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# ESTIMATION OF HEAT FLOW IN CIRCULAR BARS OF VARIABLE DIAMETER

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**Abstract.** Upper and lower bounds for the heat flow in nonhomogeneous circular bars of variable diameter are presented. The thermal properties may depend on the axial and radial coordinates and the boundary conditions of the considered heat conduction problem does not depend on the polar angle. The analysed stead-state heat conduction problem is axisymmetric. Equations of Fourier's theory is used to formulate the thermal boundary value problem of heat conduction in nonhomogeneous circular bars with nonuniform cross-section. The computation of the heat flow is based on the concept of overall heat transfer coefficient. The derivation of bounding formulae for the overall heat transfer coefficient is based on a minimum principle and the Schwarz's inequality. Six examples illustrate the applications of the derived upper and lower bound formulae how one can use to estimate the heat flow in a nonhomogeneous circular bar with nonuniform cross section.

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## 1. Introduction

The overall heat transfer coefficient in the steady-state heat conduction problem an important structural property of a solid body in which the heat is flowing between its two separated parts of its boundary surfaces. The exact (strict) value of the overall heat transfer coefficient is known only with bodies of very simple shapes, wherefore such principles and methods are of great significance with the application of which lower and upper bounds may be created to the numerical value of the overall heat transfer coefficient. From the higher temperature boundary part of body to the lower temperature boundary part of the body the process of heat flowing is characterized by the overall heat transfer coefficient according to the next equation $Q = \Lambda(T_1 - T_2)$, $T_1 > T_2$ here $Q$ is the heat flow in unit time, $T_1$ and $T_2$ are given temperature and $\Lambda$ is the overall heat transfer coefficient. There are several papers which formulates upper and lower bounds for the heat flow in the case of steady-state heat conduction problems. In paper [1] the author examines the problem of planar heat conduction through an irregularly shaped body found as an inclusion in a perfectly insulating wall between two half-planes maintained at different temperatures. He obtains upper and lower bounds for the heat flow in terms of the temperature difference, conductivity and some global properties of the body. The presented method is based on the Schwarz's inequality. Paper [2] deals with the problem of determining the temperature distribution for steady-state heat conduction in a long cylindrical pipe. The author gives upper and lower bounds of the heat flow for the case of constant parameters describing the conductivity and density. In paper [3] a heat conduction problem in hollow three-dimensional body is considered and the author derives some inequality relations by the application of which lower and upper bounds may be obtained for the numerical value of the overall heat transfer coefficient. A linear problem of the steady-state heat conduction is studied in isotropic inhomogeneous hollow rigid bodies in [4]. By the application of the Schwarz's inequality upper and lower bounds are derived for the overall heat conduction coefficient. Some mean value formula and bounds on the thermal energy for the steady-state heat conduction in anisotropic three-dimensional body are proven in [5]. The upper and lower bounds for the heat flux are derived by the application of Schwarz's inequality, avoiding the application of the minimum principles of potential thermal energy and complementary heat flux energy which were developed by Wojnar [6].

## 2. Governing equations

Let us consider a bar in the form of body of rotation. In cylindrical coordinates $(r, \varphi, z)$ the domain under consideration is $z_1 \le z \le z_2$, $0 \le r \le R(z)$, $0 \le \varphi \le 2\pi$ and the axis of the bar is taken as the axis $z$ (Figure 1). This body of rotation occupies the region $\overline{B} = B \cup \partial B$, where the inner points of $\overline{B}$ is denoted by $B$ and the set of boundary points of $\overline{B}$ is denoted by $\partial B$. $\partial B$ is divided into three parts as $\partial B_1 = A_1$, $\partial B_2 = A_2$ and $\partial B_3$. It is obvious that $\partial B = \partial B_2 \cup \partial B_2 \cup \partial B_3$.

The boundary surface $\partial B_i$ $(i = 1, 2, 3)$ is defined as

$$\partial B_3 = \left\{ (r, \varphi, z) \Big| r = R(z),\ z_1 \le z \le z_2,\ 0 \le \varphi \le 2\pi \right\},$$

$$\partial B_i = \left\{ (r, \varphi, z) \Big| z = z_i,\ 0 \le r \le R_i,\ 0 \le \varphi \le 2\pi \right\} \qquad (i = 1, 2),$$

$$R_1 = R(z_1), \qquad R_2 = R(z_2).$$

The temperature in the body is denoted by $T = T(r, \varphi, z)$ $(r, \varphi, z) \in \overline{B}$ and $k = k(r, z)$ $(r, \varphi, z) \in \overline{B}$ denotes the thermal conductivity of the material of nonuniform circular bar. The local heat transfer coefficient at cross section $z_i$ $(i = 1, 2)$ is denoted by $h_i = h_i(r, z_i)$ $(r, \varphi, z) \in \partial B_i$ $(i = 1, 2)$.

There is no distributed heat source in $B$ and no heat flux across the boundary surface segment $\partial B_3$. The boundary surface segment $\partial B_i$ is subjected to convective heat exchange and "fluid" temperature $T_i$ $(i = 1, 2)$. It is assumed that $T_1$ and $T_2$ are constants and $T_1 > T_2$.

By the use of Fourier's theory of heat conduction [7–9] it can be shown, that under the conditions prescribed above the temperature field of nonuniform circular bar can be obtained as

$$T(r, z) = (T_1 - T_2)\theta + T_2, \tag{2.1}$$

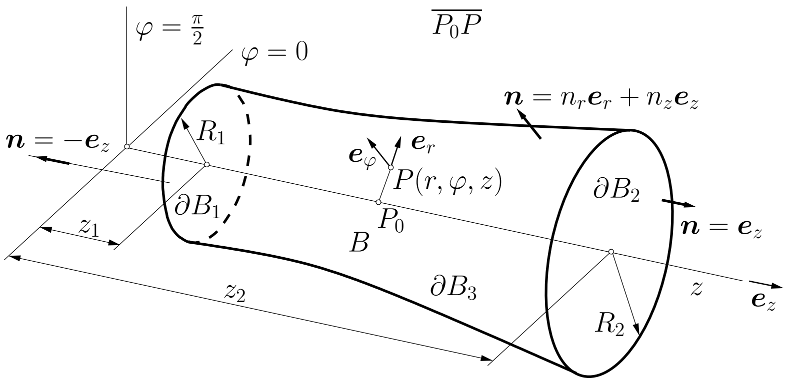


Figure 1. The body of rotation.

where the function $\theta = \theta(r, z)$ is the solution to the next boundary value problem (Figure 1)

$$\nabla \cdot (k\nabla\theta) = 0 \qquad \text{in} \quad B, \tag{2.2}$$

$$\boldsymbol{n} \cdot \nabla\theta = 0 \qquad \text{on} \quad \partial B_3, \tag{2.3}$$

$$k\,\boldsymbol{n} \cdot \nabla\theta + h_1(\theta - 1) = 0 \qquad \text{on} \quad \partial B_1, \tag{2.4}$$

$$k\,\boldsymbol{n} \cdot \nabla\theta + h_2\theta = 0 \qquad \text{on} \quad \partial B_2. \tag{2.5}$$

Here, the symbol $\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial}{\partial z}\boldsymbol{e}_z$ is the Hamilton-type vector differential operator in cylindrical coordinate frame, $\boldsymbol{n}$ is the unit outward normal vector on $\partial B$, dot between two vectors denotes their scalar product and $\boldsymbol{e}_r = \boldsymbol{e}_r(\varphi)$ is the unit vector in radial direction, $\boldsymbol{e}_z$ is the unit vector in axial direction and $\boldsymbol{e}_\varphi(\varphi) = \boldsymbol{e}_z \times \boldsymbol{e}_r(\varphi)$. The cross between two vectors denotes their vectorial product.

We note that, the boundary value problem relating to the function $\theta = \theta(r, z)$ is "axisymmetric" and on the axis of revolution the next "boundary condition" follows from the symmetry

$$\frac{\partial\theta}{\partial r} = 0 \qquad r = 0, \quad 0 \le z \le L. \tag{2.6}$$

From the cross section $A_1$ through the circular bar of variable diameter to the cross section $A_2$ heat flows. This process is characterized by the equation

$$Q = \Lambda(T_1 - T_2), \qquad T_1 > T_2. \tag{2.7}$$

In equation (2.7) $\Lambda$ is a constant, which is called the overall heat transfer coefficient and its value depends on the shape and the thermal properties of the nonuniform circular bar, $Q$ denotes the heat conducted within unit of time through the end sections. Here, we note in the book by Carslaw and Jaeger [8] the thermal resistance $\rho$ is defined by the next equation $\rho\Lambda = 1$. It is evident [7–9]

$$Q = \int_{A_1} k\,\boldsymbol{n} \cdot \nabla T \mathrm{d}A = -\int_{A_2} k\,\boldsymbol{n} \cdot \nabla T \mathrm{d}A = (T_1 - T_2)\int_{A_1} k\,\boldsymbol{n} \cdot \nabla\theta \mathrm{d}A =$$

$$(T_2 - T_1)\int\limits_{A_2} k\,\boldsymbol{n}\cdot\nabla\theta\mathrm{d}A. \quad (2.8)$$

Starting from the equation

$$\nabla\cdot(\theta k\nabla\theta) = k\left|\nabla\theta\right|^2 + \theta\nabla\cdot(k\nabla\theta) \tag{2.9}$$

by integration and by the application of the Gaussian theorem of integral transform and equations (2.2), (2.3), (2.4), (2.5) we obtain

$$\begin{gathered}0 = \int\limits_B \nabla\cdot(k\theta\nabla\theta)\mathrm{d}B - \int\limits_B k\left|\nabla\theta\right|^2\mathrm{d}B = \int\limits_{\partial B}\theta k\boldsymbol{n}\cdot\nabla\theta\mathrm{d}\partial B - \int\limits_B k\left|\nabla\theta\right|^2\mathrm{d}B = \\ \int\limits_{A_1}\theta k\boldsymbol{n}\cdot\nabla\theta\mathrm{d}A + \int\limits_{A_2}\theta k\boldsymbol{n}\cdot\nabla\theta\mathrm{d}A - \int\limits_B k\left|\nabla\theta\right|^2\mathrm{d}B = \\ -\left\{\int\limits_B k\left|\nabla\theta\right|^2\mathrm{d}B + \int\limits_{A_1}\frac{k^2}{h_1}(\boldsymbol{n}\cdot\nabla\theta)^2\mathrm{d}A + \int\limits_{A_2}\frac{k^2}{h_2}(\boldsymbol{n}\cdot\nabla\theta)^2\mathrm{d}A\right\} + \int\limits_{A_1}k\boldsymbol{n}\cdot\nabla\theta\mathrm{d}A.\end{gathered} \tag{2.10}$$

The combination of the formulae (2.7) and (2.8) with equation (2.10) gives

$$\Lambda = \int\limits_B k\left|\nabla\theta\right|^2\mathrm{d}B + \sum_{i=1}^{2}\int\limits_{A_i}\frac{k^2}{h_i}(\boldsymbol{n}\cdot\nabla\theta)^2\mathrm{d}A. \tag{2.11}$$

From equation (2.11) it follows that, $\Lambda > 0$.

The primary purpose of this paper is to derive such inequality relations by the applications of which lower and upper bounds may be performed for $\Lambda$. The exact value of the overall heat transfer coefficient $\Lambda$ might be given only with the knowledge of the solution to the boundary value problem defined by equations (2.2), (2.3), (2.4) and (2.5). The solution of the explicit form of the boundary value problem formulated in equations (2.2), (2.3), (2.4) and (2.5) is known only for bodies $B$ of very simple shapes [7–9], therefore such principles and methods are of great significance with the application of which lower and upper bounds may be produced to the numerical value of $\Lambda$. On the other hand, some of the bounding formulae of $\Lambda$ may be the theoretical framework for the different types of finite element formulation of the heat conduction problem described by the equations (2.2), (2.3), (2.4) and (2.5).

## 3. Upper bound

We introduce the symbol $E[\phi]$ by the definition

$$E[\phi] = \int\limits_B k\left|\nabla\phi\right|^2\mathrm{d}B + \int\limits_{A_1}h_1(\phi-1)^2\mathrm{d}A + \int\limits_{A_2}h_2\phi^2\mathrm{d}A, \tag{3.1}$$

where $\phi = \phi(r,\varphi,z)$ is such a function for which the integrals appear in (3.1) exist and they have finite values.

**Theorem 1.** Let $F = F\,(r, z)$ be continuous in the domain $\overline{M}$ and in the domain $M$ at least once continuously differentiable, otherwise an arbitrary function of $r$ and $z$. The inequality relations

$$\Lambda \le E[F] \tag{3.2}$$

is valid.

The domain $\overline{M} = M \cup \partial M$ is the meridian section of the body of rotation $\overline{B}$. This means that $\overline{M} = \{(r, z) \,|\, 0 \le r \le R(z),\ z_1 \le z \le z_2\}$ and $\partial M = \partial M_1 \cup \partial M_2 \cup \partial M_3 \cup \partial M_4$, where $\partial M_1 = \{(r, z) \,|\, z = z_1, 0 \le r \le R_1\}$, $\partial M_2 = \{(r, z) \,|\, z = z_2, 0 \le r \le R_2\}$, $\partial M_3 = \{(r, z) \,|\, r = R(z), z_1 \le z \le z_2\}$ and $\partial M_4 = \{(r, z) \,|\, r = 0, z_1 \le z \le z_2\}$.

**Proof.** Consider the function

$$\eta(r, z) = \theta(r, z) - F(r, z). \tag{3.3}$$

Using the expressions of $E[F]$ and $E[\phi]$ we obtain

$$E[F] = E[\theta] + \int_B k|\nabla\eta|^2 \mathrm{d}B + \sum_{i=1}^{2} \int_{A_i} h_i\eta^2 \mathrm{d}A + 2\left\{ \int_B k\nabla\theta\cdot\nabla\eta \mathrm{d}B + \int_{A_1} h_1(\theta-1)\eta \mathrm{d}A + \int_{A_2} h_2\theta\eta \mathrm{d}A \right\}. \tag{3.4}$$

By a lengthy, but elementary calculations which involve the application of the derivation of product function and Gaussian theorem of integral transformation the following relationship may be deduced

$$\begin{aligned} &\int_B k\nabla\theta\cdot\nabla\eta \mathrm{d}B + \int_{A_1} h_1(\theta-1)\eta \mathrm{d}A + \int_{A_2} h_2\theta\eta \mathrm{d}A = \\ &\quad \int_{\partial B} k\eta\boldsymbol{n}\cdot\nabla\theta \mathrm{d}\partial B - \int_B \eta\nabla\cdot(k\nabla\theta)\mathrm{d}B + \int_{A_1} h(\theta-1)\mathrm{d}A + \int_{A_2} h_2\theta\eta \mathrm{d}A = \\ &\qquad \int_{A_1} \eta\left[k\boldsymbol{n}\cdot\nabla\theta + h_1(\theta-1)\right]\mathrm{d}A + \int_{A_2} \eta\left[k\boldsymbol{n}\cdot\nabla\theta + h_2\theta\right]\mathrm{d}A = 0. \end{aligned} \tag{3.5}$$

The combination of equation (3.4) with equation (3.5) leads to inequality relation (3.2). From the demonstration it follows that equality in (3.2) can be reached only if $F = \theta$.

## 4. Lower bound

**Theorem 2.** In $\overline{B}$ the continuous vector field $\boldsymbol{b} = \boldsymbol{b}(r, \varphi, z)$ differing from the identically zero vector should satisfy the differential equation

$$\nabla\cdot\boldsymbol{b} = 0 \qquad \text{in} \quad B \tag{4.1}$$

and the boundary condition

$$\boldsymbol{n}\cdot\boldsymbol{b} = 0 \qquad \text{on} \quad \partial B_3. \tag{4.2}$$

The following inequality relation is valid

$$\Lambda \geq \frac{\left(\int\limits_{A_1} \boldsymbol{b}\cdot\boldsymbol{n}\mathrm{d}A\right)^2}{\int\limits_B \frac{\boldsymbol{b}^2}{k}\mathrm{d}B + \int\limits_{A_1} \frac{(\boldsymbol{b}\cdot\boldsymbol{n})^2}{h_1}\mathrm{d}A + \int\limits_{A_2} \frac{(\boldsymbol{b}\cdot\boldsymbol{n})^2}{h_2}\mathrm{d}A}. \tag{4.3}$$

**Proof.** Let us have

$$D(\boldsymbol{e},\boldsymbol{f}) = \int\limits_B \frac{\boldsymbol{e}\cdot\boldsymbol{f}}{k}\mathrm{d}B + \sum_{i=1}^{2}\int\limits_{A_i} \frac{(\boldsymbol{e}\cdot\boldsymbol{n})(\boldsymbol{f}\cdot\boldsymbol{n})}{h_i}\mathrm{d}A, \tag{4.4}$$

where $\boldsymbol{e} = \boldsymbol{e}(r,\varphi,z)$ and $\boldsymbol{f} = \boldsymbol{f}(r,\varphi,z)$ defined in $B$ are two arbitrary continuous vector fields. On the basis of the Schwarz's inequality it may be written that

$$D(k\nabla\theta, k\nabla\theta)\, D(\boldsymbol{b},\boldsymbol{b}) \geq \left(D(k\nabla\theta,\boldsymbol{b})\right)^2. \tag{4.5}$$

It can easily be understood that

$$\Lambda = D(k\nabla\theta, k\nabla\theta). \tag{4.6}$$

The relationship

$$\begin{aligned} D(k\nabla\theta,\boldsymbol{b}) &= \int\limits_B \nabla\theta\cdot\boldsymbol{b}\mathrm{d}B + \int\limits_{A_1}\frac{k}{h_1}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A + \\ &\int\limits_{A_2}\frac{k}{h_2}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A = \int\limits_{\partial B}\theta\boldsymbol{n}\cdot\boldsymbol{b}\mathrm{d}\partial B + \int\limits_{A_1}\frac{k}{h_1}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A + \\ &\int\limits_{A_2}\frac{k}{h_2}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A - \int\limits_B \theta\nabla\cdot\boldsymbol{b}\mathrm{d}B = \int\limits_{A_1}\boldsymbol{n}\cdot\boldsymbol{b}\left(\theta + \frac{k}{h_1}\boldsymbol{n}\cdot\nabla\theta\right)\mathrm{d}A + \\ &\int\limits_{A_2}\boldsymbol{n}\cdot\boldsymbol{b}\left(\theta + \frac{k}{h_2}\boldsymbol{n}\cdot\nabla\theta\right)\mathrm{d}A = \int\limits_{A_1}\boldsymbol{n}\cdot\boldsymbol{b}\mathrm{d}A \end{aligned} \tag{4.7}$$

further, inequality (4.5) and formula (4.6) by their combination directly yield the lower bound formula (4.3) to be proven. In deriving the relationship (4.7) the rule of differentiation of the product function as well as the Gaussian integration theorem, equations (2.4), (2.5), (4.1) and (4.2) have been applied.

By some discussion it may be pointed out that in relation (4.3) the sign of equality is valid only in the case when

$$\boldsymbol{b} = \alpha k\nabla\theta, \tag{4.8}$$

where $\alpha$ differs from zero, however, otherwise being an arbitrary real constant.

## 5. Examples

**5.1. Example for upper bound.** We assume

$$F(r,z) = C_1 \int\limits_{z_1}^{z} \frac{\mathrm{d}\zeta}{K(\zeta)} + C_2, \tag{5.1}$$

where

$$C_1 = \frac{1}{I + \frac{1}{H_1} + \frac{1}{H_2}}, \qquad C_2 = \frac{1 + \frac{1}{H_2}}{I + \frac{1}{H_1} + \frac{1}{H_2}}, \tag{5.2}$$

$$K(z) = \int\limits_0^{R(z)} rk(r,z)\,\mathrm{d}r, \quad I = \int\limits_{z_1}^{z_2} \frac{\mathrm{d}z}{K(z)}, \quad H_i = \int\limits_0^{R_i} rh_i(r)\mathrm{d}r \qquad (i = 1,2). \tag{5.3}$$

Inserting the function given by formula (5.1) into inequality relation (3.2) we obtain

$$\Lambda \leq \Lambda_U = 2\pi C_1 = \frac{2\pi}{I + \frac{1}{H_1} + \frac{1}{H_2}}. \tag{5.4}$$

**5.2. Example for lower bound.** In order to get the lower bound for $\Lambda$, we use in (4.3) the next divergence free vector field

$$\boldsymbol{b} = \frac{1}{R^2}\left[\frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}z}\boldsymbol{e}_r(\varphi) + \boldsymbol{e}_z\right]. \tag{5.5}$$

This vector field satisfies boundary condition (4.2) and the condition

$$\boldsymbol{b}\cdot\boldsymbol{e}_r = 0 \qquad r = 0, \qquad z_1 \leq z \leq z_2. \tag{5.6}$$

We introduce the following function and constants

$$M_1(z) = \int\limits_0^{R(z)} \frac{r^3}{k(r,z)}\mathrm{d}r, \qquad M_2(z) = \int\limits_0^{R(z)} \frac{r}{k(r,z)}\mathrm{d}r, \tag{5.7}$$

$$N_1 = \int\limits_{z_1}^{z_2} \frac{M_1}{(R(z))^6}\left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^2 \mathrm{d}z, \qquad N_2 = \int\limits_{z_1}^{z_2} \frac{M_2}{(R(z))^4}\mathrm{d}z, \quad N = N_1 + N_2, \tag{5.8}$$

$$\frac{1}{S_i} = \frac{1}{R_i^4}\int\limits_0^{R_i} \frac{r\mathrm{d}r}{h_i(r)} \qquad (i = 1,2). \tag{5.9}$$

Putting the vector field given by the formula (5.7) into the inequality relation (4.3) we get

$$\Lambda \geq \Lambda_L = \frac{\pi}{2(N + \frac{1}{S_1} + \frac{1}{S_2})}. \tag{5.10}$$

5.3. **Example for circular bar with uniform cross section.** Let us apply formulae (5.4) and (5.10) to the circular cylindrical bar. We assume that the thermal conductivity depends only on the axial coordinate $z$ and $h_1$, $h_2$ are constants. In this case the upper and lower bounds formulated in (5.4) and (5.10) give the same result which is the exact value of $\Lambda$. The computations yield the next value of $\Lambda$:

$$\Lambda = \frac{c^2\pi}{\int\limits_0^L \frac{\mathrm{d}z}{k(z)} + \frac{1}{h_1} + \frac{1}{h_2}} \qquad z_1 = 0, \quad z_2 = L. \tag{5.11}$$

In equation (5.11) the constant $c$ is the radius of the considered circular bar, that is $R(z) = c$, $0 \le z \le L$.

5.4. **Example for homogeneous circular cone.** In this section, we deal with the homogeneous conical bars. Setting $R(z) = a + bz$, where $a$ and $b$ are constants and $z_1 = 0$, $z_2 = L$. We find, from (5.4) and (5.10)

$$\Lambda_U = \frac{\pi}{\frac{L}{ka(a+bL)} + \frac{1}{h_1 a^2} + \frac{1}{h_2(a+bL)^2}}, \quad \Lambda_L = \frac{\pi}{\frac{(1+\frac{b^2}{2})L}{ka(a+bL)} + \frac{1}{h_1 a^2} + \frac{1}{h_2(a+bL)^2}}. \tag{5.12}$$

In equation (5.12) it has been assumed that $k$, $h_1$ and $h_2$ are constants.

In the case $h_i \to \infty$ at the end cross section $A_i$, the Robin type boundary condition will be replaced by Dirichlet type boundary condition, this means that, the cross section $A_i$ is subjected to constant temperature $T_i$ $(i = 1, 2)$.

Putting in formula (5.12) $h_1$, $h_2 \to \infty$ we obtain

$$\frac{\Lambda_U}{\Lambda_L} = 1 + \frac{b^2}{2} \tag{5.13}$$

which shows that, there is a significant difference between $\Lambda_U$ and $\Lambda_L$ for sufficiently large values of $b$.

Upper and lower bounds for $\Lambda$ may be improved by means of Rayleigh-Ritz method [10] and finite element method [11] which are based on the minimising (3.2) with respect to $F = F(r, z)$ and maximizing (4.3) with respect to $\boldsymbol{b} = \boldsymbol{b}(r, \varphi, z)$.

5.5. **Example for nonhomogeneous circular cylindrical bar of uniform cross section.** Let $c$ be the radius of boundary circle of the considered bar. The material properties are functions of the radial coordinate $r$. It is assumed that

$$k(r) = k_0 r, \quad h_i(r) = h_{0i} r, \quad (i = 1, 2), z_1 = 0, z_2 = L. \tag{5.14}$$

Let $\Lambda_0$ be defined as

$$\Lambda_0 = \frac{c^3\pi}{\frac{L}{k_0} + \frac{1}{h_{01}} + \frac{1}{h_{02}}}. \tag{5.15}$$

From the bounding formulae (5.4) and (5.10) the next result can be derived

$$\lambda_L = \frac{\Lambda_L}{\Lambda_0} = \frac{1}{2} \le \lambda = \frac{\Lambda}{\Lambda_0} \le \lambda_U = \frac{\Lambda_U}{\Lambda_0} = \frac{2}{3}. \tag{5.16}$$

Denote the mean value of $\lambda_L$ and $\lambda_U$ $\overline{\lambda} = 0.5(\lambda_U + \lambda_L)$. It is evident that

$$\left|\lambda - \overline{\lambda}\right| \leq \frac{1}{12}. \tag{5.17}$$

5.6. **Example for functionally graded circular cone.** The points of the meridian section of the circular cone are given by the next prescription

$$\overline{M} = \left\{(r, z)\middle| 0 \leq r \leq az,\, z_1 \leq z \leq z_2\right\} \text{ and } \partial M = \partial M_1 \cup \partial M_2 \cup \partial M_3 \cup \partial M_4,$$

$$\partial M_1 = \left\{(r, z)\middle| z = z_1,\, 0 \leq r \leq az_1\right\}, \quad \partial M_2 = \left\{(r, z)\middle| z = z_2,\, 0 \leq r \leq az_2\right\},$$

$$\partial M_3 = \left\{(r, z)\middle| r = az,\, z_1 \leq z \leq z_2\right\}, \quad \partial M_4 = \left\{(r, z)\middle| r = 0,\, z_1 \leq z \leq z_2\right\},$$

The thermal properties are given functions of the radial coordinate according to the next equations

$$k(r) = k_0 \exp(\nu r),\; h_1(r) = h_2(r) = h_0 \exp(\nu r), \tag{5.18}$$

where $k_0$, $h_0$ and $\nu$ are material parameters. In the numerical example the following data are used $a = 0.5$, $z_1 = 0.8\,\mathrm{m}$, $z_2 = 3\,\mathrm{m}$, $k_0 = 100\,\frac{\mathrm{W}}{\mathrm{mK}}$, $h_0 = 20\,\frac{\mathrm{W}}{\mathrm{m^2K}}$, $\nu = 0.5\,\frac{1}{\mathrm{m}}$.

Substitution of this data into equations (5.3), (5.4) and equations (5.7-5.10) gives

$$I = 0.05769894569\,\frac{\mathrm{K}}{\mathrm{W}},\; H_1 = 1.830223478\,\frac{\mathrm{W}}{\mathrm{K}},\; H_2 = 37.65999967\,\frac{\mathrm{W}}{\mathrm{K}},$$

$$N_1 = 0.001731450850\,\frac{\mathrm{K}}{\mathrm{W}},\; N_2 = 0.0145366169\,\frac{\mathrm{K}}{\mathrm{W}},\; N = 0.01626806776\,\frac{\mathrm{K}}{\mathrm{W}},$$

$$\frac{1}{S_1} = 0.1368991899\,\frac{\mathrm{K}}{\mathrm{W}},\; \frac{1}{S_2} = 0.006848732156\,\frac{\mathrm{K}}{\mathrm{W}},$$

$$\Lambda_U = 9.96328894\,\frac{\mathrm{W}}{\mathrm{K}},\; \Lambda_L = 9.816496019\,\frac{\mathrm{W}}{\mathrm{K}}.$$

If we approximate $\Lambda$ the mean value of $\Lambda_U$ and $\Lambda_L$ then the relative error is less as 0.7421 %.

## 6. Conclusions

Upper and lower bounds for the heat flux in nonhomogeneous circular bars of variable diameter are presented. Thermal properties may depend on the radial and axial coordinates. The axisymmetric nonhomogenity considered includes those cases too, when the bar is a composite of different homogeneous materials, so that the thermal conductivity and surface conductivity are piecewise constants. The discontinuities of the thermal properties should not effect the presented analysis. Here, we note that, for compound bar the function $F = F(r, z)$ is continuous on the whole meridian section and its normal derivative computed on the curves which separate the different parts of meridian section may have jump. Normal component of $\boldsymbol{b}$ remains continuous and the tangential component of $\boldsymbol{b}$ may have jump across the common boundary curves of different phases. Equations of Fourier's theory of steady-state heat conduction are used to formulate the field equations and boundary conditions of the heat transfer problem analyzed. Examples illustrate the applications of the bounding formulae derived. Rayleigh-Ritz method and finite element formulation give possibilities to improve the presented estimation of heat flux in circular bars of variable diameter.

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