, \quad w_\ell^{(3)}(\ell) = 0\,, \tag{A.3a}$$

$$\begin{gathered} w_r(b-0) = 0,\,,\; w_\ell(b+0)\,, \\ w_r^{(1)}(b-0) = w_\ell^{(1)}(b+0)\,, \\ w_r^{(2)}(b-0) = w_\ell^{(2)}(b+0)\,, \\ . \end{gathered} \tag{A.3b}$$

Differential equation (A.1), boundary and continuity conditions (A.3) determine a self adjoint eigenvalue problem with p as eigenvalue. Boundary and continuity conditions (A.3) lead to the following homogeneous equation system:

Boundary conditions if $x = 0$:

$$\begin{gathered} a_1 + a_3 = 0\,, \\ a_2 + p a_4 = 0\,. \end{gathered}$$

Continuity conditions at $x = b$:

$$\begin{gathered} a_1 + a_2 b + a_3 \cos pb + a_4 \sin pb = 0, \\ c_1 + c_2 b + c_3 \cos pb + c_4 \sin pb = 0, \\ a_2 - p a_3 \sin pb + p a_4 \cos pb - (c_2 - p c_3 \sin pb + p c_4 \cos pb) = 0, \\ -a_3 \cos pb - a_4 \sin pb - (-c_3 \cos pb - c_4 \sin pb) = 0, \end{gathered}$$

Boundary conditions at $x = \ell = 1$:

$$\begin{gathered} c_2 - p c_3 \sin p + p c_4 \cos p = 0, \\ p^3 c_3 \sin p - p^3 c_4 \cos p = 0. \end{gathered}$$

Since this equation system is homogeneous non-zero solutions for the integration constants $a_1, \ldots, a_4$ and $c_1, \ldots, c_4$ exist if and only if the determinant of the coefficient matrix vanishes, i.e., it holds that

$$\begin{aligned} -&\begin{vmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & p & 0 & 0 & 0 & 0 \\ 1 & b & \cos pb & \sin pb & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & \cos pb & \sin pb \\ 0 & 1 & -p\sin pb & p\cos pb & 0 & -1 & p\sin pb & -p\cos pb \\ 0 & 0 & -\cos pb & -\sin pb & 0 & 0 & \cos pb & \sin pb \\ 0 & 0 & 0 & 0 & 0 & 1 & -p\sin p & p\cos p \\ 0 & 0 & 0 & 0 & 0 & 0 & p^3\sin p & -p^3\cos p \end{vmatrix} \\ =&\quad \frac{1}{2}p^4\left(\cos(p-2bp) - 4\cos p\,(b-1) + 3\cos p\right) + bp^5 \sin p = 0\,. \end{aligned} \tag{A.4}$$

If $b = 1$ the solution for p is 2π. If $b \longrightarrow 0$ the solution for p is π.

As regards PrsRp boundary condition $(\text{A.3a})_2$ changes to $w_r^{(2)} = 0$. Then the characteristic equation assumes the form:

$$\frac{1}{2}\sin p - \frac{1}{2}\sin(p - 2bp) - bp\cos p = 0 \tag{A.5}$$

If $b = 1$ the solution for p is 4.4934. If $b = 0$ the solution for p is π.

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