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A HALF CIRCULAR BEAM BENDING BY RADIAL LOADS

István Ecsedi and Attila Baksa
Institute of Applied Mechanics, University of Miskolc
H-3515 Miskolc-egyetemváros, Hungary
mechecs@uni-miskolc.hu, mechab@uni-miskolc.hu

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Abstract. Under the plane strain condition a mixed type boundary value problem of a curved beam with rectangular cross section is investigated. The mixed type boundary value problem describes a bending problem of the curved beam made of linearly elastic polar orthotopic material. A minimum strain energy property is proven for the considered bending problem. The solution is based on Castigliano's principle. One- and two-layered curved beams are analysed. The results obtained are compared with those computed by commercial FEM software (Abaqus).

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1. Introduction

Figure 1 shows the linearly elastic curved beam of rectangular cross section. The governing equations and boundary conditions are formulated in the cylindrical coordinate system $Or\varphi z$. The plane $z = 0$ is the symmetry plane of the curved beam for the geometrical and loading properties. The space occupied by the curved beam is $\overline{B} = B \cup \partial B$. The points of $\overline{B}$ are given by the prescriptions:

$$B = \left\{(r,\, \varphi,\, z)\,\middle|\, a < r < b,\, 0 < \varphi < \pi,\, -t < z < t\right\}, \qquad \partial B = \bigcup_{\mathrm{i}=1}^{6} \partial B_{\mathrm{i}},$$

$$\partial B_{\mathrm{i}} = \left\{(r,\, \varphi,\, z)\,\middle|\, a \le r \le b,\, \varphi = \varphi_i,\, -t \le z \le t,\, \mathrm{i} = 1, 2,\, \varphi_1 = 0,\, \varphi_2 = \pi\right\},$$

$$\partial B_{\mathrm{i}} = \left\{(r,\, \varphi,\, z)\,\middle|\, r = r_{\mathrm{i}},\, 0 \le \varphi \le \pi,\, -t \le z \le t,\, \mathrm{i} = 3, 4,\, r_3 = a,\, r_4 = b\right\},$$

$$\partial B_{\mathrm{i}} = \left\{(r, \varphi,\, z)\,\middle|\, a \le r \le b,\, 0 \le \varphi \le \pi,\, z = z_{\mathrm{i}},\, \mathrm{i} = 5, 6,\, z_5 = -t,\, z_6 = t\right\}.$$

Unit vectors of the cylindrical coordinate system $Or\varphi z$ are denoted by $\boldsymbol{e}_{\mathrm{r}}$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_{\mathrm{z}}$ (Figure 1).

Since the beam is in plane strain the displacement vector is of the form $\boldsymbol{u} = u(r, \varphi)\boldsymbol{e}_{\mathrm{r}} + v(r, \varphi)\boldsymbol{e}_\varphi$. It is assumed that the material of the curved beam obeys Hooke's law. Its inverse is given by the equations

$$\varepsilon_{\mathrm{r}} = \frac{\partial u}{\partial r} = S_{11}\sigma_{\mathrm{r}} + S_{12}\sigma_\varphi, \tag{1.1}$$

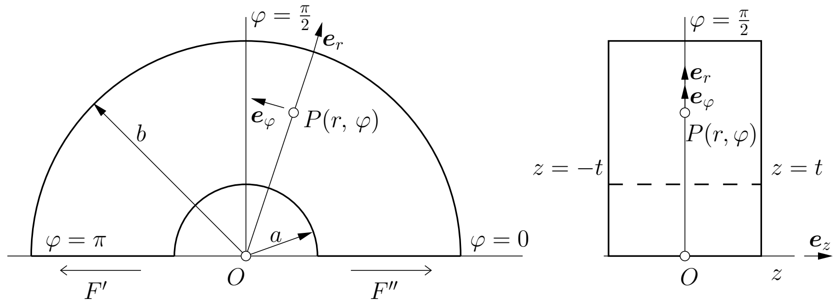

Figure 1. Bending of an orthotropic curved beam of rectangular cross section by radial loads

$$\varepsilon_\varphi = \frac{1}{r}\left(u + \frac{\partial v}{\partial \varphi}\right) = S_{12}\sigma_{\mathrm{r}} + S_{22}\sigma_\varphi, \tag{1.2}$$

$$\gamma_{\mathrm{r}\varphi} = \frac{1}{r}\left(\frac{\partial u}{\partial \varphi} - v\right) + \frac{\partial v}{\partial r} = S_{66}\tau_{\mathrm{r}\varphi}, \tag{1.3}$$

where ε_{r}, ε_φ, $\gamma_{\mathrm{r}\varphi}$ are the strains, σ_{r}, σ_φ, $\tau_{\mathrm{r}\varphi}$ are the stresses and S_{11}, S_{12}, S_{22} and S_{66} are material constants. S_{11}, S_{12}, S_{22} are called reduced flexibility coefficients. Their determination is based on the equations [1, 2]

$$S_{11} = s_{11} - \frac{s_{13}^2}{s_{33}}, \quad S_{12} = s_{12} - \frac{s_{13}s_{23}}{s_{33}}, \quad S_{22} = s_{22} - \frac{s_{23}^2}{s_{33}}$$

in which $s_{11}, \ldots, s_{33}$ are the stiffness components. We would like to emphasize that all quantities, i.e., the displacements, strains and stresses, which appear in equations (1.1), (1.2), (1.3) depend only on the polar coordinates r and φ.

We shall assume that there are no body forces. The considered bending problem is defined by the following boundary conditions (Figure 1)

$$u(r,\, 0) = 0, \quad \sigma_\varphi(r,\, 0) = 0, \quad a \le r \le b, \tag{1.4}$$

$$u(r,\, \pi) = \frac{\pi}{2}C, \quad \sigma_\varphi(r,\, \pi) = 0, \quad a \le r \le b, \tag{1.5}$$

$$\sigma_{\mathrm{r}}(a,\, \varphi) = \sigma_{\mathrm{r}}(b,\, \varphi) = \tau_{\mathrm{r}\varphi}(a,\, \varphi) = \tau_{\mathrm{r}\varphi}(b,\, \varphi) = 0, \quad 0 \le \varphi \le \pi. \tag{1.6}$$

In equation (1.5), C is a given constant ($C \neq 0$).

The stress resultants at the end cross sections $\varphi = 0$ and $\varphi = \pi$ should meet the following conditions:

$$F' = 2t\int_a^b \tau_{\mathrm{r}\varphi}(r,\, \pi)\mathrm{d}r, \qquad F'' = -2t\int_a^b \tau_{\mathrm{r}\varphi}(r,\, 0)\mathrm{d}r. \tag{1.7}$$

If the local equilibrium equations are all satisfied are then

$$F' = -F'' = F, \tag{1.8}$$

since there are no body forces and the surface segment $\partial B_3 \cup \partial B_4$ is stress free. It is also obvious that there exists a linear relationship between the stress resultant F and displacement constant C.

2. Minimum strain energy property

We consider a new boundary value problem of curved beams made of orthotopic linearly elastic material. The boundary conditions of the new problem are as follows:

$$\widetilde{u}(r,\,0) = 0, \quad \widetilde{\sigma}_\varphi(r,\,0) = \widetilde{\sigma}_\varphi(r,\,\pi) = 0, \qquad a \le r \le b, \tag{2.1}$$

$$F = 2t \int_a^b \widetilde{\tau}_{\mathrm{r}\varphi}(r,\,\pi)\mathrm{d}r, \tag{2.2}$$

$$\widetilde{\sigma}_\mathrm{r}(a,\,\varphi) = \widetilde{\sigma}_\mathrm{r}(b,\,\varphi) = \widetilde{\tau}_{\mathrm{r}\varphi}(a,\,\varphi) = \widetilde{\tau}_{\mathrm{r}\varphi}(b,\,\varphi) = 0, \qquad 0 \le \varphi \le \pi. \tag{2.3}$$

The radial displacement u at $\varphi = \pi$ is not specified but the stress resultant at the cross section $\varphi = \pi$ is fixed. This boundary value problem has many solutions, it is a relaxed version of the boundary value problem governed by equations (1.4), (1.5), (1.6),(1.7). One solution of the relaxed boundary value problem (2.1), (2.2), (2.3) is $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, where $\boldsymbol{u} = \boldsymbol{u}(r,\varphi)$ is the unique solution of the bending problem if the boundary conditions are given by equations (1.4), (1.5), (1.6) and (1.7).

Denote U the strain energy of the curved beam. The next theorem formulates a minimum strain energy property of the considered bending problem. Sternberg and Knowles [3] characterized the Saint-Venant extension bending, torsion and flexures problems in terms of certain associated minimum strain energy properties. Here, a similar characterization is formulated for the considered bending problem of the curved beam.

Theorem. For any F $(F \neq 0)$ it holds that

$$U(\boldsymbol{u}) \le U(\widetilde{\boldsymbol{u}}), \tag{2.4}$$

where $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}(r,\varphi)$ is an arbitrary solution of the plane strain boundary value problem determined by equations (2.1), (2.2) and (2.3).

Proof. From the definition of the strain energy [4] it follows that

$$U(\widetilde{\boldsymbol{u}}) = U(\boldsymbol{u}) + U(\widetilde{\boldsymbol{u}} - \boldsymbol{u},\,\boldsymbol{u}) + U(\widetilde{\boldsymbol{u}} - \boldsymbol{u}). \tag{2.5}$$

Here, $U(\widetilde{\boldsymbol{u}} - \boldsymbol{u},\,\boldsymbol{u})$ denotes the mixed strain energy defined on the equilibrium displacement fields $\hat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}} - \boldsymbol{u}$ and $\boldsymbol{u}$ (see [4]).

According to Betti's theorem [4] we have

$$\begin{aligned} U(\widetilde{\boldsymbol{u}} - \boldsymbol{u},\,\boldsymbol{u}) &= \frac{\pi}{2}\int_{\partial B_2} \left[\widetilde{\tau}_{\mathrm{r}\varphi}(r,\,\pi) - \tau_{\mathrm{r}\varphi}(r,\,\pi)\right]\, C\,\mathrm{d}r\,\mathrm{d}z = \\ &= \frac{\pi}{2}\left\{2t\int_a^b \widetilde{\tau}_{\mathrm{r}\varphi}(r,\,\pi)\,\mathrm{d}r - 2t\int_a^b \tau_{\mathrm{r}\varphi}(r,\,\pi)\,\mathrm{d}r\right\} C = \frac{\pi}{2}(F-F)C = 0. \end{aligned} \tag{2.6}$$

Combination of equation (2.5) with equation (2.6) yields

$$U(\widetilde{\boldsymbol{u}}) = U(\boldsymbol{u}) + U(\widetilde{\boldsymbol{u}} - \boldsymbol{u}). \tag{2.7}$$

Equation (2.7) is the proof of statement (2.4) since the strain energy is always non-negative [4]. Hence $U(\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \geq 0$.

3. Application of Castigliano's principle

The local equilibrium equations for our problem are given by

$$\frac{\partial \sigma_{\mathrm{r}}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\mathrm{r}\varphi}}{\partial \varphi} + \frac{\sigma_{\mathrm{r}} - \sigma_{\varphi}}{r} = 0, \qquad a < r < b, \quad 0 < \varphi < \pi, \tag{3.1}$$

$$\frac{\partial \tau_{\mathrm{r}\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi}}{\partial \varphi} + \frac{2\tau_{\mathrm{r}\varphi}}{r} = 0, \qquad a < r < b, \quad 0 < \varphi < \pi. \tag{3.2}$$

An equilibrated stress field can be obtained from formulae

$$\sigma_{\mathrm{r}} = \frac{V(r)}{r^2}\sin\varphi, \qquad \sigma_{\varphi} = \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\sin\varphi, \qquad \tau_{\mathrm{r}\varphi} = -\frac{V(r)}{r^2}\cos\varphi \tag{3.3}$$

in which $V = V(r)$ is a stress function. Note that the stress boundary conditions $(1.4)_2$, $(1.5)_2$ and the equilibrium equations (3.1), (3.2) are all satisfied. The stress boundary conditions given by (1.6) are also satisfied if

$$V(a) = V(b) = 0\,. \tag{3.4}$$

Then the stress field in terms of $V(r)$ is statically admissible. The total complementary energy of the curved beam can be written in the form [4, 5, 6]

$$\Pi_{\mathrm{c}}(V) = U(V) - W_{\mathrm{u}}, \tag{3.5}$$

where

$$U(V) = \frac{\pi t}{2}\int_a^b \left[S_{11}\left(\frac{V}{r^2}\right)^2 + 2S_{12}\frac{V}{r^3}\frac{\mathrm{d}V}{\mathrm{d}r} + S_{22}\frac{1}{r^2}\left(\frac{\mathrm{d}V}{\mathrm{d}r}\right)^2 + S_{66}\left(\frac{V}{r^2}\right)^2\right] r\,\mathrm{d}r, \tag{3.6}$$

$$W_{\mathrm{u}} = \int_{\partial B_2} u(r,\,\pi)\tau_{\mathrm{r}\varphi}(r,\,\pi)\,\mathrm{d}r\,\mathrm{d}z = C\pi t\int_a^b \frac{V}{r^2}\,\mathrm{d}r. \tag{3.7}$$

According to the well known Castigliano's principle [5, 6]

$$\delta\Pi_{\mathrm{c}} = 0 \tag{3.8}$$

where the stress function $V = V(r)$ is to be varied. We emphasize that the boundary condition (3.4) should also be satisfied.

A detailed computation leads to the following boundary value problem

$$-S_{22}r^2\frac{\mathrm{d}^2V}{\mathrm{d}r^2} + S_{22}r\frac{\mathrm{d}V}{\mathrm{d}r} + (S_{11} + 2S_{12} + S_{66})\,V = C\,r, \qquad a < r < b, \tag{3.9}$$

$$V(a) = 0, \qquad V(b) = 0. \tag{3.10}$$

The general solution of differential equation (3.9) is

$$V(r) = \alpha_1 r^{\lambda_1} + \alpha_2 r^{\lambda_2} + \frac{C}{S_{11} + 2S_{12} + S_{22} + S_{66}}\, r \tag{3.11}$$

where α_1 and α_2 are unknown integration constants and

$$\lambda_1 = 1 + \sqrt{\frac{S_{11} + 2S_{12} + S_{22} + S_{66}}{S_{22}}}, \tag{3.12}$$

$$\lambda_2 = 1 - \sqrt{\frac{S_{11} + 2S_{12} + S_{22} + S_{66}}{S_{22}}}. \tag{3.13}$$

Substitution of equation (3.11) into (3.10) yields

$$\alpha_1 = \frac{ab^{\lambda_2} - ba^{\lambda_2}}{\left(a^{\lambda_2} b^{\lambda_1} - a^{\lambda_1} b^{\lambda_2}\right)\left(S_{11} + 2S_{12} + S_{22} + S_{66}\right)}\, C\,, \tag{3.14}$$

$$\alpha_2 = \frac{a^{\lambda_1} b - ab^{\lambda_1}}{\left(a^{\lambda_2} b^{\lambda_1} - a^{\lambda_1} b^{\lambda_2}\right)\left(S_{11} + 2S_{12} + S_{22} + S_{66}\right)}\, C\,. \tag{3.15}$$

The connection between the displacement constant C and stress resultant F can be derived from the following equation:

$$F = 2t \int_a^b \tau_{\mathrm{r}\varphi}(r,\, \pi)\, \mathrm{d}r = 2t \int_a^b \frac{V}{r^2}\, \mathrm{d}r. \tag{3.16}$$

A detailed computation gives

$$\begin{aligned} F = \frac{2tC}{S_{11} + 2S_{12} + S_{22} + S_{66}} \Bigg\{ \ln \frac{b}{a} + \\ + \frac{1}{a^{\lambda_2} b^{\lambda_1} - a^{\lambda_1} b^{\lambda_2}} \Bigg[\frac{\left(ab^{\lambda_2} - a^{\lambda_2} b\right)\left(b^{\lambda_1 - 1} - a^{\lambda_1 - 1}\right)}{\lambda_1 - 1} + \\ + \frac{\left(a^{\lambda_1} b - ab^{\lambda_1}\right)\left(b^{\lambda_2 - 1} - a^{\lambda_2 - 1}\right)}{\lambda_2 - 1} \Bigg] \Bigg\}. \end{aligned} \tag{3.17}$$

Formulae for the stresses are as follows:

$$\sigma_{\mathrm{r}} = \left(\alpha_1 r^{\lambda_1 - 2} + \alpha_2 r^{\lambda_2 - 2} + \frac{C}{\left(S_{11} + 2S_{12} + S_{22} + S_{66}\right) r} \right) \sin\varphi, \tag{3.18}$$

$$\sigma_{\varphi} = \left(\alpha_1 \lambda_1 r^{\lambda_1 - 2} + \alpha_2 \lambda_2 r^{\lambda_2 - 2} + \frac{C}{\left(S_{11} + 2S_{12} + S_{22} + S_{66}\right) r} \right) \sin\varphi, \tag{3.19}$$

$$\tau_{\mathrm{r}\varphi} = -\left(\alpha_1 r^{\lambda_1 - 2} + \alpha_2 r^{\lambda_2 - 2} + \frac{C}{\left(S_{11} + 2S_{12} + S_{22} + S_{66}\right) r} \right) \cos\varphi. \tag{3.20}$$

If the beam is isotropic it holds that

$$S_{11} = S_{22} = \frac{1 - \nu^2}{E}, \qquad S_{12} = -\frac{\nu(1 + \nu)}{E}, \qquad S_{66} = \frac{2(1 + \nu)}{E}, \tag{3.21}$$

where E is the Young's modulus and ν is the Poisson number. A simple computation gives

$$S_{11} + 2S_{12} + S_{22} + S_{66} = \frac{4(1-\nu^2)}{E} \tag{3.22}$$

$$\lambda_1 = 3, \qquad \lambda_2 = -1. \tag{3.23}$$

Inserting equations (3.22) and (3.23) into expressions (3.18), (3.19) and (3.20) set up for the stresses, we obtain

$$\sigma_{\mathrm{r}} = \left(\alpha_1 r + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r}\right)\sin\varphi, \tag{3.24}$$

$$\sigma_{\varphi} = \left(3\alpha_1 r - \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r}\right)\sin\varphi, \tag{3.25}$$

$$\tau_{\mathrm{r}\varphi} = -\left(\alpha_1 r + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r}\right)\cos\varphi, \tag{3.26}$$

where

$$\alpha_3 = \frac{E}{1-\nu^2}\,C. \tag{3.27}$$

Equations (3.24), (3.25) and (3.26) are identical to those which were derived by Timoshenko and Goodier [7], and Lurje [6] for curved beams made of isotropic materials.

4. Two-layered curved beam

Figure 2 shows a two-layered curved beam made of two different linearly elastic orthotopic materials. The boundary conditions for this compound structure are given by equations (1.4), (1.5) and (1.6). The elastic constants for material i (i $= 1, 2$), which occupies the region B_i, are denoted by $S_{\mathrm{i}11}$, $S_{\mathrm{i}12}$, $S_{\mathrm{i}22}$ and $S_{\mathrm{i}66}$. The region B_i is uniquely determined by the following relations:

$$B_{\mathrm{i}} = \Big\{(r,\ \varphi,\ z)\,\Big|\, a_{\mathrm{i}} < r < b_{\mathrm{i}},\ 0 \le \varphi \le \pi,\ -t \le z \le t;\ \ \mathrm{i} = 1, 2;$$
$$a_1 = a,\ b_1 = c;\ \ a_2 = c,\ b_2 = b\Big\}.$$

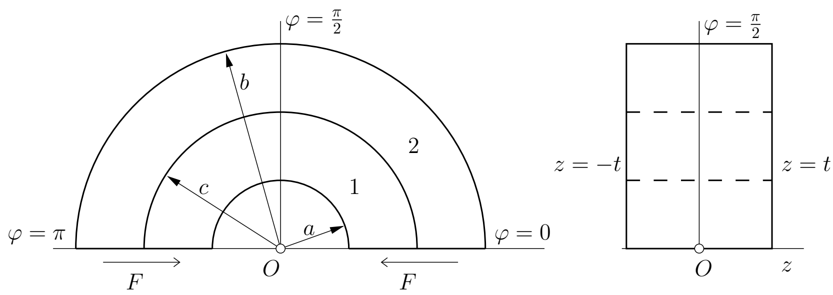

Figure 2. Two-layered curved beam of rectangular cross section.

The connection between the beam components on the common cylindrical surface $r = c$ is perfect, i.e. neither the displacements u, v nor the stresses σ_{r}, $\tau_{\mathrm{r}\varphi}$ have jumps if $r = c$. Consequently

$$u_1(c,\,\varphi) = u_2(c,\,\varphi), \quad v_1(c,\,\varphi) = v_2(c,\,\varphi), \qquad 0 \le \varphi \le \pi, \tag{4.1}$$

$$\sigma_{1\mathrm{r}}(c,\,\varphi) = \sigma_{2\mathrm{r}}(c,\,\varphi), \quad \tau_{1\mathrm{r}\varphi}(c,\,\varphi) = \tau_{2\mathrm{r}\varphi}(c,\,\varphi), \qquad 0 \le \varphi \le \pi. \tag{4.2}$$

We can obtain a solution to the boundary value problem constituted by equations (1.4), (1.5), (1.6), (4.1) and (4.2) if we apply again the principle of minimum complementary energy. Let us denote the stress function for region B_{i} by $V_{\mathrm{i}} = V_{\mathrm{i}}(r)$ $(\mathrm{i} = 1, 2)$. The statically admissible stress fields should satisfy both the equations of equilibrium (3.1), (3.2) and the stress boundary conditions $(1.4)_1$, $(1.5)_1$, (1.6). It is obvious that the traction continuity conditions given by equations (4.2) should also be fulfilled. Formulae for the statically admissible stresses are as follows:

$$\sigma_{\mathrm{ir}} = \frac{V_{\mathrm{i}}(r)}{r^2}\sin\varphi, \qquad \sigma_{\mathrm{i}\varphi} = \frac{1}{r}\frac{\mathrm{d}V_{\mathrm{i}}}{\mathrm{d}r}\sin\varphi, \qquad \tau_{\mathrm{ir}\varphi} = -\frac{V_{\mathrm{i}}(r)}{r^2}\cos\varphi, \qquad (\mathrm{i} = 1, 2), \tag{4.3}$$

where

$$V_1(a) = 0, \qquad V_2(b) = 0 \qquad V_1(c) = V_2(c). \tag{4.4}$$

The total complementary energy for the curved two-layered beam is of the form:

$$\begin{aligned} &\Pi_{\mathrm{c}}(V_1,\,V_2) = \\ &= \frac{\pi t}{2}\left\{\int_a^c \left[S_{111}\left(\frac{V_1}{r^2}\right)^2 + 2S_{112}\frac{V_1}{r^3}\frac{\mathrm{d}V_1}{\mathrm{d}r} + S_{122}\frac{1}{r^2}\left(\frac{\mathrm{d}V_1}{\mathrm{d}r}\right)^2 + S_{166}\left(\frac{V_1}{r^2}\right)^2\right] r\,\mathrm{d}r \,+\right. \\ &\quad \left. + \int_c^b \left[S_{211}\left(\frac{V_2}{r^2}\right)^2 + 2S_{212}\frac{V_2}{r^3}\frac{\mathrm{d}V_2}{\mathrm{d}r} + S_{222}\frac{1}{r^2}\left(\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)^2 + S_{266}\left(\frac{V_2}{r^2}\right)^2\right] r\,\mathrm{d}r - \right\} \\ &\qquad - C\pi t\left\{\int_a^c \frac{V_1}{r^2}\,\mathrm{d}r + \int_c^b \frac{V_2}{r^2}\,\mathrm{d}r\right\}. \end{aligned} \tag{4.5}$$

By means of Castigliano's principle [5, 6] we get from equation (4.5) that

$$\delta\Pi_c = 0 \tag{4.6}$$

where the stress functions $V_1 = V_1(r)$ and $V_2 = V_2(r)$ should be varied under conditions (4.4). After some paper and pencil calculations (details are omitted), equation (4.6) results in the following stationary conditions:

$$\begin{aligned} &-S_{\mathrm{i}22}r^2\frac{\mathrm{d}^2V_{\mathrm{i}}}{\mathrm{d}r^2} + S_{\mathrm{i}22}r\frac{\mathrm{d}V_{\mathrm{i}}}{\mathrm{d}r} + (S_{\mathrm{i}11} + 2S_{\mathrm{i}12} + S_{\mathrm{i}66})\,V_{\mathrm{i}} = Cr, \\ &\qquad a_{\mathrm{i}} \le r \le b_{\mathrm{i}}, \quad (\mathrm{i} = 1, 2), \quad a_1 = a,\ b_1 = c;\ a_2 = c,\ b_2 = b, \end{aligned} \tag{4.7}$$

$$S_{122}\frac{1}{c}\left(\frac{\mathrm{d}V_1}{\mathrm{d}r}\right)_{r=c}+S_{112}\frac{V_1(c)}{c^2}-S_{222}\frac{1}{c}\left(\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)_{r=c}-S_{212}\frac{V_2(c)}{c^2}=0. \tag{4.8}$$

The general solution of the differential equation (4.6) is

$$V_{\mathrm{i}}(r)=\alpha_{\mathrm{i}1}r^{\lambda_{\mathrm{i}1}}+\alpha_{\mathrm{i}2}r^{\lambda_{\mathrm{i}2}}+C_{\mathrm{i}}\,r, \tag{4.9}$$

where $(\mathrm{i}=1,2)$ and

$$C_{\mathrm{i}}=\frac{C}{S_{\mathrm{i}11}+2S_{\mathrm{i}12}+S_{\mathrm{i}22}+S_{\mathrm{i}66}}, \tag{4.10}$$

$$\lambda_{\mathrm{i}1}=1+\sqrt{\frac{S_{\mathrm{i}11}+2S_{\mathrm{i}12}+S_{\mathrm{i}22}+S_{\mathrm{i}66}}{S_{\mathrm{i}22}}}, \tag{4.11}$$

$$\lambda_{\mathrm{i}2}=1-\sqrt{\frac{S_{\mathrm{i}11}+2S_{\mathrm{i}12}+S_{\mathrm{i}22}+S_{\mathrm{i}66}}{S_{\mathrm{i}22}}}. \tag{4.12}$$

The unknown integration constants in the expressions for the stress functions can be computed from the following system of linear equations, which are based on boundary conditions (4.4) and (4.8):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}=\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{bmatrix}. \tag{4.13}$$

Here

$$\alpha_1=\alpha_{11},\quad \alpha_2=\alpha_{12},\quad \alpha_3=\alpha_{21},\quad \alpha_4=\alpha_{22} \tag{4.14}$$

$$\begin{gathered}\beta_1=-C_1a,\quad \beta_2=(C_2-C_1)c,\\ \beta_3=[C_2(S_{212}+S_{222})-C_1(S_{112}+S_{122})]c,\quad \beta_4=-C_2b,\end{gathered} \tag{4.15}$$

$$a_{11}=a^{\lambda_{11}},\quad a_{12}=a^{\lambda_{12}},\quad a_{13}=a_{14}=0, \tag{4.16}$$

$$a_{21}=c^{\lambda_{11}},\quad a_{22}=c^{\lambda_{12}},\quad a_{23}=-c^{\lambda_{21}},\quad a_{24}=-c^{\lambda_{22}}, \tag{4.17}$$

$$\begin{gathered}a_{31}=(S_{112}+\lambda_{11}S_{122})c^{\lambda_{11}},\quad a_{32}=(S_{112}+\lambda_{12}S_{122})c^{\lambda_{12}},\\ a_{33}=-(S_{212}+\lambda_{21}S_{222})c^{\lambda_{21}},\quad a_{34}=-(S_{212}+\lambda_{22}S_{222})c^{\lambda_{22}},\end{gathered} \tag{4.18}$$

$$a_{41}=a_{42}=0,\quad a_{43}=b^{\lambda_{21}},\quad a_{44}=b^{\lambda_{22}}. \tag{4.19}$$

The determination of the connection between the stress resultant F and displacement constant C can be obtained from

$$F=2t\left[\int_a^c\tau_{1\mathrm{r}\varphi}(r,\,\pi)\mathrm{d}r+\int_c^b\tau_{2\mathrm{r}\varphi}(r,\,\pi)\mathrm{d}r\right]=2t\left[\int_a^c\frac{V_1}{r^2}\mathrm{d}r+\int_c^b\frac{V_2}{r^2}\mathrm{d}r\right]. \tag{4.20}$$

A combination of equations (4.9), (4.14) with equation (4.20) gives the final formula for the stress resultant:

$$F=2t\left[\frac{\alpha_1}{\lambda_{11}-1}\left(c^{\lambda_{11}-1}-a^{\lambda_{11}-1}\right)+\frac{\alpha_2}{\lambda_{12}-1}\left(c^{\lambda_{12}-1}-a^{\lambda_{12}-1}\right)+C_1\ln\frac{c}{a}+\right.$$

$$+\frac{\alpha_3}{\lambda_{21}-1}\left(b^{\lambda_{21}-1}-c^{\lambda_{21}-1}\right)+\frac{\alpha_4}{\lambda_{22}-1}\left(b^{\lambda_{22}-1}-c^{\lambda_{22}-1}\right)+C_2\ln\frac{b}{c}\Bigg]. \quad (4.21)$$

Formulae for stresses σ_{ir}, $\sigma_{\mathrm{i}\varphi}$ and $\tau_{\mathrm{ir}\varphi}$ (i = 1, 2) are as follows:

$$\sigma_{\mathrm{ir}} = \left(\alpha_{\mathrm{i}1}r^{\lambda_{\mathrm{i}1}-2}+\alpha_{\mathrm{i}2}r^{\lambda_{\mathrm{i}2}-2}+\frac{C_{\mathrm{i}}}{r}\right)\sin\varphi, \quad (4.22)$$

$$\sigma_{\mathrm{i}\varphi} = \left(\alpha_{\mathrm{i}1}\lambda_{\mathrm{i}1}r^{\lambda_{\mathrm{i}1}-2}+\alpha_{\mathrm{i}2}\lambda_{\mathrm{i}2}r^{\lambda_{\mathrm{i}2}-2}+\frac{C_{\mathrm{i}}}{r}\right)\sin\varphi, \quad (4.23)$$

$$\tau_{\mathrm{ir}\varphi} = -\left(\alpha_{\mathrm{i}1}r^{\lambda_{\mathrm{i}1}-2}+\alpha_{\mathrm{i}2}r^{\lambda_{\mathrm{i}2}-2}+\frac{C_{\mathrm{i}}}{r}\right)\cos\varphi. \quad (4.24)$$

Following the method presented here we can generalize the two-layered solution for the case of more than two layers.

5. Analysis of the displacement continuity conditions at the interface

There are two independent continuity conditions the displacements should fulfill on the common cylindrical boundary surface of the two curved beam components. By the use of the displacement continuity conditions (4.1) we can derive two independent new continuity conditions that can be expressed in terms of strains and stresses. It follows from equation (1.2) that

$$\varepsilon_{1\varphi}(c,\,\varphi)=\varepsilon_{2\varphi}(c,\,\varphi), \qquad 0\le\varphi\le\pi, \quad (5.1)$$

that is

$$c\,S_{112}\,\sigma_{1\mathrm{r}}(c,\,\varphi)+c\,S_{122}\,\sigma_{1\varphi}(c,\,\varphi)=c\,S_{212}\,\sigma_{2\mathrm{r}}(c,\,\varphi)+c\,S_{222}\,\sigma_{2\varphi}(c,\,\varphi). \quad (5.2)$$

By using (4.3) we can rewrite this equation in terms of the stress functions V_1, V_2:

$$S_{112}\frac{V_1(c)}{c}+S_{122}\left(\frac{\mathrm{d}V_1}{\mathrm{d}r}\right)_{r=c}-S_{212}\frac{V_2(c)}{c}-S_{222}\left(\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)_{r=c}=0. \quad (5.3)$$

We remark that this equation is one of the stationarity conditions for the total complementary energy – see equation (4.8).

The exact solutions should satisfy the two independent displacement continuity condition if $r = c$. Next, we formulate a new displacement continuity conditions in terms of stresses. Starting from equations (1.1), (1.2) we can write

$$\frac{\partial^2 v}{\partial r\partial\varphi}=\frac{\partial}{\partial r}\left[r\left(S_{12}\,\sigma_{\mathrm{r}}+S_{22}\,\sigma_{\varphi}\right)\right]-S_{11}\,\sigma_{\mathrm{r}}-S_{12}\,\sigma_{\varphi}. \quad (5.4)$$

It follows from equation (1.3) that

$$\frac{\partial^2 v}{\partial r\partial\varphi}=S_{66}\frac{\partial\tau_{\mathrm{r}\varphi}}{\partial\varphi}-\frac{1}{r}\left(\frac{\partial^2 u}{\partial\varphi^2}-\frac{\partial v}{\partial\varphi}\right). \quad (5.5)$$

A combination of equation (4.19) with equation (5.5) yields

$$q(r,\,\varphi)=\frac{1}{r}\left(\frac{\partial^2 u}{\partial\varphi^2}-\frac{\partial v}{\partial\varphi}\right)=S_{66}\frac{\partial\tau_{\mathrm{r}\varphi}}{\partial\varphi}-\frac{\partial}{\partial r}\left[r\left(S_{12}\,\sigma_{\mathrm{r}}+S_{22}\,\sigma_{\varphi}\right)\right]+S_{11}\,\sigma_{\mathrm{r}}+S_{12}\,\sigma_{\varphi}. \tag{5.6}$$

If equation (4.1) is satisfied at every point on the common cylindrical boundary surface of the curved beam components then it follows that

$$q_1(c,\,\varphi)=\frac{1}{c}\left(\frac{\partial^2 u_1}{\partial\varphi^2}-\frac{\partial v_1}{\partial\varphi}\right)\bigg|_{r=c}=q_2(c,\,\varphi)=\frac{1}{c}\left(\frac{\partial^2 u_2}{\partial\varphi^2}-\frac{\partial v_2}{\partial\varphi}\right)\bigg|_{r=c} \tag{5.7}$$

in which $0\leq\varphi\leq\pi$.

Substitute the stress functions $V_1=V_1(r)$ and $V_2=V_2(r)$ into equation (5.7) by utilizing equations (4.7) and (5.6). After some manipulations we get

$$-\left[S_{122}\frac{1}{c}\left(\frac{\mathrm{d}V_1}{\mathrm{d}r}\right)_{r=c}+S_{112}\frac{V_1(c)}{c^2}\right]+\frac{C}{c}=-\left[S_{222}\frac{1}{c}\left(\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)_{r=c}+S_{212}\frac{V_2(c)}{c^2}\right]+\frac{C}{c}. \tag{5.8}$$

or

$$S_{112}\frac{V_1(c)}{c}+S_{122}\left(\frac{\mathrm{d}V_1}{\mathrm{d}r}\right)_{r=c}-S_{212}\frac{V_2(c)}{c}-S_{222}\left(\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)_{r=c}=0, \tag{5.9}$$

Hence we have proved that two independent displacement continuity conditions (5.1), (5.7) are all satisfied since they can be transformed into equation (4.8) which follows from the stationary condition (4.6).

6. Examples

6.1. **Example 1.** The geometrical and material data of the considered single curved beam are as follows:

$$a=35\,\mathrm{mm},\qquad b=70\,\mathrm{mm},\qquad t=10\,\mathrm{mm};$$

$$S_{11}=0.5525\cdot10^{-5}\,\frac{1}{\mathrm{MPa}},$$

$$S_{12}=S_{21}=-0.1547\cdot10^{-5}\,\frac{1}{\mathrm{MPa}},$$

$$S_{22}=0.9709\cdot10^{-5}\,\frac{1}{\mathrm{MPa}},$$

$$S_{66}=0.1359\cdot10^{-5}\,\frac{1}{\mathrm{MPa}}.$$

The displacement constant $C=-1\,\mathrm{mm}$. The force resultant which belongs to C is $F=-53.59065\,\mathrm{kN}$. This value is obtained by the application of formula (3.17).

The stresses are calculated by using equations (3.18), (3.19), (3.20). They can also be obtained from a FEM solution which is based on the application of the commercial program Abaqus. The results are listed in Table 1.

Table 1. Stresses in single curved beam. Comparison of theoretical and FEM solutions.

Position		σ_{r} [MPa]		σ_{φ} [MPa]		$\tau_{\mathrm{r}\varphi}$ [MPa]	
r [mm]	φ [rad]	Eq. (3.18)	FEM	Eq. (3.19)	FEM	Eq. (3.20)	FEM
35.0	0.0	0.0000	0.0178	0.0000	0.0843	0.0000	0.5754
40.0	0.0	0.0000	0.0094	0.0000	-0.0089	90.4089	90.6799
50.0	0.0	0.0000	0.0084	0.0000	-0.0046	115.5556	115.676
60.0	0.0	0.0000	0.0080	0.0000	-0.0025	66.9516	67.0311
70.0	0.0	0.0000	0.0124	0.0000	0.0939	0.0000	-0.0049
35.0	0.7854	0.0000	-0.4065	-683.5371	-683.1990	0.0000	0.4068
40.0	0.7854	-63.9287	-64.1205	-425.3195	-425.3140	63.9287	64.1204
50.0	0.7854	-81.7101	-81.7951	-68.1447	-68.1905	81.7101	81.7950
60.0	0.7854	-47.3419	-47.3982	169.8475	169.8050	47.3419	47.3982
70.0	0.7854	0.0000	0.0000	341.7685	341.8760	0.0000	-0.0355
35.0	2.3562	0.0000	-0.4065	-683.5371	-683.1990	0.0000	-0.4068
40.0	2.3562	-63.9287	-64.1205	-425.3195	-425.3140	-63.9287	-64.1204
50.0	2.3562	-81.7101	-81.7951	-68.1447	-68.1905	-81.7101	-81.7950
60.0	2.3562	-47.3419	-47.3982	169.8475	169.8050	-47.3419	-47.3982
70.0	2.3562	0.0000	0.0000	341.7685	341.8760	0.0000	-0.0035
35.0	3.1416	0.0000	0.0178	0.0000	0.0843	0.0000	-0.5754
40.0	3.1416	0.0000	0.0094	0.0000	-0.0089	-90.4089	-90.6799
50.0	3.1416	0.0000	0.0084	0.0000	-0.0046	-115.5556	-115.6760
60.0	3.1416	0.0000	0.0080	0.0000	-0.0025	-66.9516	-67.0311
70.0	3.1416	0.0000	0.0124	0.0000	0.0939	0.0000	0.0049

6.2. **Example 2.** Two-layered curved beam. The geometrical and material data of the considered two-layered curved beam made of two different materials are as follows:

$$a = 35\,\mathrm{mm}, \qquad b = 70\,\mathrm{mm}, \qquad c = 50\,\mathrm{mm}, \qquad t = 10\,\mathrm{mm};$$

$$S_{111} = 0.5525 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}}, \qquad S_{211} = 7.14 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}},$$

$$S_{112} = S_{121} = -0.1547 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}}, \qquad S_{212} = S_{221} = -3.19 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}},$$

$$S_{122} = 0.9709 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}}, \qquad S_{222} = 43.76 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}},$$

$$S_{166} = 0.1359 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}}, \qquad S_{266} = 50.70 \cdot 10^{-5}\,\frac{1}{\mathrm{MPa}},$$

The displacement constant is again $C = -1\,\mathrm{mm}$. The force resultant for C is $F = -17.525\,\mathrm{kN}$. For calculating F we have applied formula (4.21).

The stresses calculated with equations (4.22), (4.23), (4.24) are compared to those of a FEM solution obtained by using the commercial program Abaqus. The results are listed in Table 2.

Table 2. Stresses in two-layered curved beam. Comparison of theoretical and FEM solutions.

Position		σ_{r} [MPa]		σ_{φ} [MPa]		$\tau_{\mathrm{r}\varphi}$ [MPa]	
r [mm]	φ [rad]	Eq. (4.22)	FEM	Eq. (4.23)	FEM	Eq. (4.24)	FEM
35.0	0.0000	0.0000	0.0163	0.0000	0.0477	0.0000	0.4494
40.0	0.0000	0.0000	0.0108	0.0000	-0.0069	40.6747	0.0409
50.0	0.0000	0.0000	0.0113	0.0000	0.0440	6.5699	6.6233
50.0	0.0000	0.0000	0.0079	0.0000	0.0026	6.5699	6.5695
60.0	0.0000	0.0000	0.0004	0.0000	0.0000	2.9508	2.9528
70.0	0.0000	0.0000	0.0005	0.0000	0.0036	0.0005	-0.0007
35.0	0.7854	0.0000	-0.3167	-383.7897	-383.4140	0.0000	0.3177
40.0	0.7854	-28.7614	-28.9218	-124.6154	-124.5690	28.7614	28.9216
50.0	0.7854	-4.6456	-4.6862	237.5746	237.7650	4.6456	4.6834
50.0	0.7854	-4.6456	-4.6454	2.4758	4.9515	4.6456	4.6453
60.0	0.7854	-2.0865	-2.0879	6.9181	9.6141	2.0865	2.0879
70.0	0.7854	0.0000	0.0004	10.3726	13.2767	0.0000	-0.0004
35.0	2.3562	0.0000	-0.3167	-383.7897	-383.4140	0.0000	0.3177
40.0	2.3562	-28.7614	-28.9218	-124.6154	-124.5690	-28.7614	-28.9216
50.0	2.3562	-4.6456	-4.6862	237.5746	237.7650	-4.6456	-4.6834
50.0	2.3562	-4.6456	-4.6454	2.4758	4.9515	-4.6456	-4.6453
60.0	2.3562	-2.0865	-2.0879	6.9181	9.6141	-2.0865	-2.0879
70.0	2.3562	0.0000	0.0004	10.3726	13.2767	0.0000	0.0004
35.0	3.1416	0.0000	0.0163	0.0000	0.0477	0.0000	-0.449
40.0	3.1416	0.0000	0.0108	0.0000	-0.0069	-40.6747	-40.9013
50.0	3.1416	0.0000	0.0113	0.0000	0.0440	-6.5699	-6.6233
50.0	3.1416	0.0000	0.0079	0.0000	0.0026	-6.5699	-6.5695
60.0	3.1416	0.0000	0.0004	0.0000	0.0000	-2.9508	-2.9528
70.0	3.1416	0.0000	0.0005	0.0000	0.0036	0.0000	0.0007

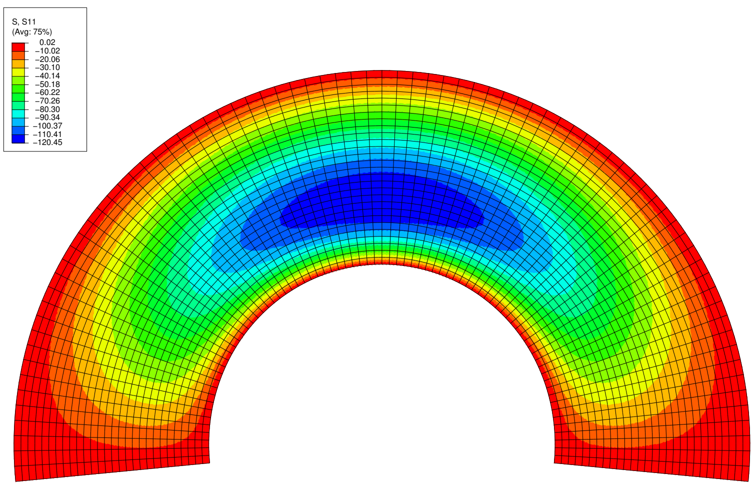

Figure 3. Stress σ_{r} in a curved beam of rectangular cross section (one layer)

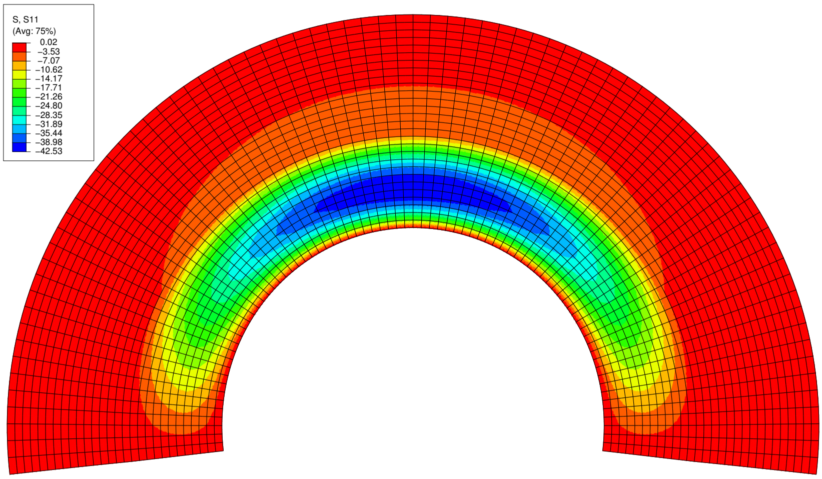

Figure 4. Stress σ_{r} in a curved beam of rectangular cross section (two-layered)

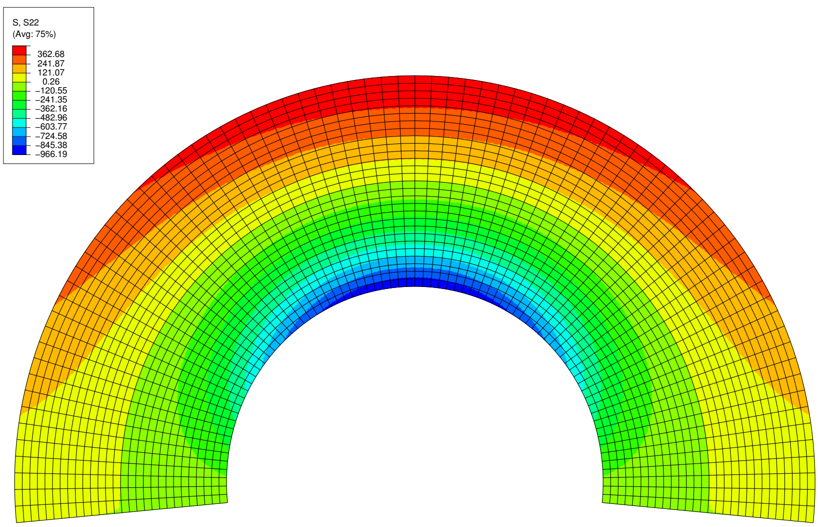

Figure 5. Stress σ_{φ} in a curved beam of rectangular cross section (one layer)

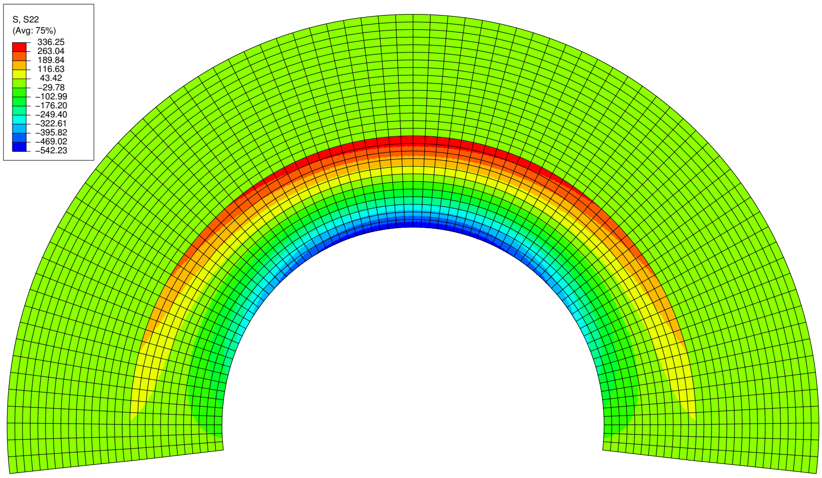

Figure 6. Stress σ_φ in a curved beam of rectangular cross section (two-layered)

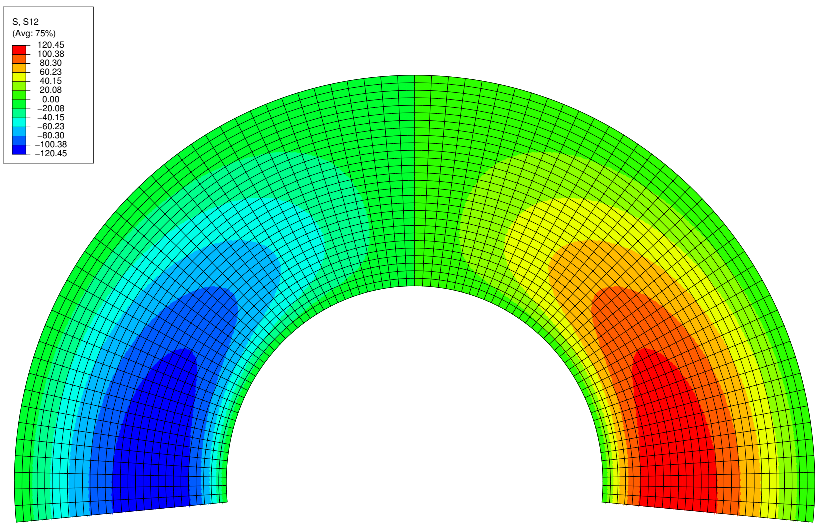

Figure 7. Stress $\tau_{\mathrm{r}\varphi}$ in a curved beam of rectangular cross section (one layer)

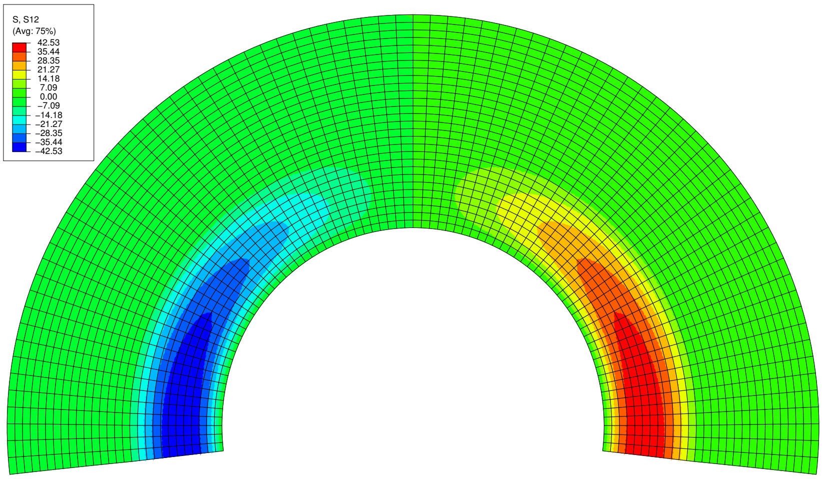

Figure 8. Stress $\tau_{r\varphi}$ in a curved beam of rectangular cross section (two-layered)

Figures 3 and 4 depict a single curved beam and a beam with two layers. Both figures show the stress distribution of the normal stress σ_R. Observe that the computed σ_φ stress distribution is discontinuous, as is expected, only on the two-layered curved beam (see Figures 5 and 6 for details).

The computed stress distribution $\tau_{r\varphi}$ is illustrated for the single curved beam in Figure 7, and for the two-layered beam in Figure 8.

According to results shown in Tables 1 and 2 the theoretical and FEM solutions are in good agreement in both examples.

7. Conclusions

Under the plane strain conditions a mixed type boundary value problem of a curved beam with rectangular cross section is analysed. One- and two-layered curved beams made of polar orthotopic materials are considered. The mixed type boundary value problems are bending problems. For isotropic, homogeneous curved beams this problem was first solved by Golovin [8].

The present paper applies a minimum strain energy property for finding the equations of the considered bending problem. Formulae for the stresses are obtained by means of Castigliano's principle. A detailed analysis is presented for the displacement continuity conditions on the common cylindrical boundary surface of the two-layered curved beam. By applying the method presented in the paper the solutions for the two-layered beam can easily be generalized for the case of beams with more than two layers. The results of the theoretical computations are in good agreement with the FEM solution.

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