

ACCELERATION ANALYSIS OF RIGID BODY MOTION

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*Dedicated to Professor Barna Szabó on the occasion of his eightieth birthday
and to Professor Imre Kozák on the occasion of his eighty-fifth birthday*

Abstract. The aim of this paper is to analyze some second order motion properties of rigid body motion. The existence of unique acceleration center is proven by means of vector-tensor algebra for the case when the vectors of angular velocity and of angular acceleration are linearly independent. The case when the vectors of angular velocity and of angular acceleration are linearly dependent is also considered. Explicit coordinate free relationships are derived for the position of the acceleration center and axis. A detailed analysis of the linear eigenvalue problem arising in the definition of acceleration axis is presented.

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1. INTRODUCTION

Consider a rigid body b moving in general spatial motion with respect to a fixed reference frame $\{0; x, y, z\}$. Let $\boldsymbol{\omega}$ be the angular velocity of body b and let the points A and B be on body b with their velocities denoted by $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. The velocity field of body b is formulated as [1, 2]

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A), \quad (1.1)$$

where $\mathbf{r}_A$ and $\mathbf{r}_B$ are the position vectors of points A and B relative to the fixed frame $\{0; x, y, z\}$ (Figure 1). In equation (1.1), the cross denotes the vectorial product of two vectors. It is known if

$$\mathbf{v}_A \cdot \boldsymbol{\omega} = 0 \quad \text{and} \quad \boldsymbol{\omega} \neq \mathbf{0}, \quad (1.2)$$

then there exists a set of points on body b which have zero velocity at the instant considered. The points which have zero velocity are called velocity center points; they are on a straight line which is parallel to the angular velocity vector $\boldsymbol{\omega}$ [1, 3]. We note that in equation (1.2) the dot between two vectors indicates their scalar product. Assuming that the condition formulated in equation (1.2) is satisfied, then

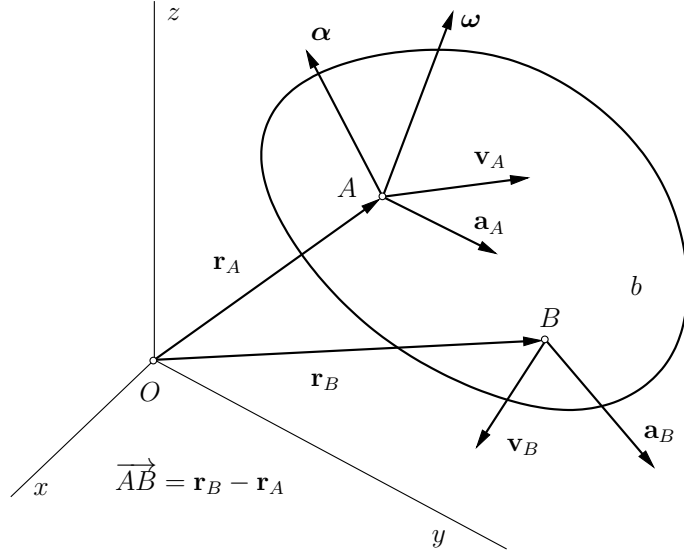


Figure 1. Rigid body in instantaneous general spatial motion.

the position of velocity center points is given by the formula

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2} + p\boldsymbol{\omega} \quad -\infty < p < \infty. \quad (1.3)$$

Equation (1.3) gives the equation of the instantaneous axis of rotation [2, 3]. The validity of formula (1.3) can be checked by a direct substitution of equation (1.3) into equation (1.1).

Differentiation of equation (1.1) with respect to time leads to the formula of acceleration field

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)) , \quad (1.4)$$

where

$$\mathbf{a}_A = \frac{d\mathbf{v}_A}{dt}, \quad \mathbf{a}_B = \frac{d\mathbf{v}_B}{dt} \quad (1.5)$$

are the accelerations of points A and B (Figure 1), t is the time, $\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$ is the angular acceleration of body b .

The aim of this paper is to analyze two second order motion properties of rigid body motion, which are the acceleration center and acceleration axis. The concepts of acceleration center and of acceleration axis are borrowed from the book by Bottema and Roth [1] and a paper by Mohamed [4]. The existence of a unique acceleration center is proven from the case when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are linearly independent vectors by the use of the method of vector and tensor algebra. An explicit coordinate-free relationships will be derived for the position of acceleration center and axis. The case when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are linearly dependent vectors is also considered.

Martinez and Duffy [5] presents a review of papers dealing with the acceleration center. Martinez and Duffy gave the inverse of acceleration matrix introduced by Mohamed [4] but they did not cite Mohamed's paper [4]. In [5], a closed form expression for the inverse of the acceleration matrix in terms of the coordinates of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ is presented. Martinez and Duffy transformed the coordinate representation of the position vector of center of acceleration into a coordinate-free vectorial formula [5].

2. ACCELERATION CENTER

Denote Q the acceleration center of body b at the considered instant and let $\boldsymbol{\rho}$ be defined as $\boldsymbol{\rho} = \mathbf{r}_Q - \mathbf{r}_A$. According to the definition of acceleration center [1, 4, 5] we can write

$$\mathbf{a}_Q = \mathbf{0} = \mathbf{a}_A + \boldsymbol{\alpha} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}). \quad (2.1)$$

Two cases are treated. At first we assume that

$$\boldsymbol{\omega} \times \boldsymbol{\alpha} \neq \mathbf{0} \quad (2.2)$$

and then we investigate the existence of the center of acceleration when

$$\boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0} \quad \text{and} \quad E = \omega^4 + \alpha^2 \neq 0. \quad (2.3)$$

If condition (2.2) holds then the vectors $\mathbf{a}_1 = \boldsymbol{\omega}$, $\mathbf{a}_2 = \boldsymbol{\alpha}$ and $\mathbf{a}_3 = \boldsymbol{\omega} \times \boldsymbol{\alpha}$ form a base of the 3D space.

In this case, we seek the position vector of point Q relative to point A as

$$\boldsymbol{\rho} = p_1 \boldsymbol{\omega} + p_2 \boldsymbol{\alpha} + p_3 \boldsymbol{\omega} \times \boldsymbol{\alpha} \quad (\boldsymbol{\omega} \times \boldsymbol{\alpha} \neq \mathbf{0}). \quad (2.4)$$

Combination of equation (2.1) with equation (2.4) yields

$$p_1 \boldsymbol{\alpha} \times \boldsymbol{\omega} + p_2 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + p_3 \{ \boldsymbol{\alpha} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha})] \} = -\mathbf{a}_A. \quad (2.5)$$

Dot products of vector equation (2.5) with the vectors $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\omega} \times \boldsymbol{\alpha}$ give

$$p_3 (\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 = -\mathbf{a}_A \cdot \boldsymbol{\omega}, \quad (2.6)$$

$$p_2 (\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 = \mathbf{a}_A \cdot \boldsymbol{\alpha}, \quad (2.7)$$

$$p_1 (\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 + p_3 \omega^2 (\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 = \mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}). \quad (2.8)$$

Substitution of equation (2.6) into equation (2.8) leads to the equation

$$p_1 (\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 = \mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\mathbf{a}_A \cdot \boldsymbol{\omega}) \omega^2. \quad (2.9)$$

The combination of equations (2.6), (2.7) and (2.9) with equation (2.4) gives the coordinate-free expression of the position vector of acceleration center Q relative to point A

$$\begin{aligned} \boldsymbol{\rho} = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [& (\mathbf{a}_A \cdot \boldsymbol{\alpha}) \boldsymbol{\alpha} - (\mathbf{a}_A \cdot \boldsymbol{\omega}) (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + \\ & + \{ \mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\boldsymbol{\omega} \cdot \mathbf{a}_A) \omega^2 \} \boldsymbol{\omega}] . \end{aligned} \quad (2.10)$$

By the use of the next identity

$$[(\boldsymbol{\omega} \times \boldsymbol{\alpha}) \times \boldsymbol{\omega}] \times \mathbf{a}_A = [\mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha})] \boldsymbol{\omega} - (\mathbf{a}_A \cdot \boldsymbol{\omega}) \boldsymbol{\omega} \times \boldsymbol{\alpha} \quad (2.11)$$

we can write into a more compact form the expression of position vector $\boldsymbol{\varrho}$ as

$$\boldsymbol{\varrho} = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [\{ (\boldsymbol{\omega} \times \boldsymbol{\alpha}) \times \boldsymbol{\omega} \} \times \mathbf{a}_A + (\boldsymbol{\alpha} \cdot \mathbf{a}_A) \boldsymbol{\alpha} + \omega^2 (\boldsymbol{\omega} \cdot \mathbf{a}_A) \boldsymbol{\omega}] . \quad (2.12)$$

Here, we note

$$(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 = \omega^2 \alpha^2 \sin^2 \gamma , \quad (2.13)$$

where γ is the angle formed by the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ and we have

$$(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 = \omega^2 \alpha^2 - \omega^2 \alpha^2 \cos^2 \gamma = \omega^2 \alpha^2 - (\boldsymbol{\omega} \cdot \boldsymbol{\alpha})^2 , \quad (2.14)$$

$$\omega^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega} , \quad \alpha^2 = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} . \quad (2.15)$$

Mohamed [4] deduced a system of linear equations for the coordinates of vector $\boldsymbol{\varrho}$ in a matrix form. The coefficient matrix of the unknown coordinates of $\boldsymbol{\varrho}$ is called the “acceleration matrix” and it is shown by Mohamed [4], its determinant is opposite to $(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2$. An explicit (closed) form of the inverse matrix of acceleration matrix was not presented by Mohamed [4].

Martinez and Duffy [5] gave the expression of inverse of acceleration matrix in terms of coordinates of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ by means of the symbolic algebra software MapleTM. Martinez and Duffy decomposed the inverse of the acceleration matrix into its symmetric and skew-symmetric parts and they interpreted these parts in the form of vector and scalar products respectively, and they obtained the vector formula (2.10) for the position of acceleration center [5]. Our approach does not use the coordinate representation of equation (2.1) in any Cartesian coordinate system to get the vector formula (2.10) for the acceleration center.

In the Appendix, the geometrical meaning of vectors

$$\boldsymbol{\varrho}_1 = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [(\mathbf{a}_A \cdot \boldsymbol{\alpha}) \boldsymbol{\alpha} - (\mathbf{a}_A \cdot \boldsymbol{\omega}) (\boldsymbol{\omega} \times \boldsymbol{\alpha})] \quad (2.16)$$

and

$$\boldsymbol{\varrho}_2 = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [\mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\boldsymbol{\omega} \cdot \mathbf{a}_A) \omega^2] \boldsymbol{\omega} \quad (2.17)$$

appearing in formula (2.10) is presented.

Next, we analyze the existence of acceleration center under the condition (2.3). In this case the angular velocity vector $\boldsymbol{\omega}$ and angular acceleration vector $\boldsymbol{\alpha}$ are not linearly independent, their vectors are parallel. We again start from equation (2.1) and equation (2.7). Let

$$\boldsymbol{\varrho} = \boldsymbol{\varrho}_0 + p \boldsymbol{\omega} , \quad (2.18)$$

be, where

$$\boldsymbol{\omega} \cdot \boldsymbol{\varrho}_0 = 0 . \quad (2.19)$$

Inserting equation (2.18) into equation (2.1) we obtain

$$\boldsymbol{\alpha} \times \boldsymbol{\varrho}_0 - \omega^2 \boldsymbol{\varrho}_0 = -\mathbf{a}_A . \quad (2.20)$$

According to the condition (2.3) we have

$$\boldsymbol{\omega} = \omega \mathbf{e} , \quad \boldsymbol{\alpha} = \alpha \mathbf{e} , \quad (2.21)$$

where $\mathbf{e}$ is the unit vector directed to parallel to the direction of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, thus we have $\omega = \boldsymbol{\omega} \cdot \mathbf{e}$, $\alpha = \boldsymbol{\alpha} \cdot \mathbf{e}$.

From equation (2.20) we get the condition of existence of acceleration center point which is

$$\mathbf{a}_A \cdot \mathbf{e} = 0. \quad (2.22)$$

The conditions formulated in equations (2.21) and (2.22) are valid for instantaneous plane motion [2, 3]. We look for $\boldsymbol{\varrho}_0$ in the form

$$\boldsymbol{\varrho}_0 = p_1 \mathbf{a}_A + p_2 \mathbf{a}_A \times \mathbf{e}. \quad (2.23)$$

Scalar product equation (2.20) with the vectors $\mathbf{a}_A$ and $\mathbf{a}_A \times \mathbf{e}$ gives the results

$$\alpha (\mathbf{a}_A \times \mathbf{e}) \cdot \boldsymbol{\varrho}_0 - \omega^2 \boldsymbol{\varrho}_0 \cdot \mathbf{a}_A = -\mathbf{a}_A^2, \quad (2.24)$$

$$\alpha \boldsymbol{\varrho}_0 \cdot \mathbf{a}_A + \omega^2 (\mathbf{a}_A \times \mathbf{e}) \cdot \boldsymbol{\varrho}_0 = 0. \quad (2.25)$$

On the other hand from equation (2.23) it follows that

$$p_1 \mathbf{a}_A^2 = \boldsymbol{\varrho}_0 \cdot \mathbf{a}_A, \quad p_2 \mathbf{a}_A^2 = \boldsymbol{\varrho}_0 \cdot (\mathbf{a}_A \times \mathbf{e}). \quad (2.26)$$

Combination of equations (2.24), (2.25) with equations (2.26)_{1,2} gives

$$\omega^2 p_1 - \alpha p_2 = 1, \quad (2.27)$$

$$\alpha p_1 + \omega^2 p_2 = 0, \quad (2.28)$$

from which we have

$$p_1 = \frac{\omega^2}{\omega^4 + \alpha^2}, \quad p_2 = -\frac{\alpha}{\omega^4 + \alpha^2}. \quad (2.29)$$

Substitution of equations (2.29)_{1,2} into equation (2.23) yields the result

$$\boldsymbol{\varrho}_0 = \frac{1}{\alpha^2 + \omega^4} [\omega^2 \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{a}_A]. \quad (2.30)$$

It is evident the location of the acceleration centers, points having zero acceleration, with respect to point A under the conditions (2.3) is given by the formula

$$\boldsymbol{\varrho} = \frac{1}{\alpha^2 + \omega^4} [\omega^2 \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{a}_A] + p \mathbf{e}, \quad -\infty < p < \infty. \quad (2.31)$$

Equation (2.31) defines the line of acceleration centers [4].

3. ACCELERATION ELLIPSOID

According to Mohamed [4] the acceleration ellipsoid is defined as

$$|\mathbf{a}(\mathbf{R})| = |\boldsymbol{\alpha} \times \mathbf{R} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})| = M = \text{constant} \quad (M \geq 0). \quad (3.1)$$

In equation (3.1), $\mathbf{R}$ is the position vector of an arbitrary point P on body b relative to the center of acceleration, that is $\mathbf{R} = \overrightarrow{QP}$ and $\mathbf{a}(\mathbf{R}) = \mathbf{a}_P$. At first, we consider the case when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are linearly independent, the condition (2.2) is valid. We introduce tensor $\mathbf{A}_0$ by the next prescription

$$\mathbf{A}_0 = \mathbf{1} \times \boldsymbol{\alpha} + \boldsymbol{\omega} \circ \boldsymbol{\omega} - \omega^2 \mathbf{1}, \quad (3.2)$$

where $\mathbf{1}$ is the unit tensor of 3D space and a circle between two vectors denotes their tensorial (dyadic) product. The definition of the scalar and vectorial product of a

vector with a tensor and the properties of dyadic product are given by Malvern [6], Mase et al. [7] and Lurje [8]. It is obvious

$$\mathbf{a}(\mathbf{R}) = \mathbf{A}_0 \cdot \mathbf{R}. \quad (3.3)$$

Here, $\mathbf{A}_0$ is a nonsingular tensor, since its determinant [6, 8]

$$\det \mathbf{A}_0 = \frac{(\mathbf{f}_1 \times \mathbf{f}_2) \cdot \mathbf{f}_3}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} \quad (3.4)$$

is non-zero, where

$$\mathbf{f}_1 = \mathbf{A}_0 \cdot \boldsymbol{\omega}, \quad \mathbf{f}_2 = \mathbf{A}_0 \cdot \boldsymbol{\alpha}, \quad \mathbf{f}_3 = \mathbf{A}_0 \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}). \quad (3.5)$$

By a simple calculation we obtain

$$\mathbf{f}_1 = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \mathbf{f}_2 = \boldsymbol{\omega} (\boldsymbol{\omega} \cdot \boldsymbol{\alpha}) - \boldsymbol{\alpha} \omega^2, \quad \mathbf{f}_3 = \boldsymbol{\alpha} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}) - \omega^2 (\boldsymbol{\omega} \times \boldsymbol{\alpha}). \quad (3.6)$$

Inserting the results above obtained into equation (3.4) we have

$$\det \mathbf{A}_0 = -(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2 \quad (3.7)$$

according to the result of Mohamed [4] since the tensor $\mathbf{A}_0$ is the tensorial representation of the acceleration matrix $\boldsymbol{\psi}$ introduced by Mohamed [4]. The tensor

$$\mathbf{A} = \mathbf{A}_0^T \cdot \mathbf{A}_0 \quad (3.8)$$

is a positive definite symmetric tensor since its determinant is positive

$$\det \mathbf{A} = (\boldsymbol{\omega} \times \boldsymbol{\alpha})^4 > 0, \quad (3.9)$$

and for an arbitrary $\mathbf{x}$ vector we have

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} \geq 0. \quad (3.10)$$

We reformulate equation (3.1) as

$$\mathbf{R} \cdot \mathbf{A} \cdot \mathbf{R} = M^2. \quad (3.11)$$

This equation shows that the points whose acceleration vector has a given magnitude lie on an ellipsoid (acceleration ellipsoid). The center point and main axes of the acceleration ellipsoid for different value of M are the same. The common center point of acceleration ellipsoids is the acceleration center.

By a lengthy but elementary calculation starting from equation (3.2) and using the definition of $\mathbf{A}$ we can derive the coordinate-free representation for $\mathbf{A}$ as

$$\mathbf{A} = -\boldsymbol{\alpha} \times \mathbf{1} \times \boldsymbol{\alpha} + \boldsymbol{\omega} \circ (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\boldsymbol{\omega} \times \boldsymbol{\alpha}) \circ \boldsymbol{\omega} - \omega^2 \boldsymbol{\omega} \circ \boldsymbol{\omega} + \omega^4 \mathbf{1}. \quad (3.12)$$

In the second case when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are not linearly independent we have $\boldsymbol{\omega} = \omega \mathbf{e}$, $\boldsymbol{\alpha} = \alpha \mathbf{e}$ and we resolve the vector $\mathbf{R}$ into two components as

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}_1, \quad \mathbf{R}_0 \cdot \mathbf{e} = 0, \quad \mathbf{R}_1 \times \mathbf{e} = \mathbf{0}. \quad (3.13)$$

A simple calculation shows that

$$\mathbf{A}_0 = \alpha \mathbf{1} \times \mathbf{e} + \omega^2 \mathbf{e} \circ \mathbf{e} - \omega^2 \mathbf{1}, \quad (3.14)$$

$$\mathbf{A} = -\alpha^2 \mathbf{e} \times \mathbf{1} \times \mathbf{e} - \omega^4 \mathbf{e} \circ \mathbf{e} + \omega^4 \mathbf{1}, \quad (3.15)$$

$$\mathbf{A}_0 \cdot \mathbf{e} = \mathbf{0} \quad \text{and} \quad \mathbf{A} \cdot \mathbf{e} = \mathbf{0}, \quad (3.16)$$

$$\mathbf{R} \cdot \mathbf{A} \cdot \mathbf{R} = \mathbf{R}_0 \cdot \mathbf{A} \cdot \mathbf{R}_0 = (\alpha^2 + \omega^4) R_0^2. \quad (3.17)$$

From equation (3.17) it follows that the points whose acceleration vector has a given magnitude M and $E = \alpha^2 + \omega^4 \neq 0$ are on a cylindrical surface.

The generators of this cylindrical surface are parallel to the vectors $\mathbf{e}$ and the center point of its base circle is one of the acceleration center given by equation (2.31); the radius of base circle is

$$|\mathbf{R}_0| = \frac{|M|}{\sqrt{\alpha^2 + \omega^4}}. \quad (3.18)$$

It is obvious the normal vector of the plane of base circle is $\mathbf{e}$.

4. ACCELERATION AXIS

At first we compute the angle between the acceleration vector $\mathbf{a}_P = \mathbf{a}(\mathbf{R})$ and the position vector $\mathbf{R} = \overrightarrow{QP} = R\mathbf{q}$. Here, $\mathbf{q}$ is a unit vector and its direction is parallel to the vector $\mathbf{R}$. It is evident that

$$\mathbf{a}(\mathbf{R}) = R\mathbf{a}(\mathbf{q}), \quad \mathbf{a}(\mathbf{q}) = \mathbf{A}_0 \cdot \mathbf{q}. \quad (4.1)$$

Equation (4.1) shows that the acceleration vectors of the points of straight line determined by point Q and unit vector $\mathbf{q}$ ($0 \leq R < \infty$) have same direction and their magnitudes are proportional with $R = |\overrightarrow{QP}|$. The angle between $\mathbf{a}(\mathbf{q})$ and $\mathbf{q}$ is denoted by δ . A simple calculation gives for $\boldsymbol{\omega} \times \boldsymbol{\alpha} \neq \mathbf{0}$

$$\cos \delta = \frac{\mathbf{q} \cdot \mathbf{a}(\mathbf{q})}{|\mathbf{a}(\mathbf{q})|} = \frac{\mathbf{q} \cdot \mathbf{A}_0 \cdot \mathbf{q}}{\sqrt{\mathbf{q} \cdot \mathbf{A} \cdot \mathbf{q}}} = -\frac{(\boldsymbol{\omega} \times \mathbf{q})^2}{\sqrt{\mathbf{q} \cdot \mathbf{A} \cdot \mathbf{q}}}. \quad (4.2)$$

From equation (4.2) it follows that

$$\frac{\pi}{2} \leq \delta \leq \pi. \quad (4.3)$$

If $\boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}$ and $E = \alpha^2 + \omega^4 \neq 0$, then we have $\boldsymbol{\omega} = \omega\mathbf{e}$, $\boldsymbol{\alpha} = \alpha\mathbf{e}$ and we resolve $\mathbf{q}$ into two components as

$$\mathbf{q} = \mathbf{q}_0 + q_1\mathbf{e}, \quad \mathbf{q}_0 \cdot \mathbf{e} = 0, \quad q_0^2 + q_1^2 = 1. \quad (4.4)$$

By a detailed computation which is based on equations (3.14) and (3.15) we obtain

$$\cos \delta = \frac{\mathbf{q} \cdot \mathbf{a}(\mathbf{q})}{|\mathbf{a}(\mathbf{q})|} = -\frac{\omega^2}{\sqrt{\alpha^2 + \omega^4}} |\mathbf{q}_0|, \quad 0 \leq |\mathbf{q}_0| \leq 1. \quad (4.5)$$

The latter case is characterised by the equation (2.21), we consider one of the acceleration centers given by equation (2.31). Let $q_1 = 0$ be in equation (4.4); in this case we have $|\mathbf{q}_0| = 1$. The angle δ for points lying in that plane whose normal vector is $\mathbf{e}$ and contains the chosen acceleration center ($q_1 = 0$) is as follows

$$\cos \delta = -\frac{\omega^2}{\sqrt{\alpha^2 + \omega^4}}. \quad (4.6)$$

Equation (4.6) it is a known result of plane kinematics [2, 3].

Next, we deal with the general 3D motion which is characterised by equation (2.2). For this case the acceleration axis is defined by the equation

$$\cos \delta = -1, \quad \delta = \pi. \quad (4.7)$$

All points P whose acceleration vectors are directed along the position vector $\mathbf{R} = \overrightarrow{QP}$ (Q is the acceleration center) are in a line which is called acceleration axis [1, 4]. Equation (4.7) can also be formulated as an eigenvalue problem

$$\mathbf{A}_0 \cdot \mathbf{R} = \mu \mathbf{R}. \quad (4.8)$$

At first we prove the real eigenvalues of tensor $\mathbf{A}_0$ are non-positive. From equation (4.8) it follows that

$$\mathbf{R} \cdot \mathbf{A}_0 \cdot \mathbf{R} = -(\boldsymbol{\omega} \times \mathbf{R})^2 = \mu R^2, \quad (4.9)$$

i.e.

$$\mu = -\frac{(\boldsymbol{\omega} \times \mathbf{R})^2}{R^2} \leq 0. \quad (4.10)$$

The next bound for the real eigenvalues of tensor $\mathbf{A}_0$ can be derived

$$-\omega^2 \leq \mu \leq 0 \quad (4.11)$$

by the use of (4.10) and

$$(\boldsymbol{\omega} \times \mathbf{R})^2 \leq \omega^2 R^2. \quad (4.12)$$

The scalar product of equation (4.8) with the vectors

$$\mathbf{a}_1 = \boldsymbol{\omega}, \quad \mathbf{a}_2 = \boldsymbol{\alpha}, \quad \mathbf{a}_3 = \boldsymbol{\omega} \times \boldsymbol{\alpha} \quad (4.13)$$

leads to a system of homogenous linear equations

$$\begin{bmatrix} -\mu & 0 & 1 \\ \boldsymbol{\alpha} \cdot \boldsymbol{\omega} & -(\mu + \omega^2) & 0 \\ -\alpha^2 & \boldsymbol{\alpha} \cdot \boldsymbol{\omega} & -(\mu + \omega^2) \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (4.14)$$

for

$$X_1 = \boldsymbol{\omega} \cdot \mathbf{R}, \quad X_2 = \boldsymbol{\alpha} \cdot \mathbf{R}, \quad X_3 = (\boldsymbol{\omega} \times \boldsymbol{\alpha}) \cdot \mathbf{R}. \quad (4.15)$$

The condition of the zero value of the determinant of coefficient matrix of equation (4.14) yields the next characteristic equation for $s = \frac{\mu}{\omega^2}$

$$s^3 + 2s^2 + (1 + h^2)s + h^2 \sin^2 \gamma = 0, \quad (4.16)$$

where

$$h = \frac{\alpha}{\omega^2}. \quad (4.17)$$

A special coordinate representation of the angular velocity and angular acceleration vectors was used by Bottema and Roth to get the characteristic equation for μ [1]. It is very easy to point out that equation (4.16) is the same as which can be obtained from Bottema and Roth's result.

The discriminant of cubic equation (4.16) can be written in the form [9]

$$D = D_1 + D_2, \quad (4.18)$$

where

$$D_1 = \frac{1}{27} \left(h^2 - \frac{1}{3} \right)^3, \quad D_2 = \frac{1}{4} \left(h^2 \sin^2 \gamma - \frac{2}{3} h^2 - \frac{2}{27} \right)^2. \quad (4.19)$$

- For $D < 0$ there are three distinct real roots of equation (4.16). In this case three real acceleration axes exist.
- For $D > 0$, equation (4.16) has only one real root and two complex roots. In this case there is only one real acceleration axis and the other two are imaginary.
- For $D = 0$ the cubic equation (4.16) has three real roots and at least two are equal. If $D_1 = D_2 = 0$ then there are three equal real roots. For the case $D = 0$ and $D_1^2 + D_2^2 \neq 0$ there are two real acceleration axes. If $D_1^2 + D_2^2 = 0$ then there is only one real acceleration axis.

The direction of the acceleration axis is determined by the eigenvector $\mathbf{R}$ corresponding to the eigenvalue $\mu = \omega^2 s$. We look for $\mathbf{R}$ as

$$\mathbf{R} = \lambda_1 \boldsymbol{\alpha} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + \lambda_2 (\boldsymbol{\omega} \times \boldsymbol{\alpha}) \times \boldsymbol{\omega} + \lambda_3 (\boldsymbol{\omega} \times \boldsymbol{\alpha}). \quad (4.20)$$

The vectors

$$\mathbf{h}_1 = \boldsymbol{\alpha} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}), \quad \mathbf{h}_2 = (\boldsymbol{\omega} \times \boldsymbol{\alpha}) \times \boldsymbol{\omega}, \quad \mathbf{h}_3 = \boldsymbol{\omega} \times \boldsymbol{\alpha} \quad (4.21)$$

are linearly independent since

$$(\mathbf{h}_1 \times \mathbf{h}_2) \cdot \mathbf{h}_3 = |(\boldsymbol{\omega} \times \boldsymbol{\alpha})|^4 \neq 0. \quad (4.22)$$

Combination of equation (4.15) with equation (4.20) gives

$$\lambda_1 = \frac{X_1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2}, \quad \lambda_2 = \frac{X_2}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2}, \quad \lambda_3 = \frac{X_3}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2}. \quad (4.23)$$

The relationships between X_2 , X_3 and X_1 for $\mu \neq -\omega^2$ are as follows

$$X_2 = \frac{\boldsymbol{\alpha} \cdot \boldsymbol{\omega}}{\mu + \omega^2} X_1, \quad X_3 = \mu X_1. \quad (4.24)$$

In (4.24) X_1 is an arbitrary constant.

Let

$$X = \frac{X_1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2}. \quad (4.25)$$

The combination of equation (4.20) with equations (4.23), (4.24) and (4.25) gives the equation of acceleration axis for $\mu \neq -\omega^2$

$$\mathbf{R} = X \left[\boldsymbol{\alpha} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + \frac{\boldsymbol{\alpha} \cdot \boldsymbol{\omega}}{\mu + \omega^2} (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + \mu (\boldsymbol{\omega} \times \boldsymbol{\alpha}) \right], \quad -\infty < X < \infty. \quad (4.26)$$

Next, the case of $\mu = -\omega^2$ is analyzed. For this case from equation (4.14) we obtain

$$\omega^2 X_1 + X_3 = 0 \quad (4.27)$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\omega} X_1 = 0 \quad (4.28)$$

$$-\alpha^2 X_1 + \boldsymbol{\alpha} \times \boldsymbol{\omega} X_2 = 0. \quad (4.29)$$

The existence of a nontrivial solution for X_1 , X_2 and X_3 which means that $X_1^2 + X_2^2 + X_3^2 \neq 0$, is

$$\boldsymbol{\alpha} \cdot \boldsymbol{\omega} = 0. \quad (4.30)$$

From equation (4.14) it follows that

$$X_1 = 0, \quad X_3 = 0, \quad X_2 = \boldsymbol{\alpha} \cdot \mathbf{R} = \text{arbitrary}, \quad (4.31)$$

and the equation of acceleration axis has the form

$$\mathbf{R} = \lambda \boldsymbol{\alpha} \quad -\infty < \lambda < \infty \quad (4.32)$$

according to equations (4.15) and (4.31).

The existence of $\mu = -\omega^2$ under the conditions

$$\boldsymbol{\alpha} \times \boldsymbol{\omega} \neq 0 \quad \text{and} \quad \boldsymbol{\alpha} \cdot \boldsymbol{\omega} = 0 \quad (4.33)$$

can be derived from the solution of cubic equation (4.16) with substitution for $\gamma = \frac{\pi}{2}$. In this case we have following solution for s

$$s_1 = -1, \quad \mu_1 = -\omega^2, \quad (4.34)$$

$$s_2 = \frac{-1 + \sqrt{1 - 4h^2}}{2}, \quad \mu_2 = \frac{-\omega^2 + \sqrt{\omega^4 - 4\alpha^2}}{2}, \quad (4.35)$$

$$s_3 = \frac{-1 - \sqrt{1 - 4h^2}}{2}, \quad \mu_3 = \frac{-\omega^2 - \sqrt{\omega^4 - 4\alpha^2}}{2}, \quad (4.36)$$

In what follows the case of $D_1^2 + D_2^2 = 0$ will be considered. For this case we have

$$h^2 = \frac{1}{3}, \quad \sin^2 \gamma = \frac{8}{9}. \quad (4.37)$$

Substitution of results obtained above into the cubic equation (4.16) gives

$$s^3 + 2s^2 + \frac{4}{3}s + \frac{8}{27} = 0, \quad (4.38)$$

i.e.

$$(3s + 2)^3 = 0. \quad (4.39)$$

It is evident that if the conditions formulated in equation (4.37) are satisfied then

$$s_1 = s_2 = s_3 = -\frac{2}{3}, \quad \mu_1 = \mu_2 = \mu_3 = -\frac{2}{3}\omega^2. \quad (4.40)$$

Substitution of equation (4.40)₂ into equation (4.14) gives

$$X_3 = -\frac{2}{3}\omega^2 X_1, \quad X_2 = \pm \frac{\alpha}{\omega} X_1. \quad (4.41)$$

From the equations above obtained it follows that

$$\mathbf{n}_1 \cdot \mathbf{R} = 0, \quad \mathbf{n}_2 \cdot \mathbf{R} = 0, \quad (4.42)$$

where

$$\mathbf{n}_1 = \boldsymbol{\omega} \times \boldsymbol{\alpha} + \frac{2}{3}\omega^2 \boldsymbol{\omega}, \quad (4.43)$$

$$\mathbf{n}_2 = \boldsymbol{\alpha} - \frac{\alpha}{\omega} \boldsymbol{\omega} \quad \text{for} \quad \cos \gamma = \frac{1}{3}, \quad (4.44)$$

$$\mathbf{n}_2 = \boldsymbol{\alpha} + \frac{\alpha}{\omega} \boldsymbol{\omega} \quad \text{for} \quad \cos \gamma = -\frac{1}{3}. \quad (4.45)$$

The intersection of planes whose equations are given by (4.41) is a straight line, which is the real acceleration axis in the considered case. From equation (4.41) it follows that the equation of acceleration axis has the form

$$\mathbf{R} = \lambda(\mathbf{n}_1 \times \mathbf{n}_2) \quad -\infty < \lambda < \infty. \quad (4.46)$$

It is also worth noting that the eigenvalue μ cannot be zero or an imaginary number if $\boldsymbol{\omega} \neq \boldsymbol{\alpha} \neq 0$. The validity of this statement follows from equation (4.16). It is evident, if

$$\boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0} \quad \text{and} \quad \alpha \neq 0 \quad (4.47)$$

then the acceleration axis does not exist, but for the case

$$\boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0} \quad \text{and} \quad \alpha = 0, \quad \omega \neq 0 \quad (4.48)$$

we have

$$\mathbf{a}(\mathbf{q}) = \omega^2 \mathbf{e} \times [\mathbf{e} \times (\mathbf{q}_0 + q_1 \mathbf{e})] = -\omega^2 \mathbf{q}_0 \quad (4.49)$$

i.e., all the line segments $\overline{QP}$ in the plane $q_1 = 0$ can be considered as an acceleration axis.

5. CONCLUSION

This paper deals with the analysis of the acceleration field of a rigid body in a general 3D spatial motion. Some results derived by Mohamed [4] are reformulated and analyzed in detailed forms. Explicit coordinate free relationships are presented for the position of the acceleration center and acceleration axis. A new proof is given to the vectorial formula of the acceleration center which was derived by Martinez and Duffy. Formulation and solution of problems are based on the well known relationships of rigid-body kinematics and the tools of vector–tensor algebra.

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APPENDIX A. REMARK TO THE FORMULA OF ACCELERATION CENTER

Let us consider the point X whose position vector relative to point A is (Figure 2)

$$\overrightarrow{AX} = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [(\boldsymbol{\alpha} \cdot \mathbf{a}_A) \boldsymbol{\alpha} - (\boldsymbol{\omega} \cdot \mathbf{a}_A)(\boldsymbol{\omega} \times \boldsymbol{\alpha})] . \quad (A.1)$$

Application of equation (1.4) gives

$$\begin{aligned} \mathbf{a}_X &= \mathbf{a}_A + \boldsymbol{\alpha} \times \overrightarrow{AX} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \overrightarrow{AX}) = \\ &= \mathbf{a}_A + \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [-(\boldsymbol{\omega} \cdot \mathbf{a}_A) \alpha^2 \boldsymbol{\omega} + (\boldsymbol{\alpha} \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \cdot \mathbf{a}_A) \boldsymbol{\alpha} + \\ &\quad + (\boldsymbol{\omega} \cdot \boldsymbol{\alpha})(\boldsymbol{\alpha} \cdot \mathbf{a}_A) \boldsymbol{\omega} - \omega^2 (\mathbf{a}_A \cdot \boldsymbol{\alpha}) \boldsymbol{\alpha} + \omega^2 (\boldsymbol{\omega} \cdot \mathbf{a}_A) \boldsymbol{\omega} \times \boldsymbol{\alpha}] . \quad (A.2) \end{aligned}$$

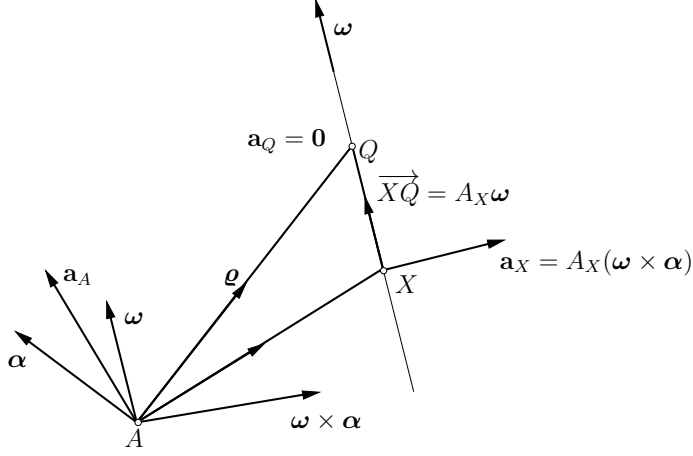


Figure 2. Determination of position of acceleration center

A simple computation shows that

$$\mathbf{a}_X \cdot \boldsymbol{\omega} = 0, \quad \mathbf{a}_X \cdot \boldsymbol{\alpha} = 0, \quad (\text{A.3})$$

$$\mathbf{a}_X \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) = \mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\mathbf{a}_A \cdot \boldsymbol{\omega})\omega^2. \quad (\text{A.4})$$

From equations (A.2), (A.3) and (A.4) it follows that

$$\mathbf{a}_X = A_X(\boldsymbol{\omega} \times \boldsymbol{\alpha}), \quad (\text{A.5})$$

where

$$A_X = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [\mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\mathbf{a}_A \cdot \boldsymbol{\omega})\omega^2], \quad (\text{A.6})$$

The position vector of point Q relative to point X according to formula (2.11) is

$$\begin{aligned} \overrightarrow{XQ} &= \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [\{(\boldsymbol{\omega} \times \boldsymbol{\alpha}) \times \boldsymbol{\omega}\} \times \mathbf{a}_X + (\boldsymbol{\alpha} \cdot \mathbf{a}_X)\boldsymbol{\alpha} + \omega^2(\boldsymbol{\omega} \cdot \mathbf{a}_X)\boldsymbol{\omega}] = \\ &= \frac{A_X}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} \{(\boldsymbol{\omega} \times \boldsymbol{\alpha}) \times \boldsymbol{\omega}\} \times (\boldsymbol{\omega} \times \boldsymbol{\alpha}) = A_X \boldsymbol{\omega}, \end{aligned} \quad (\text{A.7})$$

that is

$$\overrightarrow{XQ} = A_X \boldsymbol{\omega} = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\alpha})^2} [\mathbf{a}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\alpha}) + (\mathbf{a}_A \cdot \boldsymbol{\omega})\omega^2]. \quad (\text{A.8})$$

Comparing equation (2.10) with equations (A.1) and (A.8) we can write

$$\boldsymbol{\varrho} = \overrightarrow{AQ} = \overrightarrow{AX} + \overrightarrow{XQ}, \quad (\text{A.9})$$

where $\boldsymbol{\varrho}_1 = \overrightarrow{AX}$ and $\boldsymbol{\varrho}_2 = \overrightarrow{XQ}$ are given by (A.1) and (A.8), respectively (Figure 2).

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