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SOME REMARKS ON THE APPLICATION OF TREFFTZ'S METHOD

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Abstract. The purpose of the present paper is to determine the Green function matrix of some simple rod structures for the extended Trefftz's method. An interesting proof is given for the sum of a well-known series.

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1. Introduction

The estimation of natural frequencies for elastic continuous beam structures is possible in several ways, for example with the method of Trefftz. The application of this procedure essentially leads to the explicit construction of the Green function matrix. Usually a lot of work is needed to determine the elements of the Green function matrix. Nowadays, the application of some software – Maple, Matematica, Derive, etc. – can help us to solve this problem. Our goal is to show the construction of the elements of the Green function matrix in the case of a simple frame structure. In connection with our example we are going to deal with the sum of a well-known series. We remark that the concept of the Green function matrices using a bit different approach can be attributed to G. Obádovics and G. Szeidl [1–3].

2. Summary of an extension of Trefftz's method

Consider the interval $[a,b] = \{t \in \Re : a \leq t \leq b \;\; a,b \in \Re\}$ denote the set of all Lebesgue integrable n-vector functions with real-valued coordinates by $L^2(a,b)$ where $\Re$ is the set of all real numbers. If $\mathbf{x}, \mathbf{y} \in L^2(a,b)$ are n-vector functions with the property that the product $\mathbf{y}^T\mathbf{x}$ is integrable, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{y}(t)^T \mathbf{x}(t) \, \mathrm{d}t \tag{1}$$

is the inner product. If $\mathbb{A}$ is an invertible differential operator then, on the basis of [4], its inversion is given by the formula

$$\mathbf{x}(t) = (\mathbb{A}^{-1}\mathbf{y})(t) = \int_a^b \mathbf{G}(t,s)\mathbf{y}(s)\,\mathrm{d}s\,. \tag{2}$$

Its kernel is the function matrix $\mathbf{G}(\mathbf{t},\mathbf{s})$, which we can find by using

$$(\mathbb{A}\mathbf{x})(t) = \int_a^b (\mathbb{A}\mathbf{G})(t,s)\mathbf{y}(s)ds = \mathbf{y}(t)\,. \tag{3}$$

According to [4] and (3) we have the following equations:

$$\mathbb{A}\,\mathbf{G} = \boldsymbol{\delta}_{\mathbb{A}}\,, \qquad\qquad \mathbf{M}\,\mathbf{G} = \mathbf{0}\,, \tag{4}$$

where the boundary conditions are given by the relation $\boldsymbol{M}\,\boldsymbol{\xi} = \mathbf{0}$, $\boldsymbol{\delta}_{\mathbb{A}} = \delta\mathbf{E}$, while $\mathbf{E}$ is a unit matrix and δ is Dirac's distribution [5].

Let $\mathbb{A}^{-1}$ be a real symmetric positive compact self-adjoint operator. Then $\mathbf{G}(t,s) = \mathbf{G}(s,t)$, $\forall t,s \in [a,b]$ and the eigenvalues λ_i of $\mathbb{A}^{-1}$ are all real and nonnegative. Moreover we have [4, 5]

$$\int_a^b \mathrm{Spur}\,\mathbf{G}(t,t)\mathrm{d}t = \sum_k \lambda_{\,k} \tag{5}$$

and if $j \le k$ then, according to Trefftz [6] and (5) we obtain an estimation of the j^{th} eigenvalue

$$\lambda_{\,j} \le \mu_{\,j} + \int_a^b \mathrm{Spur}\,\mathbf{G}(t,t)\mathrm{d}t - \sum_{i=1}^{k} \mu_{\,i} := \nu_{\,j} \tag{6}$$

in which $\mu_{\,j}$ is a lower bound for the j^{th} eigenvalue. Equation (6) is a new extension of Trefftz's method - see equation (3.5) in [4] for further details.

3. A simple beam structure

Let us consider the bending vibrations of a simply supported uniform beam divided mentally into two parts (Figure 1).

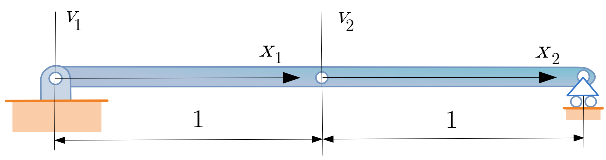

Figure 1. Simply supported beam

If $x_1, x_2 \in [0,1]$ then the boundary value problem [7] is

$$\mathbb{A}\mathbf{v} - \alpha^{\mathbf{2}}\mathbb{B}\mathbf{v} = \mathbf{0} \tag{7}$$

where

$$\mathbf{v} = \begin{bmatrix} v_1(x_1) \\ v_2(x_2) \end{bmatrix}, \quad \mathbb{A}\mathbf{v} = \begin{bmatrix} \frac{d^4 \cdot}{dx_1^4} & 0 \\ 0 & \frac{d^4 \cdot}{dx_2^4} \end{bmatrix} \begin{bmatrix} v_1(x_1) \\ v_2(x_2) \end{bmatrix}, \quad \mathbb{B}\mathbf{v} = \frac{q^2}{p^2} \begin{bmatrix} v_1(x_1) \\ v_2(x_2) \end{bmatrix}, \tag{8}$$

α is the natural circular frequency, while p and q depend on the material and geometrical parameters of the beam. Equation (7) is associated with the following boundary conditions:

$$\begin{gathered} v_1(0) = \frac{d^2 v_1}{dx_1^2}(0) = v_2(1) = \frac{d^2 v_2}{dx_2^2}(1) = 0\,, \qquad v_1(1) = v_2(0)\,, \\ \frac{dv_1}{dx_1}(1) = \frac{dv_2}{dx_2}(0)\,, \quad \frac{d^2 v_1}{dx_1^2}(1) = \frac{d^2 v_2}{dx_2^2}(0)\,, \quad \frac{d^3 v_1}{dx_1^3}(1) = \frac{d^3 v_2}{dx_2^3}(0)\,. \end{gathered} \tag{9}$$

The operator $\mathbb{A}$ is self-adjoint. This statement is a consequence of the transformation

$$\begin{aligned} \langle \mathbb{A}\mathbf{u}, \mathbf{v} \rangle &= \int_0^1 \begin{bmatrix} \frac{d^4 u_1(x)}{dx^4} & \frac{d^4 u_2(x)}{dx^4} \end{bmatrix}^T \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} \mathrm{d}x = \\ &= \int_0^1 \left[v_1(x) \frac{d^4 u_1(x)}{dx^4} + v_2(x) \frac{d^4 u_2(x)}{dx^4} \right] dx = \\ &= [\cdots]_0^1 + \int_0^1 \left[u_1(x) \frac{d^4 v_1(x)}{dx^4} + u_2(x) \frac{d^4 v_2(x)}{dx^4} \right] \mathrm{d}x = \\ &= \int_0^1 \begin{bmatrix} u_1(x) & u_2(x) \end{bmatrix}^T \begin{bmatrix} \frac{d^4 v_1(x)}{dx^4} \\ \frac{d^4 v_2(x)}{dx^4} \end{bmatrix} \mathrm{d}x = \langle \mathbf{u}, \mathbb{A}\mathbf{v} \rangle \qquad \forall \mathbf{u},\mathbf{v} \in \mathrm{L}^2(0,1) \end{aligned}$$

in which $[\cdots]_0^1 = 0$ due to the boundary conditions.

According to (4) we have the following equations:

$$\begin{aligned} \frac{d^4 G_{ii}(x,t)}{dx^4} &= \delta(x-t)\,, \qquad i = 1,2\,, \\ \frac{d^4 G_{ij}(x,t)}{dx^4} &= 0 \qquad\qquad i,j = 1,2 \ \wedge i \neq j\,. \end{aligned} \tag{10}$$

After solving differential equations (10) the elements of the Green function matrix for the beam shown in Figure 1 are given by

$$\begin{aligned} G_{ii}(x,t) = H(x-t) \int_t^x \int_t^u (s-t) \mathrm{d}s\mathrm{d}u + \\ + \frac{x^3}{6}\beta_{ii}^1(t) + \frac{x^2}{2}\beta_{ii}^2(t) + x\beta_{ii}^3(t) + \beta_{ii}^4(t)\,, \qquad i = 1,2 \end{aligned}$$

and

$$G_{ij}(x,t) = \frac{x^3}{6}\beta_{ij}^1(t) + \frac{x^2}{2}\beta_{ij}^2(t) + x\beta_{ij}^3(t) + \beta_{ij}^4(t)\,, \qquad i,j = 1,2 \wedge i \neq j$$

where H is Heaviside's distribution and the functions β_{ij}^k are to be determined by using the boundary conditions [5, 7].

Since $\mathbb{A}^{-1}$ is also self-adjoint, it can easily be seen that $G_{12}(x,t) = G_{2,1}(x,t)$ since

$$G_{12}(x,t) = \frac{1}{12}x\left((t^3 - 3t^2 + (x^2-1)t - x^2 + 3\right)$$

and

$$G_{21}(x,t) = \frac{1}{12}t\left(x^3 - 3x^2 + (t^2-1)x - t^2 + 3\right)\,.$$

It should be noted that

$$G_{11}(x,t) = H(x-t)\int_t^x\int_t^u (s-t)\mathrm{dsdu} + \frac{1}{12}\mathrm{x}^3(\mathrm{t}-2) + \frac{1}{12}\mathrm{xt}(\mathrm{t}^2 - 6\mathrm{t} + 8)\,, \quad (11)$$

$$G_{22}(x,t) = H(x-t)\int_t^x\int_t^u (s-t)\mathrm{dsdu} + \frac{1}{12}\mathrm{x}^3(\mathrm{t}-1) + \frac{1}{4}\mathrm{x}^2(\mathrm{t}-1) + \\ + \frac{1}{12}xt(t^2 - 3t + 2) + \frac{1}{12}(t^3 - 3t^2 + 2)\,. \quad (12)$$

Making use of (11) and (12) we can get for (5) that

$$\int_0^1 \mathrm{Spur}\,\mathbf{G}(\mathrm{x},\mathrm{x})\mathrm{dx} = \int_0^1 \mathrm{G}_{11}(\mathrm{x},\mathrm{x})\mathrm{dx} + \int_0^1 \mathrm{G}_{22}(\mathrm{x},\mathrm{x})\mathrm{dx} = \frac{8}{45}\,. \quad (13)$$

Consequently, there is no further objection to our applying Trefftz's method in this simple example, because the estimation (6) can now be easily used. To proceed we determine the Green function [$G(x,t)$ $x,t \in [0,2]$ of the simply supported uniform beam:

$$G(x,t) = H(x-t)\int_t^x\int_t^u (s-t)\mathrm{dsdu} + \frac{1}{12}\mathrm{x}^3(\mathrm{t}-2) + \frac{1}{12}\mathrm{xt}(\mathrm{t}^2 - 6\mathrm{t} + 8)\,.$$

Since $G(x,t) = G_{11}(x,t)$ we shall also find that

$$\int_0^2 G(x,x)\mathrm{dx} = \frac{8}{45}\,.$$

4. The sum of a well-known series

Solving the frequency equation of the simply supported uniform beam shown in Figure 1, natural circular frequencies can given by the equation

$$\alpha_n = \frac{n^2\pi^2 p}{4q} \tag{14}$$

where n is an integer[3, 8]. Setting p and q to one, the sum of the series $\sum_n 1/n^4$ is obtained from (5):

$$\sum_n \lambda_n = \sum_n \frac{1}{\alpha_n^2} = \sum_n \frac{16}{n^4\pi^4} = \frac{16}{\pi^4}\sum_n \frac{1}{n^4} = \frac{8}{45} = \int_0^1 \mathrm{Spur}\mathbf{G}(x,x)dx$$

thus

$$\sum_n \frac{1}{n^4} = \frac{\pi^4}{90}\,. \tag{15}$$

5. Concluding remarks

The purpose of the above discussion has been to show the determination of the Green function matrix for a simple beam structure. With the procedure shown above we can use estimation (6) in similar structures. In addition, we have obtained an interesting proof for the sum of the series (15).

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