Journal of Computational and Applied Mechanics, Vol. 17, No. 1, (2022), pp. 45–56
DOI: 10.32973/jcam.2022.003

TWO THEOREMS ON THE TORSIONAL RIGIDITY OF PIEZOELECTRIC BEAMS

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[Received: June 30, 2022; Accepted: September 18, 2022]

Abstract. In this paper two inequalities are presented for the torsional rigidity of homogeneous monoclinic piezoelectric beams. All results of the paper are based on the Saint-Venant theory of uniform torsion. The cross section of the considered elastic and piezoelectric beams may be simply connected or multiply connected two-dimensional bounded plane domain. Examples illustrate the proven inequality relations.

Mathematical Subject Classification: 74E10, 74F15, 74Q20
Keywords: piezoelectric beam, Saint-Venant's torsion, torsional rigidity

1. Introduction

Piezoelectric materials have several applications because of their unique behaviour. Piezoelectricity law is a constitute model that describes how the mechanical and electric fields are coupled within a material. Sensors and actuators are examples of active components made of linear piezoelectric materials which are widely used in smart structures. These structural components are often subjected to mechanical loading. The torsional deformation of these structural members is an important task. The present paper mainly deals with the torsional rigidity of monoclinic piezoelectric beams subjected to uniform torsion. Two theorems will be proven for the torsional rigidities of elastic beam and piezoelectric beam with the same cross sections, elastic stiffnesses and elastic flexibility (compliance) matrices. The proven inequality relations are illustrated in two examples.

The Saint-Venant torsion of linear piezoelectric beams has been considered by several researchers. Bisegna [1, 2] formulated and solved the Saint-Venant torsion problem for piezoelectric beams with a solid cross section by the use of Prandtl's stress function and electric displacement potential functions formulation. Yang studied the torsion of a solid circular ceramic cylinder which is polarized in tangential direction [3]. Rovenski et al. [4, 5] gave a torsion and electric potential functions formulation of the Saint-Venant torsion for monoclinic homogeneous piezoelectric beams. In the two studies a coupled Neumann problem is derived for the torsion and electric potential functions. Exact and numerical solutions for elliptical and rectangular cross sections are presented in [4, 5]. Daví [6] obtained a coupled boundary-value problem for the torsion function and electric potential function from a three-dimensional linear static problem of a piezoelectric body by the application of the usual assumptions of the Saint-Venant's theory. Ecsedi and Baksa [7] gave a formulation of the Saint-Venant torsion for homogeneous monoclinic piezoelectric beams in terms of Prandtl's stress function and electric displacement potential function. The Prandtl stress function and electric displacement potential function satisfy a coupled Dirichlet problem in the multiply connected cross section of the piezoelectric beam [7]. In another paper by Ecsedi and Baksa a variational formulation is developed to solve the torsional deformation of homogeneous linear piezoelectric beams [8]. The variational formulation presented in [8] uses the torsion and electric potential functions as the independent quantities of the considered variational formulation. Rovenski and Abramovich [9] applied a linear analysis to non-homogeneous beams that consists of various monoclinic piezoelectric and elastic materials. Hassani and Faal [10] presented a work which deals with the torsional analysis of a cracked isotropic bar subjected to torsion. An outer piezoelectric coating layer is under both mechanical and electrical loading. The loading includes torque and electric displacement on the coating layer. The isotropic bar contains only cracks [10]. In a paper by Talebanpour and Hematiyan an approximate analytical method was formulated to solve the Saint-Venant torsion of orthotropic piezoelectric hollow bars [11]. Saint-Venant's torsion of an orthotropic non-homogeneous piezoelectric circular cylinder was studied in a paper of Ecsedi and Baksa [12].

The present paper deals with the Saint-Venant torsion of monoclinic homogeneous piezoelectric beams, focusing on torsional rigidity. Two inequality relations are proven for the torsional rigidity of the piezoelectric beams.

2. Formulation of the torsion problem in terms of torsion function and electric potential functions

The analytical approach presented by Rovenski et al. [4, 5] to the Saint-Venant torsion of piezoelectric beams is founded on Saint-Venant's semi-inverse method of uniform torsion. The analytical solution of the torsional problem for linearly electroelastic beams is based on the next displacement field according to the Saint-Venant theory and the assumption that the electric potential does not depend on the axial coordinate z

$$u = -\vartheta yz, \quad v = \vartheta xz, \quad w = \vartheta \omega(x, y), \tag{2.1}$$

$$\varphi = \vartheta \phi(x, y). \tag{2.2}$$

Here, ϑ is the rate of twist, u, v, w are the displacements in x, y and z directions, respectively, $\omega = \omega(x, y)$ is the torsion function, and $\varphi = \varphi(x, y)$ is the electric potential field. The origin of the coordinate system $Oxyz$ is placed at the left end cross section of the beam (Figure 1). It is not necessary that O coincides with the centre of the cross section at $z = 0$. The cross section A and its geometry are shown

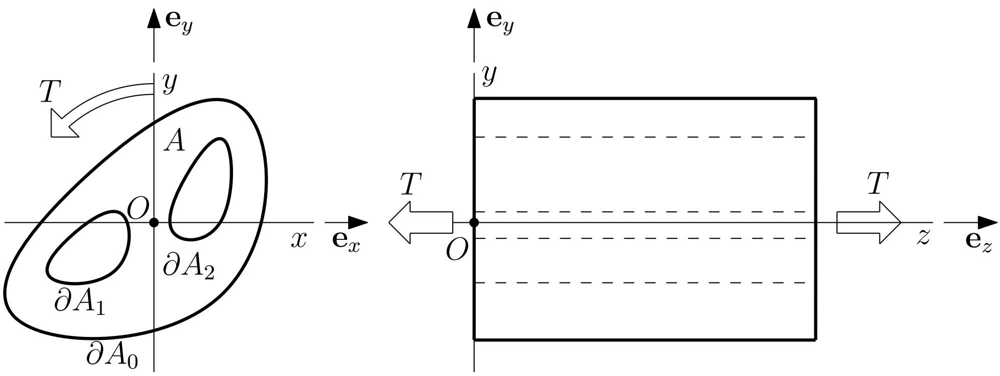

Figure 1. Piezoelectric beam with torsional load

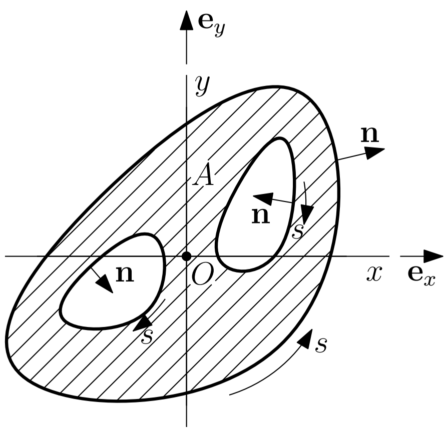

Figure 2. Multiply connected cross section

in Figure 2. The outer unit normal vector to the boundary curve ∂A is denoted by $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y$, where $\mathbf{e}_x$, $\mathbf{e}_y$ are the unit vectors in x and y directions, respectively. The arc-length defined on the boundary curve ∂A of the cross section A is denoted by s (Figure 2). The stress field, electric displacement and electric strength fields for a monoclinic piezoelectric beam in the case of Saint-Venant's torsion can be represented as [4, 5]

$$\boldsymbol{\tau}_z = \tau_{xz}\mathbf{e}_x + \tau_{yz}\mathbf{e}_y = \vartheta \mathbf{A} \cdot (\boldsymbol{\Omega} + \mathbf{e} \cdot \nabla\phi)\,, \tag{2.3}$$

$$\mathbf{D} = \vartheta\left(\mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\Omega} - \boldsymbol{\kappa} \cdot \nabla\phi\right), \tag{2.4}$$

$$\mathbf{E} = -\vartheta\nabla\phi, \quad \nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y, \tag{2.5}$$

where τ_{xz}, τ_{yz} are the shearing stresses, $\mathbf{D} = D_x\mathbf{e}_x + D_y\mathbf{e}_y$ is the electric displacement vector, $\mathbf{E} = E_x\mathbf{e}_x + E_y\mathbf{e}_y$ is the electric field vector, $\boldsymbol{\Omega} = \nabla\omega + \mathbf{e}_z \times \mathbf{R}$, $\mathbf{R} = x\mathbf{e}_x + y\mathbf{e}_y$,

$\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$, cross between two vectors denotes their vector product,

$$\mathbf{A} = \begin{bmatrix} A_{55} & A_{54} \\ A_{45} & A_{44} \end{bmatrix}$$

is the matrix of elastic stiffness tensor ($A_{45} = A_{54}$),

$$\mathbf{e} = \begin{bmatrix} e_{15} & e_{25} \\ e_{14} & e_{24} \end{bmatrix}$$

is the matrix of the tensor of piezoelectric constants,

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}$$

is the matrix of the dielectric tensor ($\kappa_{12} = \kappa_{21}$), and the upper T in equation (2.4) indicates the operation of transpose and the scalar product is indicated by a dot.

From the equation of mechanical equilibrium, Gauss-Maxwell's equation, the stress boundary condition and electrical boundary conditions it follows that $\omega = \omega(x, y)$ and $\phi = \phi(x, y)$ is a solution of the next coupled Neumann's boundary-value problem [4, 5]

$$\nabla \cdot (\mathbf{A} \cdot \boldsymbol{\Omega} + \mathbf{e} \cdot \nabla \phi) = 0 \text{ in } A, \tag{2.6}$$

$$\nabla \cdot \left(\mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\Omega} - \boldsymbol{\kappa} \cdot \nabla \phi\right) = 0 \text{ in } A, \tag{2.7}$$

$$\mathbf{n} \cdot (\mathbf{A} \cdot \boldsymbol{\Omega} + \mathbf{e} \cdot \nabla \phi) = 0 \text{ on } \partial A, \tag{2.8}$$

$$\mathbf{n} \cdot \left(\mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\Omega} - \boldsymbol{\kappa} \cdot \nabla \phi\right) = 0 \text{ on } \partial A. \tag{2.9}$$

In equations (2.8) and (2.9)

$$\partial A = \sum_{i=1}^{p-1} \partial A_i \tag{2.10}$$

is the whole boundary curve of the hollow cross section A which consists of $p-1$ holes, inner boundary curves.

The connection between the applied mechanical torque T and the rate of twist is [4, 5]

$$T = \vartheta S, \tag{2.11}$$

where S is the torsional rigidity of the electroelastic beam. Its value is obtained as

$$S = \int_A (\mathbf{e} \times \mathbf{R}) \cdot [\mathbf{A} \cdot \boldsymbol{\Omega} + \mathbf{e} \cdot \nabla \phi] \, \mathrm{d}A. \tag{2.12}$$

3. Formulation of the torsion problem in terms of Prandtl's stress function and electric displacement function

In terms of Prandtl's stress function $U = U(x, y)$ and the electric displacement function $F = F(x, y)$ the following coupled Dirichlet boundary-value problem can be derived [1, 2, 7]

$$\nabla \cdot (\mathbf{S} \cdot \nabla U + \mathbf{G} \cdot \nabla F) = -2 \text{ in } A, \tag{3.1}$$

$$\nabla \cdot \left(\mathbf{G}^{\mathrm{T}} \cdot \nabla U - \mathbf{H} \cdot \nabla F\right) = 0 \text{ in } A, \tag{3.2}$$

$$U = 0 \text{ and } F = 0 \text{ on } \partial A_0, \tag{3.3}$$

$$U = U_i = \text{constant on } \partial A_i \quad (i = 1, 2, \ldots, p), \tag{3.4}$$

$$F = F_i = \text{constant on } \partial A_i \quad (i = 1, 2, \ldots, p), \tag{3.5}$$

$$\oint_{\partial A_i} \mathbf{n} \cdot (\mathbf{S} \cdot \nabla U + \mathbf{G} \cdot \nabla F)\, \mathrm{d}s = 2A_i, \quad (i = 1, 2, \ldots, p-1), \tag{3.6}$$

$$\oint_{\partial A_i} \mathbf{n} \cdot \left(\mathbf{G}^{\mathrm{T}} \cdot \nabla U - \mathbf{H} \cdot \nabla F\right) \mathrm{d}s = 0, \quad (i = 1, 2, \ldots, p-1), \tag{3.7}$$

where

$$\mathbf{S} = \begin{bmatrix} s_{44} & -s_{45} \\ -s_{45} & s_{55} \end{bmatrix}$$

is the matrix of the elastic flexibility (compliance) tensor,

$$\mathbf{G} = \begin{bmatrix} g_{24} & -g_{14} \\ -g_{25} & g_{24} \end{bmatrix}$$

is the matrix of the piezoelectric impermeability tensor,

$$\mathbf{H} = \begin{bmatrix} \eta_{22} & -\eta_{12} \\ -\eta_{21} & \eta_{22} \end{bmatrix}$$

is the matrix of the dielectric impermeability tensor ($\eta_{12} = \eta_{21}$), and A_i is the area of the plane domain closed by the boundary curve ∂A_i ($i = 1, 2, \ldots, p$). The stress field $\boldsymbol{\tau}_z$ and electric displacement vector $\mathbf{D}$ can be represented as

$$\boldsymbol{\tau}_z = \vartheta \nabla U \times \mathbf{e}_z, \quad \mathbf{D} = \vartheta \nabla F \times \mathbf{e}_z. \tag{3.8}$$

The torsional rigidity in terms of Prandtl's stress function is as follows:

$$S = -\int_A \mathbf{R} \cdot \nabla U \mathrm{d}A = 2\left(\int_A U \mathrm{d}A + \sum_{i=1}^{p} U_i A_i\right). \tag{3.9}$$

4. Formulae for the torsional rigidity

In paper [8] it was proven that

$$S = \int_A (\boldsymbol{\Omega} \cdot \mathbf{A} \cdot \boldsymbol{\Omega} + 2\boldsymbol{\Omega} \cdot \mathbf{e} \cdot \nabla\phi - \nabla\phi \cdot \boldsymbol{\kappa} \cdot \nabla\phi)\, \mathrm{d}A. \tag{4.1}$$

Starting from the following equation

$$\begin{aligned} 0 = \int_{\partial A} \phi \mathbf{n} \cdot \left(\mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\Omega} - \boldsymbol{\kappa} \cdot \nabla\phi\right) \mathrm{d}s = \int_A \left(\nabla\phi \cdot \mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\Omega} - \nabla\phi \cdot \boldsymbol{\kappa} \cdot \nabla\phi\right) \mathrm{d}A + \\ + \int_A \phi \nabla \cdot \left(\mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\Omega} - \boldsymbol{\kappa} \cdot \nabla\phi\right) \mathrm{d}A = \\ = \int_A \boldsymbol{\Omega} \cdot \mathbf{e} \cdot \nabla\phi \mathrm{d}A - \int_A \nabla\phi \cdot \boldsymbol{\kappa} \cdot \nabla\phi \mathrm{d}A = 0 \end{aligned} \tag{4.2}$$

we can derive that

$$\int_A \mathbf{\Omega} \cdot \mathbf{e} \cdot \nabla\phi \mathrm{d}A = \int_A \nabla\phi \cdot \boldsymbol{\kappa} \cdot \nabla\phi \mathrm{d}A. \tag{4.3}$$

Combination of equation (4.1) with equation (4.3) gives

$$S = \int_A (\mathbf{\Omega} \cdot \mathbf{A} \cdot \mathbf{\Omega} + \nabla\phi \cdot \boldsymbol{\kappa} \cdot \nabla\phi)\, \mathrm{d}A, \tag{4.4}$$

which shows that $S > 0$ since $\mathbf{A}$ and $\boldsymbol{\kappa}$ are positive definite symmetric tensors [5, 7, 9]. In paper [7] it was proven that

$$S = \int_A (\nabla U \cdot \mathbf{S} \cdot \nabla U + \nabla F \cdot \mathbf{H} \cdot \nabla F)\, \mathrm{d}A. \tag{4.5}$$

Here, we note $\mathbf{S}$ and $\mathbf{H}$ are positive definite symmetric second order tensors.

5. Saint-Venant's torsion of elastic beam

The governing equations of Saint-Venant's torsion of elastic beam in terms of torsion function $\omega_0 = \omega_0(x, y)$ is obtained from equations (2.6–2.9) by the substitution

$$\mathbf{e} = \mathbf{0}. \tag{5.1}$$

In this case we have

$$\nabla \cdot (\mathbf{A} \cdot \mathbf{\Omega}_0) = 0 \text{ in } A, \tag{5.2}$$

$$\mathbf{n} \cdot \mathbf{A} \cdot \mathbf{\Omega}_0 = 0 \text{ on } \partial A, \tag{5.3}$$

$$\nabla \cdot (\boldsymbol{\kappa} \cdot \nabla\phi_0) = 0 \text{ in } A, \tag{5.4}$$

$$\mathbf{n} \cdot \boldsymbol{\kappa} \cdot \nabla\phi_0 = 0 \text{ on } \partial A. \tag{5.5}$$

From equations (5.4) and (5.5) it follows that

$$\nabla\phi_0 = \mathbf{0}, \quad \phi_0 = \text{constant}. \tag{5.6}$$

To prove this statement we consider the next equation

$$\int_A \phi_0 \nabla \cdot (\boldsymbol{\kappa} \cdot \nabla\phi_0)\, \mathrm{d}A = \int_{\partial A} \phi_0 \mathbf{n} \cdot \boldsymbol{\kappa} \cdot \nabla\phi_0 \mathrm{d}s - \int_A \nabla\phi_0 \cdot \boldsymbol{\kappa} \cdot \nabla\phi_0 \mathrm{d}A = 0, \tag{5.7}$$

that is

$$\int_A \nabla\phi_0 \cdot \boldsymbol{\kappa} \cdot \nabla\phi_0 \mathrm{d}A = 0. \tag{5.8}$$

Since $\boldsymbol{\kappa}$ is the positive definite symmetric tensor from equation (5.8) it follows that

$$\nabla\phi_0 = \mathbf{0}, \quad \phi_0 = \text{constant in } A \cup \partial A. \tag{5.9}$$

The expression of the torsional rigidity for $\mathbf{e} = \mathbf{0}$ is as follows:

$$S_0 = \int_A (\mathbf{e} \times \mathbf{R}) \cdot \mathbf{\Omega}_0 \mathrm{d}A = \int_A \mathbf{\Omega}_0 \cdot \mathbf{A} \cdot \mathbf{\Omega}_0 \mathrm{d}A; \tag{5.10}$$

this expression follows from equation (4.4). The governing equations of Saint-Venant's torsion for elastic beams in terms of Prandtl's stress function are obtained from equations (3.1–3.7) by the substitution

$$\mathbf{G} = \mathbf{0}. \tag{5.11}$$

This substitution leads to the result

$$\nabla \cdot (\mathbf{S} \cdot \nabla U_0) = -2 \text{ in } A, \tag{5.12}$$

$$\nabla \cdot (\mathbf{H} \cdot \nabla F_0) = 0 \text{ in } A, \tag{5.13}$$

$$U_0 = 0 \text{ and } F_0 = 0 \text{ on } \partial A_0, \tag{5.14}$$

$$U_0 = U_{0i} = \text{constant on } \partial A_i, \quad (i = 1, 2, \ldots, p), \tag{5.15}$$

$$F_0 = F_{0i} = \text{constant on } \partial A_i, \quad (i = 1, 2, \ldots, p), \tag{5.16}$$

$$\oint_{\partial A_i} \mathbf{n} \cdot \mathbf{S} \cdot \nabla U_0 \mathrm{d}s = 2A_i, \quad (i = 1, 2, \ldots, p), \tag{5.17}$$

$$\int_{\partial A_i} \mathbf{n} \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}s = 0, \quad (i = 1, 2, \ldots, p). \tag{5.18}$$

It is very easy to prove that if $F_0 = F_0(x, y)$ satisfies equations (5.13), $(5.14)_2$, (5.16) and (5.18) then we have

$$F_0 = F_0(x, y) = 0 \text{ in } A \cup \partial A. \tag{5.19}$$

The following equation is used to prove the above statement:

$$\begin{gathered}\int_A F_0 \nabla \cdot (\mathbf{H} \cdot \nabla F_0)\, \mathrm{d}A = \int_{\partial A} F_0 \mathbf{n} \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}s - \int_A \nabla F_0 \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}A = \\ = \oint_{\partial A_0} F_0 \mathbf{n} \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}s + \sum_{i=1}^{p} F_{0i} \oint_{\partial A_i} \mathbf{n} \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}s - \int_A \nabla F_0 \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}A = 0.\end{gathered} \tag{5.20}$$

From equation (3.4) it follows that

$$\int_A \nabla F_0 \cdot \mathbf{H} \cdot \nabla F_0 \mathrm{d}A = 0, \tag{5.21}$$

and since $\mathbf{H}$ is a positive definite two-dimensional second order tensor, this means that

$$\nabla F_0 = \text{constant}, \quad F_0 = 0 \text{ in } A \cup \partial A \tag{5.22}$$

according to the boundary condition $(5.14)_2$. The torsional rigidity of elastic beams in terms of $U_0 = U_0(x, y)$ can be represented as

$$S_0 = 2\left(\int_A U_0(x, y)\mathrm{d}A + U_{0i}A_i\right) = \int_A \nabla U_0 \cdot \mathbf{S} \cdot \nabla U_0 \mathrm{d}A, \tag{5.23}$$

which follows from equation (4.5).

6. Inequality relations for the torsional rigidity

The proof of the inequality relations for the torsional rigidity is based on the Schwarz inequality

$$\int_A (\mathbf{P}\cdot\mathbf{M}\cdot\mathbf{P}+\mathbf{p}\cdot\mathbf{m}\cdot\mathbf{p})\,\mathrm{d}A\int_A (\mathbf{Q}\cdot\mathbf{M}\cdot\mathbf{Q}+\mathbf{q}\cdot\mathbf{m}\cdot\mathbf{q})\,\mathrm{d}A \geq \geq \left(\int_A (\mathbf{P}\cdot\mathbf{M}\cdot\mathbf{Q}+\mathbf{p}\cdot\mathbf{m}\cdot\mathbf{q})\,\mathrm{d}A\right)^2, \tag{6.1}$$

where $\mathbf{P}$, $\mathbf{p}$, $\mathbf{Q}$, $\mathbf{q}$ are arbitrary two-dimensional vectors

$$\mathbf{P}=P_x\mathbf{e}_x+P_y\mathbf{e}_y,\quad \mathbf{p}=p_x\mathbf{e}_x+p_y\mathbf{e}_y, \tag{6.2}$$

$$\mathbf{Q}=Q_x\mathbf{e}_x+Q_y\mathbf{e}_y,\quad \mathbf{q}=q_x\mathbf{e}_x+q_y\mathbf{e}_y, \tag{6.3}$$

$\mathbf{M}$, $\mathbf{m}$ are arbitrary two-dimensional positive definite symmetric second order tensors. Let

$$\mathbf{M}=\mathbf{A},\quad \mathbf{m}=\boldsymbol{\kappa},\quad \mathbf{P}=\boldsymbol{\Omega},\quad \mathbf{p}=\nabla\phi,\quad \mathbf{Q}=\boldsymbol{\Omega}_0,\quad \mathbf{q}=\mathbf{0} \tag{6.4}$$

in inequality (6.1). We have

$$\int_A (\boldsymbol{\Omega}\cdot\mathbf{A}\cdot\boldsymbol{\Omega}+\nabla\phi\cdot\boldsymbol{\kappa}\cdot\nabla\phi)\,\mathrm{d}A\int_A (\boldsymbol{\Omega}_0\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0)\,\mathrm{d}A \geq \geq \left(\int_A (\boldsymbol{\Omega}\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0)\,\mathrm{d}A\right)^2, \tag{6.5}$$

A simple computation gives

$$\begin{aligned}\int_A (\boldsymbol{\Omega}\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0)\,\mathrm{d}A &= \int_A (\nabla\omega+\mathbf{e}_z\times\mathbf{R})\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0\mathrm{d}A = \\ &= \int_A \nabla\omega\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0\mathrm{d}A+\int_A (\mathbf{e}_z\times\mathbf{R})\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0\mathrm{d}A = \\ &\int_{\partial A}\omega\mathbf{n}\cdot\mathbf{A}\cdot\boldsymbol{\Omega}_0\mathrm{d}s-\int_A \omega\nabla\cdot(\mathbf{A}\cdot\boldsymbol{\Omega}_0)\,\mathrm{d}A+S_0=S_0.\end{aligned} \tag{6.6}$$

From inequality (6.5) and equation (6.6) it follows that

$$S(\mathbf{A},\boldsymbol{\kappa},\mathbf{e})\geq S_0(\mathbf{A},\boldsymbol{\kappa}). \tag{6.7}$$

This inequality relation is valid if the cross section of the piezoelastic beam and elastic beam ($\mathbf{e}=\mathbf{0}$) have the same cross section. By the use of Prandtl's stress function and elastic displacement formulation we can derive another inequality relation for the torsional rigidity. Let

$$\mathbf{M}=\mathbf{S},\quad \mathbf{m}=\mathbf{H},\quad \mathbf{P}=\nabla U,\quad \mathbf{p}=\nabla F,\quad \mathbf{Q}=\nabla U_0,\quad \mathbf{q}=\mathbf{0} \tag{6.8}$$

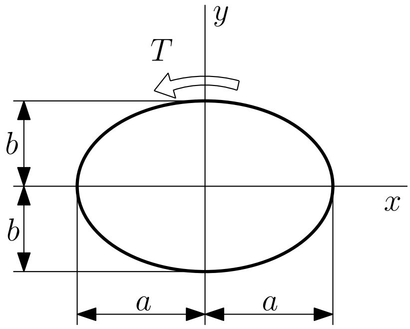

Figure 3. Solid elliptical cross section

in inequality (6.1). It is obvious that

$$\begin{aligned}\int_A (\nabla U\cdot\mathbf{S}\cdot\nabla U+\nabla F\cdot\mathbf{H}\cdot\nabla F)\,\mathrm{d}A\int_A(\nabla U_0\cdot\mathbf{S}\cdot\nabla U_0)\,\mathrm{d}A\geq\\ \geq\left(\int_A(\nabla U\cdot\mathbf{S}\cdot\nabla U_0)\,\mathrm{d}A\right)^2,\end{aligned}\tag{6.9}$$

and we have

$$\begin{aligned}\int_A\nabla U\cdot\mathbf{S}\cdot\nabla U_0\mathrm{d}A=\int_{\partial A}U\mathbf{n}\cdot\mathbf{S}\cdot\nabla U_0\mathrm{d}s-\int_A U\nabla\cdot(\mathbf{S}\cdot\nabla U_0)\,\mathrm{d}A=\\ =\int_{\partial A_0}U\mathbf{n}\cdot\mathbf{S}\cdot\nabla U_0\mathrm{d}s+\sum_{i=1}^p U_i\oint_{\partial A_i}\mathbf{n}\cdot\mathbf{S}\cdot\nabla U_0\mathrm{d}s+2\int_A U\mathrm{d}A=\\ =2\left(\int_A U\mathrm{d}A+\sum_{i=1}^p U_iA_i\right)=S.\end{aligned}\tag{6.10}$$

Substitution of equation (6.10) into inequality relation (6.9) gives the following inequality relation for the torsional rigidities $S(\mathbf{S},\mathbf{G},\mathbf{H})$ and $S_0(\mathbf{S},\mathbf{H})$

$$S_0(\mathbf{S},\mathbf{H})\geq S(\mathbf{S},\mathbf{G},\mathbf{H}).\tag{6.11}$$

7. Illustration of inequality relations

The application of the inequalities (6.7) and (6.11) is illustrated by the example of a twisted orthotropic piezoelectric beam with a solid elliptical cross section (figure 3). The solution of the torsion problem of elliptical cross section shown in Figure 3 in terms of $\omega=\omega(x,y)$ and $\phi=\phi(x,y)$ is as follows:

$$\omega(x,y)=-\frac{c_\omega}{c}xy,\quad \phi(x,y)=-\frac{c_\varphi}{c}xy,\tag{7.1}$$

where

$$c_\omega = \left(a_{44}\kappa_{22} + e_{24}^2\right) a^4 + \left(a_{44}\kappa_{11} - a_{55}\kappa_{22}\right) a^2 b^2 - \left(a_{55}\kappa_{11} + e_{15}^2\right) b^4, \tag{7.2}$$

$$c_\varphi = 2a^2 b^2 \left(-a_{55} e_{24} + a_{44} e_{15}\right), \tag{7.3}$$

$$c = \left(a_{44}\kappa_{22} + e_{24}^2\right) a^4 + \left(a_{55}\kappa_{22} + a_{44}\kappa_{11} + 2e_{15}e_{24}\right) a^2 b^2 + \left(a_{55}\kappa_{11} + e_{15}^2\right) b^4. \tag{7.4}$$

Here we note that for orthotropic piezoelectric material $a_{45} = a_{54} = 0$ and $e_{25} = e_{14} = 0$. The expression of the torsional rigidity is

$$S\left(\mathbf{A}, \mathbf{e}, \boldsymbol{\kappa}\right) = \frac{a_{55} a^2 \left(a_{44}\kappa_{22} + e_{24}^2\right) + a_{44} b^2 \left(a_{55}\kappa_{11} + e_{15}^2\right)}{c} a^3 b^3 \pi. \tag{7.5}$$

From equation (7.5) we get

$$S_0\left(\mathbf{A}, \boldsymbol{\kappa}\right) = \frac{a_{55} a_{44} a^3 b^3 \pi}{a_{44} a^2 + a_{55} b^2}. \tag{7.6}$$

Combination of equation (7.5) with equation (7.6) gives

$$S\left(\mathbf{A}, \mathbf{e}, \boldsymbol{\kappa}\right) - S_0\left(\mathbf{A}, \boldsymbol{\kappa}\right) = \frac{\left(a_{44} e_{15} - a_{55} e_{24}\right)^2}{a^2 a_{44} + b^2 a_{55}} \frac{a^5 b^5 \pi}{K} \geq 0 \tag{7.7}$$

according to inequality (6.7) where

$$K = b^4 a_{55}\kappa_{11} + b^2 a^2 a_{55}\kappa_{22} + b^2 a^2 a_{44}\kappa_{11} + a^4 a_{44}\kappa_{22} + \left(e_{15} b^2 + e_{24} a^2\right)^2 \tag{7.8}$$

The Prandtl stress function and elastic displacement function formulation for the orthotropic piezoelectric beam when $s_{45} = s_{54} = 0$ and $g_{25} = g_{14} = 0$ can be represented as

$$U(x, y) = \frac{c_u}{C}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{7.9}$$

$$F(x, y) = \frac{c_f}{C}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{7.10}$$

Here,

$$c_u = a^2 b^2 \left(a^2 \eta_{11} + b^2 \eta_{22}\right), \tag{7.11}$$

$$c_f = a^2 b^2 \left(a_{55} g_{15} + b^2 g_{24}\right), \tag{7.12}$$

$$C = \left(a^2 s_{55} + b^2 s_{44}\right)\left(a^2 \eta_{11} + b^2 \eta_{22}\right) + \left(g_{15} a^2 + g_{24} b^2\right)^2. \tag{7.13}$$

The expression of torsional rigidity of the orthotropic piezoelectric solid elliptical cross section is as follows:

$$S\left(\mathbf{S}, \mathbf{G}, \mathbf{H}\right) = a^3 b^3 \pi \frac{\eta_{11} a^2 + \eta_{22} b^2}{\left(a^2 s_{55} + b^2 s_{44}\right)\left(a^2 \eta_{11} + b^2 \eta_{22}\right) + \left(g_{15} a^2 + g_{24} b^2\right)^2}. \tag{7.14}$$

From the formula (7.14) we can obtain the torsional rigidity of the elastic beam with the substitution $g_{15} = g_{24} = 0$

$$S_0\left(\mathbf{S}, \mathbf{H}\right) = \frac{\pi a^3 b^3}{s_{55} a^2 + s_{44} b^2}. \tag{7.15}$$

From formulae (7.14) and (7.15) it follows that

$$S_0(\mathbf{S},\mathbf{H}) - S(\mathbf{S},\mathbf{G},\mathbf{H}) =$$

$$= a^3b^3\pi\left(\frac{1}{s_{55}a^2+s_{44}b^2} - \frac{1}{\frac{(g_{15}a^2+g_{24}b^2)^2}{\eta_{11}a^2-\eta_{22}b^2}+s_{55}a^2+s_{44}b^2}\right) \geq 0 \tag{7.16}$$

according to the inequality relation (6.11). It must be noted that for a double connected elliptical cross section whose main axes are a, λa and b, λb $(0 < \lambda < 1)$ formulae (7.5) and (7.6) can be represented as

$$S(\mathbf{A},\mathbf{e},\boldsymbol{\kappa}) = \frac{a^3b^3\pi}{c}\left(1-\lambda^4\right)\left[a_{55}a^2\left(a_{44}\kappa_{22}+e_{24}^2\right)+a_{44}b^2\left(a_{55}\kappa_{11}+e_{15}^2\right)\right], \tag{7.17}$$

$$S_0(\mathbf{A},\boldsymbol{\kappa}) = a^3b^3\pi\left(1-\lambda^4\right)\frac{a_{55}a_{44}}{a_{44}a^2+a_{55}b^2}. \tag{7.18}$$

These equations show that equation (7.7) is valid for a hollow elliptical cross as well. A similar remark is valid for equation (7.16) since in this case

$$S(\mathbf{S},\mathbf{G},\mathbf{H}) = \frac{\pi a^3b^3}{\frac{(a^2g_{15}+b^2g_{24})^2}{\eta_{11}a^2+\eta_{22}b^2}+s_{55}a^2+s_{44}b^2}\left(1-\lambda^4\right), \tag{7.19}$$

$$S_0(\mathbf{S},\mathbf{H}) = \frac{\pi a^3b^3}{s_{55}a^2+s_{44}b^2}\left(1-\lambda^4\right). \tag{7.20}$$

8. Conclusion

This paper deals with the torsional rigidity of monoclinic piezoelectric beams subjected to uniform torsion. Two inequalities are presented for the torsional rigidity which concern the connection of torsional rigidities of piezoelectric and elastic beams. Both beams have the same cross sections and their elastic properties are also the same. This means that the elastic stiffness and elastic flexibility (compliance) matrices for the two beams are equal. An example illustrates the proven statements.

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