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GREEN'S FUNCTIONS FOR NONHOMOGENEOUS CURVED BEAMS WITH APPLICATIONS TO VIBRATION PROBLEMS

László Péter Kiss
Institute of Applied Mechanics, University of Miskolc
H-3515 Miskolc-Egyetemváros
mechkiss@uni-miskolc.hu

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Abstract. In the open literature we have found no report on the Green function matrices of curved beams except papers [1, 2, 3] by Szeidl at al. These works assume that the material of the beam is homogeneous and isotropic. In the present paper we assume that the beam is made of heterogeneous material in such a way that the material properties depend on the cross-sectional coordinates. Under this condition we have the following aims: (1) we would like to determine the Green function matrices in a closed-form for (a) fixed-fixed, (b) pinned-pinned and (c) pinned-fixed circular beams. (2) With the knowledge of the Green function matrices we can reduce those eigenvalue problems which provide the natural frequencies of the free vibrations to eigenvalue problems governed by homogeneous Fredholm integral equations. Our goal in this respect is to solve the latter eigenvalue problems numerically and compare the results obtained with the results of finite element (FE) computations. Our numerical solutions show a good agreement with the commercial FE computations.

Mathematical Subject Classification: 74G60, 74B15
Keywords: Circular beam, nonhomogeneous material, Green function matrices, problem of free vibrations

1. Introduction

Curved beams are often used as various structural elements because of their favourable load-carrying capabilities. We mention, for instance, arch bridges and the role of curved beams as stiffener elements in roof- and shell structures. Nowadays, it is gradually getting cheaper to manufacture nonhomogeneous (heterogeneous or inhomogeneous) curved beams, such as composites, laminates and sandwich structures. The benefits of such structural members can be the reduced weight and the higher strength. A class of inhomogeneity (heterogeneity) is called cross-sectional inhomogeneity which means that the material parameters (the Young modulus E and the Poisson number) are functions of the cross-sectional coordinates – these material parameters can change continuously on the cross-section, or can be constant over each segment of the cross-section.

In the present paper we will focus on the free vibrations of heterogeneous curved beams using a Green function matrix technique.

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As regards the preliminaries it is worthy to mention that Den Hartog [4] is known to be the first to have dealt with the free vibrations of curved beams. Other early but relevant results, considering the inextensibility of the centerline, were achieved in [5, 6, 7]. A more recent research by Qatu and Elsharkawy [8] presents an exact model and numerical solutions for the free vibrations of laminated arches. With the differential quadrature method, Kang et al. [9] determine the eigenfrequencies for the in- and out-of-plane vibrations of circular Timoshenko arches with rotatory inertia and shear deformations included. Tüfekçi and Arpaci [10] obtain exact solutions for the differential equations which describe the in-plane free harmonic vibrations of extensible curved beams. Krishnan and Suresh [11] tackle the very same issue with a shear-deformable finite element (FE) model. Paper [12] by Ecsedi and Dluhi analyse some dynamic features of nonhomogeneous curved beams and closed rings assuming cross-sectional heterogeneity. Elastic foundation is taken into account in [13]. Survey paper [14] by Hajianmaleki and Qatu collects a bunch of references up until the early 2010s in the topic investigated. Kovács [15] considers layered arches with both perfect and even imperfect bonding between any two adjacent layers. Article [16] by Juna et al. uses the trigonometric shear deformation theory. The dynamic stiffness matrix is obtained from the exact solutions of the related differential equations.

It seems that, meanwhile the Green function is commonly used for various straight beam problems [17, 18, 19, 20, 21], it is somehow not preferred for the free vibrations of curved beams. There are really a few exceptions. Szeidl in his PhD [1] investigates how the extensibility of the centerline affects the free vibrations of planar, radially loaded circular beams. One of the developed numerical techniques is based on the use of the Green function matrix since its knowledge makes it possible to transform the eigenvalue problem set up for the eigenfrequencies into an eigenvalue problem governed by a system of homogeneous Fredholm integral equations where the Green function matrix is the kernel. Similarly in [2], the authors determine the natural frequencies of pinned-pinned and fixed-fixed circular arches under distributed load. Kelemen [3] seeks how the natural frequencies are related to a constant distributed external force system.

On the base of all that has been said the present paper has two main objectives. First, to determine the Green function matrices for heterogeneous curved beams for three support arrangements, i.e., for (a) fixed-fixed, (b) pinned-pinned and (c) pinned-fixed circular beams. Then to investigate the vibratory behaviour of such beams. The paper is organized into seven sections. After the introduction, the most important hypotheses and assumptions are presented with the governing equations of the vibratory issue. Then, the properties and the definition of the Green function matrices are given in Section 3. This is followed by the numerical results and evaluation. The article is closed by some conclusions, an appendix and the list of references. The appendix contains the Green function matrices in closed-form.

2. Kinematical, constitutive and motion equations

Here we summarize the most important hypotheses and governing equations of the model we have established to tackle the vibratory problem. A thorough description is available, e.g., in [22, 23, 24].

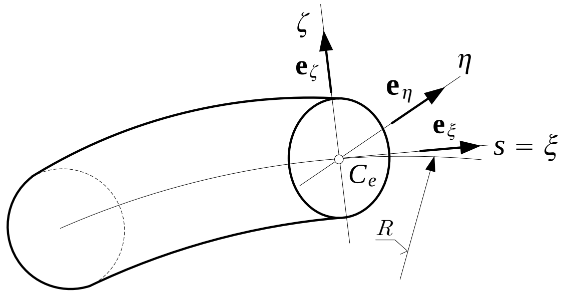

Figure 1. Coordinate system

We use a curvilinear coordinate system whose orthogonal unit vectors $\mathbf{e}_\xi; \mathbf{e}_\eta; \mathbf{e}_\zeta$ are attached to the E-weighted centerline which intersects the cross-section at the point C_e. R is the initial (constant) radius of the centerline and the included angle of the beam is $\bar{\vartheta} = 2\vartheta$. The infinitesimal line element is $\mathrm{d}s = R\mathrm{d}\varphi$ where φ is the angle coordinate. The cross-sections are uniform and symmetric with respect to the axis η not only in the geometry but also in the material composition. Hence, the Young modulus fulfills the condition $E(\eta, \zeta) = E(-\eta, \zeta)$. It is obvious that the axis ζ is a principal axis of inertia. The axis η is selected in such a way that the E-weighted first moment with respect to this axis vanishes:

$$Q_{e\eta} = \int_A E(\eta, \zeta)\zeta \mathrm{d}A = 0.$$

Under the conditions of the Euler-Bernoulli hypothesis the axial strain [8, 25] is given by

$$\varepsilon_\xi = \frac{R}{R+\zeta}\left(\varepsilon_{o\xi} + \zeta\kappa_o\right), \tag{1}$$

where

$$\varepsilon_{o\xi} = \frac{\mathrm{d}u_o}{\mathrm{d}s} + \frac{w_o}{R}, \qquad \psi_{o\eta} = \frac{u_o}{R} - \frac{\mathrm{d}w_o}{\mathrm{d}s} \quad \text{and} \quad \kappa_o = \frac{\mathrm{d}\psi_{o\eta}}{\mathrm{d}s}. \tag{2}$$

In these equations $\varepsilon_{o\xi}$, u_o and w_o are the axial strain as well as the tangential and normal displacement components on the centerline. Besides, $\psi_{o\eta}$ and κ_o are the rotation and the curvature of the centerline.

The material is linearly elastic, isotropic and, by assumption, it holds that $\sigma_\xi \gg \sigma_\eta, \sigma_\zeta$. Thus, the constitutive equation is Hooke's law in the following form: $\sigma_\xi = E(\eta, \zeta)\varepsilon_\xi$. Making use of Hooke's law and the kinematic relations (1)-(2) we can set up the following equations for the axial force N and the bending moment M – as regards the details the reader is referred to [22, 23]:

$$N=\int_A \sigma_\xi \mathrm{d}A=\frac{I_{e\eta}}{R^2}m\varepsilon_{o\xi}-\frac{M}{R}, \qquad M=\int_A \sigma_\xi \zeta \mathrm{d}A=-I_{e\eta}\left(\frac{\mathrm{d}^2 w_o}{\mathrm{d}s^2}+\frac{w_o}{R^2}\right). \tag{3}$$

Here [12]

$$A_e=\int_A E(\eta,\zeta)\,\mathrm{d}A\,, \qquad I_{e\eta}=\int_A E(\eta,\zeta)\,\zeta^2\mathrm{d}A; \quad m=\frac{A_eR^2}{I_{e\eta}}-1 \tag{4}$$

are the E-weighted area and the E-weighted moment of inertia. Moreover, m is a dimensionless geometry-heterogeneity parameter.

The equilibrium equations can be obtained from the principle of virtual work [22]. It can be given in the following form:

$$\int_V \sigma_\xi \delta\varepsilon_\xi \mathrm{d}V=\int_{\mathcal{L}}(f_n\delta w_o+f_t\delta u_o)\,\mathrm{d}s. \tag{5}$$

Here the virtual quantities are denoted by the symbol δ while f_n, f_t are distributed loads in the normal and tangential direction. The principle of virtual work yields two non-linear equilibrium equations [23]:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(N+\frac{M}{R}\right)-\frac{1}{R}\left(N+\frac{M}{R}\right)\psi_{o\eta}+f_t=0\,, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}M}{\mathrm{d}s}-\left(N+\frac{M}{R}\right)\psi_{o\eta}\right]-\frac{N}{R}+f_n=0\,. \tag{7}$$

Let us drop the non-linear terms and substitute $(3)_{1,2}$ for N and M then $(2)_1$ for $\varepsilon_{o\xi}$. If, in addition to this, we introduce the dimensionless displacements

$$U_o=\frac{u_o}{R}\,, \qquad W_o=\frac{w_o}{R}$$

we get the following differential equations [22, 24]:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(4)}+\begin{bmatrix} -m & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(2)}+$$
$$+\begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(1)}+\begin{bmatrix} 0 & 0 \\ 0 & m+1 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(0)}=\frac{R^3}{I_{e\eta}}\begin{bmatrix} f_t \\ f_n \end{bmatrix}. \tag{8}$$

Here and now on, the n-th derivative of a quantity $(\ldots)$ in terms of φ is denoted by $(\ldots)^{(n)}$. For the problem of free vibrations the distributed loads are forces of inertia. Thus

$$f_t=-\rho_a A\frac{\partial^2 u_o}{\partial t^2}; \quad f_n=-\rho_a A\frac{\partial^2 w_o}{\partial t^2}, \tag{9}$$

where ρ_a is the average density over the cross-section of area A and t denotes time.

For time-harmonic and undamped vibrations equation system (8) assumes the following form:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(2)} + \\ + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & m+1 \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(0)} = \lambda \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix} \tag{10}$$

where $\hat{U}$ and $\hat{W}$ are the dimensionless vibration amplitudes and

$$\lambda = \rho_a A \frac{R^3}{I_{e\eta}} \alpha^2 \tag{11}$$

is the unknown eigenvalue which belongs to the eigenfrequency α of the free vibrations. As regards the left side of equation (10), the effect of cross-sectional heterogeneity appears through the parameter m.

3. The Green function matrix

This section presents the definition of the Green function matrix for a class of boundary value problems governed by a system of degenerated ordinary differential equations. The definition is taken from a thesis – see [1] or paper [3] for details. First, we shall rewrite equation (10) into the following matrix form:

$$\mathbf{K}(\mathbf{y}) = \sum_{\nu=0}^{4} \overset{\nu}{\mathbf{P}}(\varphi)\mathbf{y}^{(\nu)}(\varphi) = \mathbf{r}(\varphi) =$$

$$= \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\overset{4}{\mathbf{P}}} \underbrace{\begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(4)}}_{\mathbf{y}^{(4)}} + \underbrace{\begin{bmatrix} -m & 0 \\ 0 & 2 \end{bmatrix}}_{\overset{2}{\mathbf{P}}} \underbrace{\begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(2)}}_{\mathbf{y}^{(2)}} + \underbrace{\begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}}_{\overset{1}{\mathbf{P}}} \underbrace{\begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}^{(1)}}_{\mathbf{y}^{(1)}} +$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & m+1 \end{bmatrix}}_{\overset{0}{\mathbf{P}}} \underbrace{\begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}}_{\mathbf{y}^{(0)}} = \lambda \underbrace{\begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}}_{\mathbf{r}} \qquad \overset{3}{\mathbf{P}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

Remark 1.: Differential equation (12) is called degenerated since the matrix $\overset{3}{\mathbf{P}}$ has no inverse.
Remark 2.: For equilibrium problems, the right side $\mathbf{r}$ is given by equation (10) which means that $\mathbf{r}$ represents a dimensionless distributed load.

We shall assume that (12) is associated with homogeneous linear boundary conditions of the form

$$\mathbf{U}_\mu(\mathbf{y}) = \sum_{\nu=0}^{3} \left[\mathbf{A}_{\nu\mu}\mathbf{y}^{(\nu)}(-\vartheta) + \mathbf{B}_{\nu\mu}\mathbf{y}^{(\nu)}(\vartheta) \right] =$$

$$= \sum_{\nu=0}^{3} \left\{ \begin{bmatrix} \overset{11}{A}_{\nu\mu} & \overset{12}{A}_{\nu\mu} \\ \overset{21}{A}_{\nu\mu} & \overset{22}{A}_{\nu\mu} \end{bmatrix} \begin{bmatrix} y_1(-\vartheta) \\ y_2(-\vartheta) \end{bmatrix}^{(v)} + \begin{bmatrix} \overset{11}{B}_{\nu\mu} & \overset{12}{B}_{\nu\mu} \\ \overset{21}{B}_{\nu\mu} & \overset{22}{B}_{\nu\mu} \end{bmatrix} \begin{bmatrix} y_1(\vartheta) \\ y_2(\vartheta) \end{bmatrix}^{(v)} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

where $\mu = 1, \ldots, 4$. The constant matrices $A_{\nu\mu}$ and $B_{\nu\mu}$ fulfill the conditions

$$\overset{11}{A}_{\nu\mu} = \overset{21}{A}_{\nu\mu} = \overset{11}{B}_{\nu\mu} = \overset{21}{B}_{\nu\mu} = 0.$$

For equilibrium problems equations (12), (13) constitute a boundary value problem. With the knowledge of the Green function matrix the solution sought can be given by the integral:

$$\mathbf{y}(\varphi) = \int_{-\vartheta}^{\vartheta} \mathbf{G}(\varphi, \gamma)\mathbf{r}(\gamma)\mathrm{d}\gamma \tag{14}$$

in which $\mathbf{G}(\varphi, \gamma)$ is the Green function matrix and φ, γ are angle coordinates. The Green function matrix is defined by the following four properties [1, 23]:

1. $\mathbf{G}(\varphi, \gamma)$ is a continuous function of the angle coordinates φ and γ in each of the triangular domains $-\vartheta \leq \varphi \leq \gamma \leq \vartheta$ and $-\vartheta \leq \gamma \leq \varphi \leq \vartheta$. Moreover,
$$G_{11}(\varphi, \gamma), G_{12}(\varphi, \gamma) \;\; [G_{21}(\varphi, \gamma), G_{22}(\varphi, \gamma)]$$
are 2 [4] times continuously differentiable with respect to φ. The derivatives
$$\frac{\partial^\nu \mathbf{G}(\varphi, \gamma)}{\partial \varphi^\nu} = \mathbf{G}^{(\nu)}(\varphi, \gamma) \qquad (\nu = 1, 2)$$
$$\frac{\partial^\nu G_{2i}(\varphi, \gamma)}{\partial \varphi^\nu} = G_{2i}^{(\nu)}(\varphi, \gamma) \qquad (\nu = 1, 2, \ldots, 4;\ i = 1, 2)$$
are also continuous in φ and γ.
2. For any γ in $-\vartheta, \ldots, \vartheta$ the derivatives
$$G_{11}(\varphi, \gamma);\ \ G_{12}^{(1)}(\varphi, \gamma);\ \ G_{21}^{(\nu)}(\varphi, \gamma) \qquad (\nu = 1, 2, 3);\ \ G_{22}^{(\nu)}(\varphi, \gamma) \qquad (\nu = 1, 2)$$
are continuous at $\varphi = \gamma$, except $G_{11}^{(1)}(\varphi, \gamma)$ and $G_{22}^{(3)}(\varphi, \gamma)$ – these later two have a jump that is
$$\lim_{\varepsilon \to 0} \left[G_{11}^{(1)}(\varphi + \varepsilon, \varphi) - G_{11}^{(1)}(\varphi - \varepsilon, \varphi) \right] = \overset{1}{P}{}_{11}^{-1}(\varphi), \tag{15a}$$
$$\lim_{\varepsilon \to 0} \left[G_{22}^{(3)}(\varphi + \varepsilon, \varphi) - G_{22}^{(3)}(\varphi - \varepsilon, \varphi) \right] = \overset{4}{P}{}_{22}^{-1}(\varphi). \tag{15b}$$
3. Let $\boldsymbol{\alpha}$ be an arbitrary constant vector. Then, for any γ, $\mathbf{G}(\varphi, \gamma)\boldsymbol{\alpha}$ as a function of φ ($\varphi \neq \gamma$) satisfies the equation
$$\mathbf{K}\left[\mathbf{G}(\varphi, \gamma)\boldsymbol{\alpha}\right] = \mathbf{0}.$$
4. The vector $\mathbf{G}(\varphi, \gamma)\boldsymbol{\alpha}$ as function of φ should satisfy the boundary conditions as well:
$$\mathbf{U}_\mu\left[\mathbf{G}(\varphi, \gamma)\boldsymbol{\alpha}\right] = \mathbf{0}, \quad \mu = 1, \ldots, 4.$$

If the Green function matrix exists (an existence proof can be found in [1]) the column vector (14) satisfies differential equation (12) and the boundary conditions (13), i.e., integral (14) is really the solution of the boundary value problem (12), (13).

The general solution to the homogeneous part of equation (12) is given by the equation

$$\mathbf{y} = \left[\sum_{i=1}^{4} \underset{(2\times2)}{\mathbf{Y}}{}_i \underset{(2\times2)}{\mathbf{C}}{}_i\right] \underset{(2\times1)}{\mathbf{e}} \tag{16}$$

where $\mathbf{C}_i$ is a constant non-singular matrix, $\mathbf{e}$ is a constant column matrix and

$$\mathbf{Y}_1 = \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix}, \qquad \mathbf{Y}_2 = \begin{bmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \end{bmatrix}, \tag{17a}$$

$$\mathbf{Y}_3 = \begin{bmatrix} -\sin\varphi + \varphi\cos\varphi & (m+1)\varphi \\ \varphi\sin\varphi & -m \end{bmatrix}, \qquad \mathbf{Y}_4 = \begin{bmatrix} -\cos\varphi - \varphi\sin\varphi & 1 \\ \varphi\cos\varphi & 0 \end{bmatrix}. \tag{17b}$$

It follows from Property 3 that the Green function matrix has the following mathematical form:

$$\mathbf{G}(\varphi,\gamma) = \sum_{i=1}^{4} \mathbf{Y}_i(\varphi)\left[\mathbf{A}_i(\gamma) + \mathbf{B}_i(\gamma)\right] \qquad \varphi \le \gamma, \tag{18a}$$

$$\mathbf{G}(\varphi,\gamma) = \sum_{i=1}^{4} \mathbf{Y}_i(\varphi)\left[\mathbf{A}_i(\gamma) - \mathbf{B}_i(\gamma)\right] \qquad \varphi \ge \gamma. \tag{18b}$$

Here $\mathbf{A}_i(\gamma)$ and $\mathbf{B}_i(\gamma)$ are 2×2 matrices. We remark that the coefficients $\mathbf{B}_i(\gamma)$ can be determined by using Properties 1 and 2 of the definition while the coefficients $\mathbf{A}_i(\gamma)$ can be obtained by using Property 4 of the definition, i.e., the boundary conditions

$$\mathbf{U}_\mu\left[\sum_{i=1}^{4} \mathbf{Y}_i(\varphi)\mathbf{A}_i(\gamma)\boldsymbol{\alpha}\right] = \mp\mathbf{U}_\mu\left[\sum_{i=1}^{4} \mathbf{Y}_i(\varphi)\mathbf{B}_i(\gamma)\boldsymbol{\alpha}\right]. \tag{19}$$

Calculation of the Green function matrix and the results of the calculations for fixed-fixed, pinned-pinned and pinned-fixed beams are presented in Appendix A.1 and A.4.

Consider now the system of differential equations

$$\mathbf{K}[\mathbf{y}] = \lambda\mathbf{y} \tag{20}$$

where λ is a parameter (the unknown eigenvalue). Assume that differential equations (20) are associated with the homogeneous linear boundary conditions (13). Equations (20) and (13) constitute an eigenvalue problem with λ as the eigenvalue. Since the eigenvalue problem (20) and (13) is self-adjoint [1, 23] it follows that the Green function matrix is cross-symmetric [1]:

$$\mathbf{G}(\varphi,\gamma) = \mathbf{G}^T(\gamma,\varphi).$$

Recalling (14) the eigenvalue problem (20), (13) can be replaced by the homogeneous Fredholm integral equation

$$\mathbf{y}(\varphi) = \lambda\int_{-\vartheta}^{\vartheta} \mathbf{G}(\varphi,\gamma)\mathbf{y}(\gamma)\mathrm{d}\gamma\,. \tag{21}$$

Eigenvalue problem (21) can be solved numerically if we follow the solution procedure detailed in [23, 26].

4. Numerical results – free vibrations

We have developed a Fortran 90 program for solving numerically the algebraic eigenvalue problem derived from eigenvalue problem (21). Three support arrangements were considered: (a) fixed-fixed beams, (b) pinned-pinned beams and (c) pinned-fixed beams. The results are the same as those in thesis [22] obtained under the condition that the central force acting on the beam is equal to zero.

Table 1. Typical values of $C_{i,\mathrm{char}}$

	$i=1$	$i=2$	$i=3$	$i=4$
Fixed-fixed beams	2.266	6.243	12.23	20.25
Pinned-pinned beams	1	4	9	16
Pinned-fixed beams	1.556	5.078	10.541	17.97

Consider a straight beam with the same length ℓ as that of the curved beam we deal with. The eigenfrequencies $\alpha_{i\ \mathrm{str.}}$ $(i = 1, 2, ...)$ of the straight beam are well-known – see for instance [1, 22] – and are given by

$$\alpha_{i\ \mathrm{str.}} = \frac{C_{i,\,\mathrm{char}}\pi^2}{\sqrt{\frac{\rho_a A}{I_{e\eta}}}\ell^2}\,. \tag{22}$$

The constant $C_{i,\,\mathrm{char}}$ depends on the ordinal number of the frequency – see Table 1 — and $\ell = 2R\vartheta$ is the length of the beam. Recalling now equation (11), as detailed in [23], we get

$$C_{i,\mathrm{char}}\frac{\alpha_i}{\alpha_{i\ \mathrm{str.}}} = \frac{\frac{\sqrt{\lambda_i}}{\sqrt{\frac{\rho_a A}{I_{e\eta}}}R^2}}{\frac{\pi^2}{\sqrt{\frac{\rho_a A}{I_{e\eta}}}\ell^2}} = \frac{\bar{\vartheta}^{\,2}\sqrt{\lambda_i}}{\pi^2}. \tag{23}$$

This quotient is plotted in the next three diagrams for the above mentioned three support conditions. The eigenfrequencies of the curved beams we have considered are therefore compared to the first eigenfrequency of straight beams with the same length and same material composition. It is worth emphasizing that the material composition is incorporated into the model via the parameter m.

4.1. **Fixed-fixed supports.** The quotient (23) is plotted in Figure 2 against the central angle $\bar{\vartheta}$ of the beam. The picked values of m are 750, 1 000, 1 300, 1 750, 2 400, 3 400, 5 000, 7 500, 12 000, 20 000, 35 000, 60 000, 100 000 and 200 000. The outcomes are identical to those of [1] valid for homogeneous beams. Thus, it turns out that the ratio of the even frequencies do not depend on m. Another important property is that a frequency shift can be observed: in terms of magnitude, the first/third frequency becomes the second/fourth one if the central angle is sufficiently great.

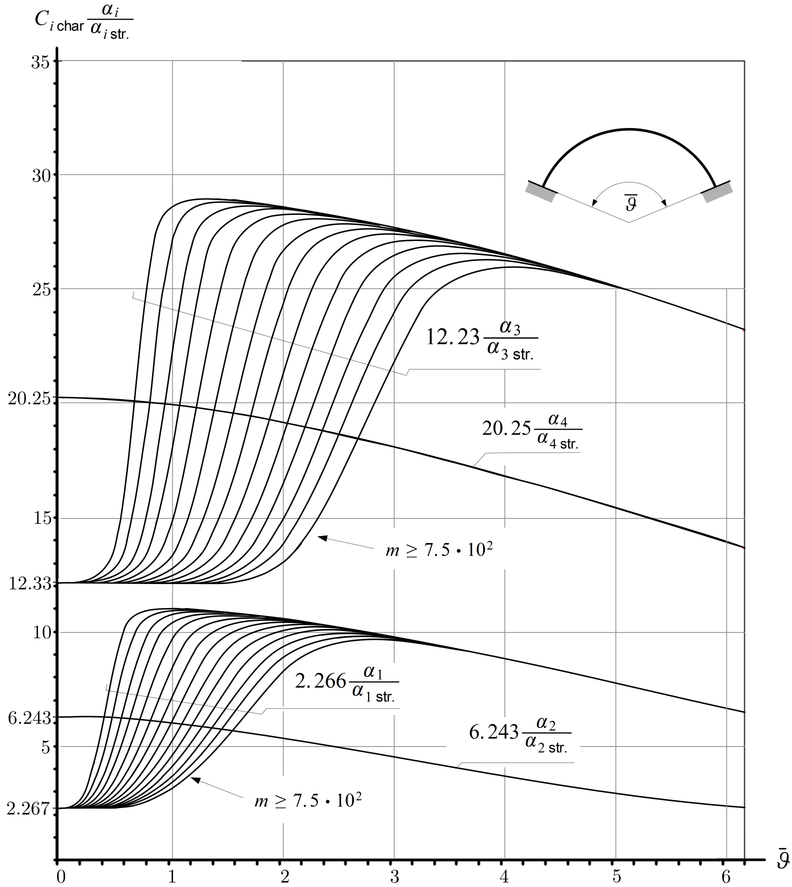

Figure 2. Eigenfrequencies for fixed-fixed beams [27]

Table 2. FE verifications, fixed-fixed beams, $m = 1\,200$, $R/b = 10$

ϑ	$\dfrac{\alpha_1 \text{ New model}}{\alpha_1 \text{ Abaqus}}$	$\dfrac{\alpha_2 \text{ New model}}{\alpha_2 \text{ Abaqus}}$	$\dfrac{\alpha_3 \text{ New model}}{\alpha_3 \text{ Abaqus}}$	$\dfrac{\alpha_4 \text{ New model}}{\alpha_4 \text{ Abaqus}}$
0.5	1.019	1.115	1.193	1.314
1	1.031	1.037	1.021	1.075
1.5	1.014	1.025	1.039	1.037
2	1.008	1.015	1.022	1.032
2.5	0.971	1.010	1.015	1.022

Some finite element computations were carried out for verification reasons using the commercial software Abaqus. In Abaqus 6.7 we used the Linear Perturbation, Frequency Step. The model consisted of $B22$ (3-node Timoshenko beam) elements. Further, we chose $E = 2 \cdot 10^{11}$ Pa and $\rho_a = 7800$ kg/m^3. R/b is the centerline radius/cross-sectional height ratio. The frequency ratios of the new model ($\alpha_{i\,\text{New model}}$) and Abaqus ($\alpha_{i\,\text{Abaqus}}$) are gathered in Tables 2 and 3. Generally, there is a very good agreement.

Table 3. FE verifications, fixed-fixed beams, $m = 10\,800$, $R/b = 30$

ϑ	$\frac{\alpha_{1\,\text{New model}}}{\alpha_{1\,\text{Abaqus}}}$	$\frac{\alpha_{2\,\text{New model}}}{\alpha_{2\,\text{Abaqus}}}$	$\frac{\alpha_{3\,\text{New model}}}{\alpha_{3\,\text{Abaqus}}}$	$\frac{\alpha_{4\,\text{New model}}}{\alpha_{4\,\text{Abaqus}}}$
0.5	1.014	1.007	1.018	1.039
1	1.004	1.006	1.010	1.014
1.5	1.002	1.003	1.006	1.009
2	1.001	1.002	1.003	1.005
2.5	1.000	1.001	1.002	1.004
3	1.000	1.001	1.002	1.004

Recalling the results gathered in Tables 1 and 4 in article [10], we can make some additional comparisons as shown in Tables 4 and 5. We assume a rectangular cross-section ($A = 0.01$ m^2; $I_\eta = 8.33\cdot 10^{-6}$ m^4) and that $E = 2\cdot 10^{11}$ Pa, $\rho_a = 7\,800$ kg/m^3. In Tables 4 and 5 Ref. [10] col. 1 and Ref. [28] consider axial extension and rotatory inertia effects, while in Ref. [10] col. 2, none of these is incorporated. Moreover, Ref. [10] col 5. is the most accurate model: axial extension, rotatory inertia and transverse shear effects are all assumed. In general, the agreement is quite good between the current and even with the most accurate model.

Table 4. Comparison of the eigenfrequencies, $2\vartheta = \pi/2$, fixed-fixed supports

m		Ref. [28]	Ref. [10] col. 1	Ref. [10] col. 2	Ref. [10] col. 5	New model
10 000	α_1	63.07	63.06	63.16	62.62	63.1
10 000	α_2	117.22	117.19	120.76	115.85	117.5
10 000	α_3	217.13	217.08	218.41	213.28	218.2
10 000	α_4	249.26	345.21	322.26	247.96	249.8
2 500	α_1	251	251	252.66	244.24	251.89
2 500	α_2	399.68	399.65	483.04	390.09	401.16
2 500	α_3	613.25	613.33	873.64	600.7	617.25
2 500	α_4	847.24	847.07	1289.06	795.82	859.02

Table 5. Comparison of the eigenfrequencies, $2\vartheta = \pi$, fixed-fixed supports

m		Ref. [28]	Ref. [10] col. 1	Ref. [10] col. 2	Ref. [10] col. 5	New model
10 000	α_1	12.23	12.23	12.24	12.21	12.24
10 000	α_2	26.89	26.89	26.95	26.80	26.92
10 000	α_3	49.93	49.93	50.03	49.70	50.07
10 000	α_4	76.43	76.44	76.84	75.95	76.85
2 500	α_1	48.87	48.86	48.96	48.51	48.9
2 500	α_2	106.85	106.85	107.78	105.53	107.1
2 500	α_3	198.57	198.51	200.13	194.94	199.5
2 500	α_4	299.61	299.59	307.37	292.46	302.13

4.2. **Pinned-pinned supports.** In Figure 3 the ratio (23) is plotted against the central angle $\bar{\vartheta}$ of the circular beam. The curves run similarly as for fixed-fixed beams and the character of the curves plotted are the same. The quotients are generally smaller for the same parameters meaning that the pinned supports are softer.

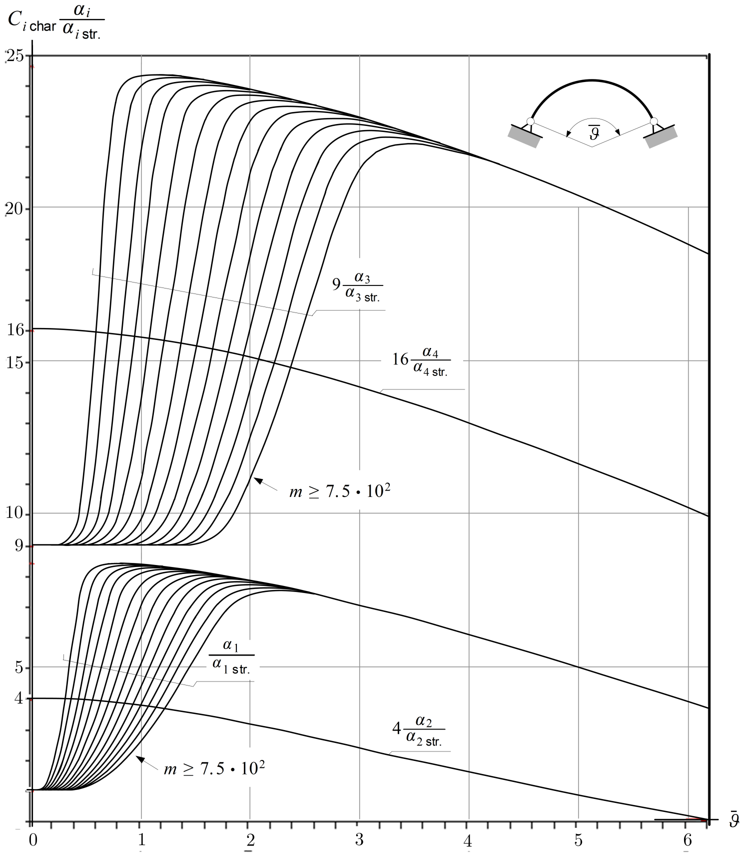

Figure 3. Eigenfrequencies for pinned-pinned beams [26]

Table 6. FE verifications, $R/b = 10$; $m = 1\ 200$

ϑ	$\dfrac{\alpha_1 \text{ New model}}{\alpha_1 \text{ Abaqus}}$	$\dfrac{\alpha_2 \text{ New model}}{\alpha_2 \text{ Abaqus}}$	$\dfrac{\alpha_3 \text{ New model}}{\alpha_3 \text{ Abaqus}}$	$\dfrac{\alpha_4 \text{ New model}}{\alpha_4 \text{ Abaqus}}$
0.5	1.001	1.053	1.109	1.179
1	1.014	1.029	1.004	1.053
1.5	1.007	1.014	1.028	1.006
2	1.004	1.008	1.014	1.022
2.5	1.003	1.005	1.010	1.015

When comparing these numerical results to the Abaqus computations (the settings were the same as mentioned in relation with fixed-fixed beam) once again, we find a really good agreement. See Tables 6 and 7 for the computational results.

Table 7. FE verifications, $R/b = 30$, $m = 10\ 800$

ϑ	$\frac{\alpha_1 \text{ New model}}{\alpha_1 \text{ Abaqus}}$	$\frac{\alpha_2 \text{ New model}}{\alpha_2 \text{ Abaqus}}$	$\frac{\alpha_3 \text{ New model}}{\alpha_3 \text{ Abaqus}}$	$\frac{\alpha_4 \text{ New model}}{\alpha_4 \text{ Abaqus}}$
0.5	1.006	1.010	1.005	1.025
1	1.002	1.004	1.007	1.011
1.5	1.001	1.002	1.003	1.006
2	1.000	1.001	1.002	1.003
2.5	1.000	1.001	1.002	1.003
3	1.001	1.001	1.001	1.002

Some further comparisons with Tables 5 and 8 in [10] are provided hereinafter. The data are the same as for fixed-fixed members. The results are presented in Tables 8 and 9.

Table 8. Comparison of the eigenfrequencies, $2\vartheta = \pi/2$, pinned-pinned supports

m		Ref. [28]	Ref. [10] col. 1	Ref. [10] col. 2	Ref. [10] col. 5	New model
10 000	α_1	38.38	38.38	38.42	38.28	38.41
10 000	α_2	89.57	89.56	90.46	89.08	89.77
10 000	α_3	171.42	171.41	172.17	169.75	172.18
10 000	α_4	244.96	244.94	269.26	243.05	245.82
2 500	α_1	152.93	152.93	153.7	151.45	153.48
2 500	α_2	343.01	342.76	361.85	336.46	345.31
2 500	α_3	552.15	552.17	688.7	549.84	552.28
2 500	α_4	675.71	675.83	1077.01	651.82	685.38

Table 9. Comparison of the eigenfrequencies, $2\vartheta = \pi$, pinned-pinned supports

m		Ref. [28]	Ref. [10] col. 1	Ref. [10] col. 2	Ref. [10] col. 5	New model
10 000	α_1	6.33	6.33	6.33	6.32	6.33
10 000	α_2	19.31	19.31	19.33	19.28	19.32
10 000	α_3	38.98	38.97	39.02	38.87	39.05
10 000	α_4	63.53	63.53	63.71	63.29	63.79
2 500	α_1	25.28	25.28	25.31	25.21	25.3
2 500	α_2	77.01	76.99	77.31	76.57	77.18
2 500	α_3	155.24	155.25	156.09	153.75	155.96
2 500	α_4	251.86	251.82	254.83	248.12	253.81

4.3. **Pinned-fixed supports.** The curves are similar to the two previous cases and the frequencies are always between the typical values valid for pinned-pinned and fixed-fixed members. Abaqus computations [29] verified the validity of these numerical results just for the two previous support arrangements.

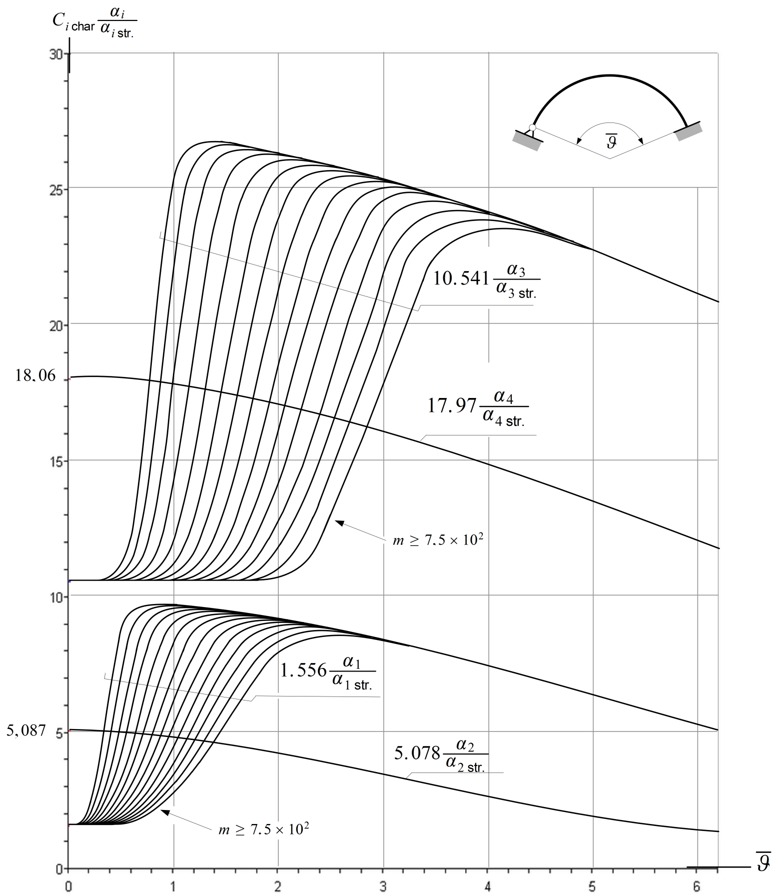

Figure 4. Eigenfrequencies for pinned-fixed beams [23]

5. Conclusions

We list our conclusions below:

1. We have investigated the free vibrations of circular beams with cross-sectional heterogeneity. For the three support arrangements, i.e., for fixed-fixed, pinned-pinned and pinned-fixed curved beams we have determined the Green function matrices in closed form.

2. With the knowledge of the Green function matrices we have reduced the self-adjoint eigenvalue problems, the solution of which results in the natural frequencies sought, to eigenvalue problems governed by homogeneous Fredholm integral equations. These integral equations were solved numerically.
3. It has turned out that, for any support arrangement, the even natural frequencies are independent of the heterogeneity-geometry parameter m while the odd ones do depend on it for smaller central angles.
4. The numerical results were verified by commercial finite element calculations and by comparing them to other models from the literature. A good agreement is found.
5. Let $\mathbf{r}$ be a dimensionless distributed load of the beams. With the knowledge of the Green function matrices the corresponding equilibrium problems can be solved in a closed form which is given by equation (14).

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Appendix A. Elements of the Green function matrix

The definition of the Green function matrix in Section 3 was published in thesis [1]. It is worth mentioning that Lin [30] introduced the concept of a generalized Green function for a class of ordinary differential equations for finding particular solutions to nonhomogeneous boundary value problems (equilibrium problems). In contrast to this work, integral (14) provides the complete solution to the boundary value problem considered if we know the corresponding Green function matrix. In the sequel we detail its calculation.

Recalling (18) and (17) we can give the Green function matrix in the following form:

$$\begin{aligned}\underbrace{\mathbf{G}(\varphi,\gamma)}_{(2\times 2)} = &\begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix} \left\{ \begin{bmatrix} \overset{1}{A}_{11} & \overset{1}{A}_{12} \\ 0 & 0 \end{bmatrix} \pm \begin{bmatrix} \overset{1}{B}_{11} & \overset{1}{B}_{12} \\ 0 & 0 \end{bmatrix} \right\} + \\ &\begin{bmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \end{bmatrix} \left\{ \begin{bmatrix} \overset{2}{A}_{11} & \overset{2}{A}_{12} \\ 0 & 0 \end{bmatrix} \pm \begin{bmatrix} \overset{2}{B}_{11} & \overset{2}{B}_{12} \\ 0 & 0 \end{bmatrix} \right\} + \\ &\begin{bmatrix} -\sin\varphi+\varphi\cos\varphi & (m+1)\varphi \\ \varphi\sin\varphi & -m \end{bmatrix} \left\{ \begin{bmatrix} \overset{3}{A}_{11} & \overset{3}{A}_{12} \\ \overset{2}{A}_{21} & \overset{2}{A}_{22} \end{bmatrix} \pm \begin{bmatrix} \overset{3}{B}_{11} & \overset{3}{B}_{12} \\ \overset{3}{B}_{21} & \overset{3}{B}_{22} \end{bmatrix} \right\} + \\ &\begin{bmatrix} -\cos\varphi-\varphi\sin\varphi & 1 \\ \varphi\cos\varphi & 0 \end{bmatrix} \left\{ \begin{bmatrix} \overset{3}{A}_{11} & \overset{3}{A}_{12} \\ \overset{3}{A}_{21} & \overset{3}{A}_{22} \end{bmatrix} \pm \begin{bmatrix} \overset{3}{B}_{11} & \overset{3}{B}_{12} \\ \overset{3}{B}_{21} & \overset{3}{B}_{22} \end{bmatrix} \right\}.\end{aligned}$$

The sign is [positive](negative) if $[\varphi \leq \psi]\,(\varphi \geq \psi)$.

A.1. **Solutions for the matrices $\mathbf{B}_i$.** Fulfillment of Properties 1 and 2 yields the unknown elements of the matrices $\mathbf{B}_i$. The discontinuities are taken from a comparison of (12) and (15a). The equation system to be solved is obviously independent of the boundary conditions:

$$\begin{bmatrix} \cos\gamma & -\sin\gamma & -\sin\gamma+\gamma\cos\gamma & (1+m)\gamma & -\cos\gamma-\gamma\sin\gamma & 1 \\ \sin\gamma & \cos\gamma & \gamma\sin\gamma & -m & \gamma\cos\gamma & 0 \\ -\sin\gamma & -\cos\gamma & -\gamma\sin\gamma & 1+m & -\gamma\cos\gamma & 0 \\ \cos\gamma & -\sin\gamma & \gamma\cos\gamma+\sin\gamma & 0 & -\gamma\sin\gamma+\cos\gamma & 0 \\ -\sin\gamma & -\cos\gamma & -\gamma\sin\gamma+2\cos\gamma & 0 & -\gamma\cos\gamma-2\sin\gamma & 0 \\ -\cos\gamma & \sin\gamma & -\gamma\cos\gamma-3\sin\gamma & 0 & \gamma\sin\gamma-3\cos\gamma & 0 \end{bmatrix}$$

$$\begin{bmatrix} \overset{1}{B}_{11} & \overset{1}{B}_{12} \\ \overset{2}{B}_{11} & \overset{2}{B}_{12} \\ \overset{3}{B}_{11} & \overset{3}{B}_{12} \\ \overset{3}{B}_{21} & \overset{3}{B}_{22} \\ \overset{4}{B}_{11} & \overset{4}{B}_{12} \\ \overset{4}{B}_{21} & \overset{4}{B}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{2m} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\dfrac{1}{2} \end{bmatrix}. \tag{24}$$

The solutions are given by the following equations:

$$\begin{aligned} \overset{1}{B}_{11} &= \frac{1}{2}\sin\gamma - \frac{1}{4}\gamma\cos\gamma \\ \overset{2}{B}_{11} &= \frac{1}{4}\gamma\sin\gamma + \frac{1}{2}\cos\gamma \\ \overset{3}{B}_{11} &= \frac{1}{4}\cos\gamma \\ \overset{3}{B}_{21} &= \frac{1}{2m} \\ \overset{4}{B}_{11} &= -\frac{1}{4}\sin\gamma \\ \overset{4}{B}_{21} &= -\frac{1}{2}(1+m)\frac{\gamma}{m} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overset{1}{B}_{12} &= -\frac{1}{4}\cos\gamma - \frac{1}{4}\gamma\sin\gamma \\ \overset{2}{B}_{12} &= \frac{1}{4}\sin\gamma - \frac{1}{4}\gamma\cos\gamma \\ \overset{3}{B}_{12} &= \frac{1}{4}\sin\gamma \\ \overset{3}{B}_{22} &= 0 \\ \overset{4}{B}_{12} &= \frac{1}{4}\cos\gamma \\ \overset{4}{B}_{21} &= \frac{1}{2}. \end{aligned} \tag{25}$$

In what follows, let us introduce simplified notations as shown

$$a = \overset{1}{B}_{1i}; \quad b = \overset{2}{B}_{1i}; \quad c = \overset{3}{B}_{1i}; \quad d = \overset{3}{B}_{2i}; \quad e = \overset{4}{B}_{1i}; \quad f = \overset{4}{B}_{2i}.$$

A.2. **The matrices $\mathbf{A}_i$ – fixed-fixed supports.** The boundary conditions are of the form

$$\hat{U}\Big|_{\pm\vartheta} = \hat{W}\Big|_{\pm\vartheta} = \hat{W}^{(1)}\Big|_{\pm\vartheta} = 0$$

thus, Property 3 yields the equations

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \sin\vartheta-\vartheta\cos\vartheta & -(m+1)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & -\sin\vartheta+\vartheta\cos\vartheta & (m+1)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c\,(\sin\vartheta-\vartheta\cos\vartheta) + d(m+1)\vartheta + e\,(\cos\vartheta+\vartheta\sin\vartheta) - f \\ a\cos\vartheta - b\sin\vartheta + c\,(-\sin\vartheta+\vartheta\cos\vartheta) + d(m+1)\vartheta + e\,(-\cos\vartheta-\vartheta\sin\vartheta) + f \end{bmatrix},$$

$$\begin{bmatrix} -\sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & -\vartheta\cos\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & \vartheta\cos\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} a\sin\vartheta - b\cos\vartheta - c\vartheta\sin\vartheta + dm + e\vartheta\cos\vartheta \\ a\sin\vartheta + b\cos\vartheta + c\vartheta\sin\vartheta - dm + e\vartheta\cos\vartheta \end{bmatrix},$$

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & -\sin\vartheta-\vartheta\cos\vartheta & 0 & \cos\vartheta-\vartheta\sin\vartheta & 0 \\ \cos\vartheta & -\sin\vartheta & \sin\vartheta+\vartheta\cos\vartheta & 0 & \cos\vartheta-\vartheta\sin\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta + c\,(\sin\vartheta+\vartheta\cos\vartheta) - e\,(\cos\vartheta-\vartheta\sin\vartheta) \\ a\cos\vartheta - b\sin\vartheta + c\,(\sin\vartheta+\vartheta\cos\vartheta) + e\,(\cos\vartheta-\vartheta\sin\vartheta) \end{bmatrix}.$$

Hence, the equation system to be solved is

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \sin\vartheta-\vartheta\cos\vartheta & -(m+1)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & -\sin\vartheta+\vartheta\cos\vartheta & (m+1)\vartheta & -\cos\vartheta-\vartheta\sin\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & -\vartheta\cos\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & \vartheta\cos\vartheta & 0 \\ \cos\vartheta & \sin\vartheta & -\sin\vartheta-\vartheta\cos\vartheta & 0 & \cos\vartheta-\vartheta\sin\vartheta & 0 \\ \cos\vartheta & -\sin\vartheta & \sin\vartheta+\vartheta\cos\vartheta & 0 & \cos\vartheta-\vartheta\sin\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta-b\sin\vartheta-c\left(\sin\vartheta-\vartheta\cos\vartheta\right)+d(m+1)\vartheta+e\left(\cos\vartheta+\vartheta\sin\vartheta\right)-f \\ a\cos\vartheta-b\sin\vartheta+c\left(-\sin\vartheta+\vartheta\cos\vartheta\right)+d(m+1)\vartheta-e\left(\cos\vartheta+\vartheta\sin\vartheta\right)+f \\ a\sin\vartheta-b\cos\vartheta-c\vartheta\sin\vartheta+dm+e\vartheta\cos\vartheta \\ a\sin\vartheta+b\cos\vartheta+c\vartheta\sin\vartheta-dm+e\vartheta\cos\vartheta \\ -a\cos\vartheta-b\sin\vartheta+c\left(\sin\vartheta+\vartheta\cos\vartheta\right)-e\left(\cos\vartheta-\vartheta\sin\vartheta\right) \\ a\cos\vartheta-b\sin\vartheta+c\left(\sin\vartheta+\vartheta\cos\vartheta\right)+e\left(\cos\vartheta-\vartheta\sin\vartheta\right) \end{bmatrix}.$$

By introducing the notations

$$D_1=\vartheta\cos^2\vartheta-\sin\vartheta\cos\vartheta+\vartheta\sin^2\vartheta=\vartheta-\sin\vartheta\cos\vartheta \tag{26a}$$

and

$$D_2=m\sin\vartheta\left(\vartheta\cos\vartheta-2\sin\vartheta\right)+(1+m)\,\vartheta^2+\vartheta\cos\vartheta\sin\vartheta \tag{26b}$$

the solutions are as follows:

$$\overset{1}{A}_{1i}=\frac{1}{D_1}\left[-b\cos^2\vartheta+c\vartheta^2+dm\left(\cos\vartheta-\vartheta\sin\vartheta\right)\right], \tag{27a}$$

$$\begin{aligned}\overset{2}{A}_{1i}=\frac{1}{D_2}\Big[&a\,(1+m)\,\vartheta\sin^2\vartheta+2am\sin\vartheta\cos\vartheta-2em\vartheta+\\ &+e\,(1+m)\,\vartheta^3+fm\left(\vartheta\cos\vartheta+\sin\vartheta\right)\Big],\end{aligned} \tag{27b}$$

$$\overset{3}{A}_{1i}=\frac{1}{D_2}\left[a\,(1+m)\,\vartheta+e\,(1+m)\,\vartheta\cos^2\vartheta-2em\sin\vartheta\cos\vartheta+fm\sin\vartheta\right], \tag{27c}$$

$$\overset{3}{A}_{2i}=\frac{1}{D_2}\left[2a\sin\vartheta-2e\vartheta\cos\vartheta+f\left(\vartheta+\sin\vartheta\cos\vartheta\right)\right], \tag{27d}$$

$$\overset{4}{A}_{1i}=\frac{1}{D_1}\left(b-c\sin^2\vartheta-dm\cos\vartheta\right), \tag{27e}$$

$$\overset{4}{A}_{2i}=\frac{1}{D_1}\left[2b\cos\vartheta-2c\vartheta\sin\vartheta+d\,(1+m)\,\vartheta^2-d\,(1+m)\,\vartheta\sin\vartheta\cos\vartheta-2dm\cos^2\vartheta\right]. \tag{27f}$$

A.3. **The matrices $\mathbf{A}_i$ – pinned-pinned supports.** The boundary conditions are

$$\hat{U}\Big|_{\pm\vartheta} = \hat{W}\Big|_{\pm\vartheta} = \hat{W}^{(2)}\Big|_{\pm\vartheta} = 0.$$

Since only the last two boundary conditions are different in contrast to the case of the fixed-fixed beams, the equation system to be solved is

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \sin\vartheta - \vartheta\cos\vartheta & -(m+1)\vartheta & -\cos\vartheta - \vartheta\sin\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & -\sin\vartheta + \vartheta\cos\vartheta & (m+1)\vartheta & -\cos\vartheta - \vartheta\sin\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & -\vartheta\cos\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & \vartheta\cos\vartheta & 0 \\ \sin\vartheta & -\cos\vartheta & 2\cos\vartheta - \vartheta\sin\vartheta & 0 & 2\sin\vartheta + \vartheta\cos\vartheta & 0 \\ -\sin\vartheta & -\cos\vartheta & 2\cos\vartheta - \vartheta\sin\vartheta & 0 & -2\sin\vartheta - \vartheta\cos\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix}$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c\left(\sin\vartheta - \vartheta\cos\vartheta\right) + d(m+1)\vartheta + e\left(\cos\vartheta + \vartheta\sin\vartheta\right) - f \\ a\cos\vartheta - b\sin\vartheta + c\left(-\sin\vartheta + \vartheta\cos\vartheta\right) + d(m+1)\vartheta - e\left(\cos\vartheta + \vartheta\sin\vartheta\right) + f \\ a\sin\vartheta - b\cos\vartheta - c\vartheta\sin\vartheta + dm + e\vartheta\cos\vartheta \\ a\sin\vartheta + b\cos\vartheta + c\vartheta\sin\vartheta - dm + e\vartheta\cos\vartheta \\ -a\sin\vartheta + b\cos\vartheta - c\left(2\cos\vartheta - \vartheta\sin\vartheta\right) - e\left(2\sin\vartheta + \vartheta\cos\vartheta\right) \\ -a\sin\vartheta - b\cos\vartheta + c\left(2\cos\vartheta - \vartheta\sin\vartheta\right) - e\left(2\sin\vartheta + \vartheta\cos\vartheta\right) \end{bmatrix}.$$

If we define D_1 and D_2 by the equations

$$D_1 = \sin^2\vartheta \tag{28a}$$

and

$$D_2 = m\vartheta + 2\left(1+m\right)\vartheta\cos^2\vartheta - 3m\sin\vartheta\cos\vartheta \tag{28b}$$

the solutions are

$$\overset{1}{A}_{1i} = \frac{1}{2D_1}\left[2b\sin\vartheta\cos\vartheta + 2c\vartheta - dm\left(2\sin\vartheta + \vartheta\cos\vartheta\right)\right], \tag{29a}$$

$$\overset{2}{A}_{1i} = \frac{1}{D_2}\Big[a\left(2(1+m)\vartheta\sin\vartheta\cos\vartheta - m\sin^2\vartheta + 2m\cos^2\vartheta\right) + \\ + e\left(3m\vartheta^2 + 2\vartheta^2 - 2m\right) - fm\left(\vartheta\sin\vartheta - 2\cos\vartheta\right)\Big], \tag{29b}$$

$$\overset{3}{A}_{1i} = \frac{1}{D_2}\left(am - e\left(m\cos^2\vartheta - 2m\sin^2\vartheta + 2(1+m)\vartheta\sin\vartheta\cos\vartheta\right) + fm\cos\vartheta\right), \tag{29c}$$

$$\overset{3}{A}_{2i} = \frac{2}{D_2}\left[a\cos\vartheta + e\left(\vartheta\sin\vartheta - \cos\vartheta\right) + f\cos^2\vartheta\right], \tag{29d}$$

$$\overset{4}{A}_{1i} = \frac{1}{2D_1}\left(-2c\sin\vartheta\cos\vartheta + dm\sin\vartheta\right), \tag{29e}$$

$$\overset{4}{A}_{2i} = \frac{1}{2D_1}\Big[-2b\sin\vartheta - 2c\left(\sin\vartheta + \vartheta\cos\vartheta\right) + \\ + d\left(m\vartheta\cos^2\vartheta + 3m\sin\vartheta\left(\cos\vartheta + \vartheta\sin\vartheta\right) + 2\vartheta\sin^2\vartheta\right)\Big]. \tag{29f}$$

A.4. **The matrices $\mathbf{A}_i$ – pinned-fixed supports.** Finally, for the third support arrangements

$$\hat{U}\Big|_{\pm\vartheta} = \hat{W}\Big|_{\pm\vartheta} = \hat{W}^{(1)}\Big|_{\vartheta} = \hat{W}^{(2)}\Big|_{-\vartheta} = 0$$

are the boundary conditions. Hence

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \sin\vartheta - \vartheta\cos\vartheta & -(m+1)\vartheta & -\cos\vartheta - \vartheta\sin\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & -\sin\vartheta + \vartheta\cos\vartheta & (m+1)\vartheta & -\cos\vartheta - \vartheta\sin\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & -\vartheta\cos\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \vartheta\sin\vartheta & -m & \vartheta\cos\vartheta & 0 \\ \sin\vartheta & -\cos\vartheta & 2\cos\vartheta - \vartheta\sin\vartheta & 0 & 2\sin\vartheta + \vartheta\cos\vartheta & 0 \\ \cos\vartheta & -\sin\vartheta & \sin\vartheta + \vartheta\cos\vartheta & 0 & \cos\vartheta - \vartheta\sin\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c(\sin\vartheta - \vartheta\cos\vartheta) + d(m+1)\vartheta + e(\cos\vartheta + \vartheta\sin\vartheta) - f \\ a\cos\vartheta - b\sin\vartheta + c(-\sin\vartheta + \vartheta\cos\vartheta) + d(m+1)\vartheta - e(\cos\vartheta + \vartheta\sin\vartheta) + f \\ a\sin\vartheta - b\cos\vartheta - c\vartheta\sin\vartheta + dm + e\vartheta\cos\vartheta \\ a\sin\vartheta + b\cos\vartheta + c\vartheta\sin\vartheta - dm + e\vartheta\cos\vartheta \\ -a\sin\vartheta + b\cos\vartheta - c(2\cos\vartheta - \vartheta\sin\vartheta) - e(2\sin\vartheta + \vartheta\cos\vartheta) \\ a\cos\vartheta - b\sin\vartheta + c(\sin\vartheta + \vartheta\cos\vartheta) + e(\cos\vartheta - \vartheta\sin\vartheta) \end{bmatrix}$$

is the equation system to be solved. With

$$D = -4m + 11m\cos^2\vartheta - 7m\cos^4\vartheta - 4m\vartheta\sin\vartheta\cos^3\vartheta - \\ - 2m\vartheta\sin\vartheta\cos\vartheta + 2\vartheta\cos\vartheta\sin\vartheta - 4\vartheta\cos^3\vartheta\sin\vartheta + 3m\vartheta^2 + 2\vartheta^2 \tag{30}$$

the solutions are as follows:

$$\overset{1}{A}_{1i} = -\frac{1}{D}\left\{a\left[-2\vartheta^2(m+1)\cos^2\vartheta + 2m\vartheta\cos\vartheta\sin\vartheta\right] + \right. \\ + b\left[-2\vartheta^2 m\sin\vartheta\cos\vartheta - 2\vartheta^2\sin\vartheta\cos\vartheta - m\vartheta\cos^2\vartheta + 4m\vartheta\cos^4\vartheta - \right. \\ \left. - 7m\sin\vartheta\cos^3\vartheta + 4m\sin\vartheta\cos\vartheta - 2\vartheta\cos^2\vartheta + 4\vartheta\cos^4\vartheta\right] + \\ c\left[-2\vartheta^3 - 2\vartheta^3(m+1)\cos^2\vartheta - 3\vartheta^3 m + 4m\vartheta\sin^2\vartheta + m\vartheta^2\sin\vartheta\cos\vartheta - 2\vartheta^2\sin\vartheta\cos\vartheta\right] \\ + d\left[m(m+1)\vartheta^3\cos\vartheta - 4m^2\vartheta\cos\vartheta + m^2\vartheta\cos^3\vartheta + 2\vartheta m\cos\vartheta - \right. \\ - 4\vartheta m\cos^3\vartheta - 4m^2\sin\vartheta + 7m^2\sin\vartheta\cos^2\vartheta + \\ \left. + 3m(m+1)\vartheta^2\sin\vartheta\cos^2\vartheta + 2m\vartheta^2\sin\vartheta + 3m^2\vartheta^2\sin\vartheta\right] + \\ + e\left[-2(m+1)\vartheta^3\sin\vartheta\cos\vartheta - 4\vartheta m\cos\vartheta(\vartheta\cos\vartheta - \sin\vartheta) - 2\vartheta^2\cos^2\vartheta\right] + \\ \left. + fm\vartheta\cos\vartheta\left[\vartheta - \sin\vartheta\cos\vartheta\right]\right\}, \tag{31a}$$

$$\overset{2}{A}_{1i} = -\frac{1}{D}\left\{a\left[-2\vartheta - m\vartheta + 6\vartheta\cos^2\vartheta - 4(m+1)\vartheta\cos^4\vartheta + 3m\vartheta\cos^2\vartheta - \right.\right. \\ \left. - 5m\cos\vartheta\sin\vartheta + 7m\sin\vartheta\cos^3\vartheta - 2\vartheta^2(m+1)\sin\vartheta\cos\vartheta\right] + \\ + b\left[2m\sin^2\vartheta - 2m\vartheta\sin\vartheta\cos\vartheta - 2(m+1)\vartheta^2\sin^2\vartheta\right] +$$

$$+ c \left[4m\vartheta \sin\vartheta \cos\vartheta + 2m \cos^2\vartheta - 2\vartheta^2 \cos^2\vartheta - 2mc - \right.$$
$$\left. 2\,(m+1)\,\vartheta^3 \sin\vartheta \cos\vartheta + 2m\vartheta^2 - 4m\vartheta^2 \cos^2\vartheta + 2\vartheta^2\right] +$$
$$+ d \left[m\vartheta^2 \cos\vartheta \sin^2\vartheta + 2m^2\vartheta^2 \cos\vartheta \sin^2\vartheta - m^2\vartheta \sin\vartheta \sin^2\vartheta - \right.$$
$$\left. - m^2 \cos\vartheta \sin^2\vartheta + m\,(m+1)\,\vartheta^3 \sin\vartheta\right] +$$
$$+ e \left[6m\vartheta - 5m\vartheta^3 + 2\vartheta^2 \sin\vartheta \cos\vartheta + 2m\vartheta^3 \cos^2\vartheta - 4\vartheta^3 - \right.$$
$$\left. - 2m \sin\vartheta \cos\vartheta + 3m\vartheta^2 \cos\vartheta \sin\vartheta - 4m\vartheta \cos^2\vartheta + 2\vartheta^3 \cos^2\vartheta\right] +$$
$$\left. + f \left[-2m \sin\vartheta + 4m \sin\vartheta \cos^2\vartheta - 5m\vartheta \cos\vartheta + 3m\vartheta \cos^3\vartheta + \vartheta^2 m \sin\vartheta\right]\right\}, \quad \text{(31b)}$$

$$\overset{3}{A}_{1i} = \frac{1}{D} \left\{a \left[2\vartheta - 2\vartheta\,(m+1) \cos^2\vartheta - m \cos\vartheta \sin\vartheta + 3m\vartheta\right] + \right.$$
$$+ b \left[2m \sin^2\vartheta - 2\,(m+1)\,\vartheta \sin\vartheta \cos\vartheta\right] +$$
$$+ c \left[4m\vartheta \sin\vartheta \cos\vartheta - 2m \sin^2\vartheta - 2\,(m+1)\,\vartheta^2 \cos^2\vartheta + 2\vartheta \sin\vartheta \cos\vartheta\right] +$$
$$+ d \left[m\,(m+1)\,\vartheta \sin\vartheta \cos^2\vartheta + m\,(m+1)\,\vartheta^2 \cos\vartheta - m^2\vartheta \sin\vartheta - m^2 \sin^2\vartheta \cos\vartheta\right] +$$
$$+ e \left[m\vartheta \cos^2\vartheta - 4m\vartheta \cos^4\vartheta - 2\,(m+1)\,\vartheta^2 \sin\vartheta \cos\vartheta - \right.$$
$$\left. - 6m \sin\vartheta \cos\vartheta + 7m \sin\vartheta \cos^3\vartheta + 2m\vartheta + 4\vartheta \cos^2\vartheta \sin^2\vartheta\right] +$$
$$\left. + f \left[2m \sin\vartheta - 3m \sin\vartheta \cos^2\vartheta + m\vartheta \cos\vartheta\right]\right\}, \quad \text{(31c)}$$

$$\overset{3}{A}_{2i} = \frac{1}{D} \left\{2a \left(-3 \sin\vartheta \cos^2\vartheta + 2 \sin\vartheta + \vartheta \cos\vartheta\right) + 2b \left(-\cos\vartheta + \cos^3\vartheta + \vartheta \sin\vartheta\right) + \right.$$
$$+ 2c \left(-\vartheta \sin\vartheta \cos^2\vartheta + \vartheta^2 \cos\vartheta - \vartheta \sin\vartheta - \cos^3\vartheta + \cos\vartheta\right) -$$
$$- dm \left(\vartheta^2 - \cos^2\vartheta + \cos^4\vartheta\right) + 2e \left(3\vartheta \cos^3\vartheta - 4\vartheta \cos\vartheta + \vartheta^2 \sin\vartheta + \sin\vartheta \cos^2\vartheta\right) +$$
$$\left. + 2f \left(-2 \sin\vartheta \cos^3\vartheta + \sin\vartheta \cos\vartheta + \vartheta\right)\right\}, \quad \text{(31d)}$$

$$\overset{4}{A}_{1i} = \frac{1}{D} \left\{-2a \left(-m \sin^2\vartheta + (1+m)\,\vartheta \sin\vartheta \cos\vartheta\right) + \right.$$
$$+ b \left(2m\vartheta \cos^2\vartheta - 3m \sin\vartheta \cos\vartheta + m\vartheta + 2\vartheta \cos^2\vartheta\right) -$$
$$- c \left[m\vartheta - 7m \sin^3\vartheta \cos\vartheta + 3m\vartheta \cos^2\vartheta - 4m\vartheta \cos^4\vartheta + \right.$$
$$\left. + 2\,(m+1)\,\vartheta^2 \sin\vartheta \cos\vartheta + 4\vartheta \cos^2\vartheta \sin^2\vartheta\right] +$$
$$+ dm \left(\vartheta \cos\vartheta - 2m \sin\vartheta + 5m \sin\vartheta \cos^2\vartheta + (m+1)\,\vartheta^2 \sin\vartheta - 3\,(m+1)\,\vartheta \cos^3\vartheta\right) +$$
$$+ 2e \left(-\vartheta^2 \sin^2\vartheta + 2m \sin^2\vartheta - m\vartheta^2 \sin^2\vartheta - \vartheta \sin\vartheta \cos\vartheta - 2m\vartheta \sin\vartheta \cos\vartheta\right) +$$
$$\left. + fm \left(\vartheta \sin\vartheta - \cos\vartheta + \cos^3\vartheta\right)\right\}, \quad \text{(31e)}$$

$$\overset{4}{A}_{2i} = \frac{1}{D} a \left[2m \sin^2 \cos\vartheta - 2\vartheta^2\,(m+1) \cos\vartheta - 2\vartheta^2 \cos\vartheta + \right.$$
$$\left. + 4m\vartheta^2 \cos^5\vartheta + 2m\vartheta \sin\vartheta - 2\,(m+1)\,\vartheta \sin\vartheta \cos^2\vartheta\right] +$$
$$+ b \left[-2\vartheta \cos\vartheta + 6b\vartheta \cos^3\vartheta - 10m \sin\vartheta \cos^2\vartheta - \right.$$

$$-2m\vartheta^2\sin\vartheta+6m\vartheta\cos^3\vartheta-2\vartheta^2\sin\vartheta+4m\sin\vartheta\big]+$$
$$+c\left[-2\left(m+1\right)\vartheta^3\cos\vartheta-2\vartheta\sin^2\vartheta\cos\vartheta-2\vartheta^2\sin\vartheta-4m\vartheta^2\sin\vartheta+4m\sin\vartheta-\right.$$
$$\left.-4m\sin\vartheta\cos^2\vartheta+8m\vartheta\cos\vartheta\sin^2\vartheta-6\left(m+1\right)\vartheta^2\sin\vartheta\cos^2\vartheta\right]+$$
$$+d\left[2\vartheta^3+4m\vartheta\left(m\vartheta^2-1\right)+6m\vartheta\left(\vartheta^2-m\right)+14m\left(m+1\right)\vartheta\sin^2\vartheta\cos^2\vartheta+\right.$$
$$+12m^2\sin\vartheta\cos^3\vartheta-4\vartheta^2\left(m+1\right)^2\sin\vartheta\cos^3\vartheta+$$
$$\left.+2\left(\vartheta^2m^2-3m^2+\vartheta^2+2m\vartheta^2\right)\cos\vartheta\sin\vartheta\right]+$$
$$+e\left[2\vartheta^3\sin\vartheta+2\left(m+1\right)\vartheta^2\cos^3\vartheta-4\vartheta^2\cos\vartheta+4m\cos\vartheta\sin^2\vartheta-\right.$$
$$\left.-4m\vartheta\sin\vartheta\cos^2\vartheta-2\vartheta\sin\vartheta\cos^2\vartheta+4m\vartheta\sin\vartheta-6m\vartheta^2\cos\vartheta-2\vartheta^3m\sin\vartheta\right]+$$
$$+fm\left(\vartheta^2-\cos^2\vartheta+\cos^4\vartheta\right). \quad (31f)$$

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